PARABOLA

Definition: The locus of a point which moves in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line is called a conic section or conic. The fixed point is called focus, the fixed straight line is called directrix and the constant ratio 'e' is called eccentricity of the conic.

If e = 1, then the conic is called a **Parabola.**

If e < 1, then the conic is called an **Ellipse.**

If e > 1, then the conic is called a **Hyperbola.**

Note: The equation of a conic is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Directrix of the Conic: A line L = 0 passing through the focus of a conic is said to be the principal axis of the conic if it is perpendicular to the directrix of the conic.

Vertices: The points of intersection of a conic and its principal axis are called **vertices** of the conic.

Centre: The midpoint o the line segment joining the vertices of a conic is called **centre** of the conic.

Note 1: If a conic has only one vertex then its centre coincides with the vertex.

Note 2: If a conic has two vertices then its centre does not coincide either of the vertices. In this case the conic is called a central conic.

Standard Form: A conic is said to be in the standard form if the principal axis of the conic is x-axis and the centre of the conic is the origin.

Equation of a Parabola in Standard Form:

The equation of a parabola in the standard form is $y^2 = 4ax$.

Proof:

Let S be the focus and L = 0 be the directrix of the parabola.

Let P be a point on the parabola.

Let M, Z be the projections of P, S on the directrix L = 0 respectively.

Let N be the projection of P on SZ.

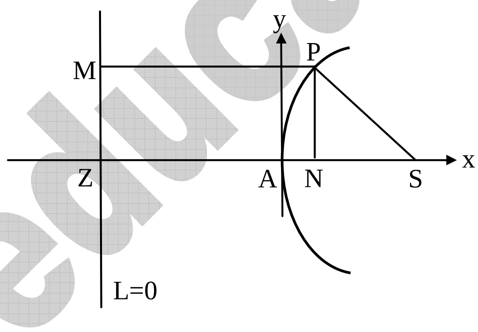
Let A be the midpoint of SZ.

Therefore, SA = AZ, $\Rightarrow$ A lies on the parabola. Let AS = a.

Let AS, the principal axis of the parabola as x-axis and Ay perpendicular to SZ as y-axis.

Then S = (a, 0) and the parabola is in the standard form.

Let P = (x_1, y_1).



Now PM = NZ = NA + AZ = $x_1 + a$

P lies on the parabola $\Rightarrow \dfrac{PS}{PM} = 1 \Rightarrow PS = PM$

$$\Rightarrow \sqrt{(x_1 - a)^2 + (y_1 - 0)^2} = x_1 + a$$
$$\Rightarrow (x_1 - a)^2 + y_1^2 = (x_1 + a)^2$$

$$\Rightarrow y_1^2 = (x_1 + a)^2 - (x_1 - a)^2 \Rightarrow y_1^2 = 4ax_1$$

The locus of P is $y^2 = 4ax$.

$\therefore$ The equation to the parabola is $y^2 = 4ax$.

Nature of the Curve $y^2 = 4ax$.

i) The curve is symmetric with respect to the x-axis.

$\therefore$ The principal axis (x-axis) is an axis of the parabola.

ii) $y = 0 \Rightarrow x = 0$. Thus the curve meets x-axis at only one point (0, 0).

Hence the parabola has only one vertex.

iii) If $x<0$ then there exists no $y \in R$. Thus the parabola does not lie in the second and third quadrants.

iv) If $x > 0$ then $y^2 > 0$ and hence y has two real values (positive and negative). Thus the parabola lies in the first and fourth quadrants.

v) $x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0, 0$. Thus y-axis meets the parabola in two coincident points and hence y-axis touches the parabola at (0, 0).

vi) As $x \to \infty \Rightarrow y^2 \to \infty \Rightarrow y \to \pm\infty$

Thus the curve is not bounded (closed) on the right side of the y-axis.

Double Ordinate: A chord passing through a point P on the parabola and perpendicular to the principal axis of the parabola is called the double ordinate of the point P.

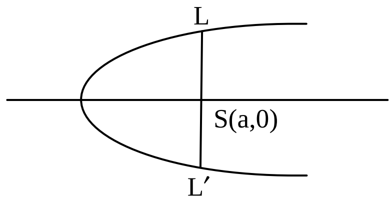
Focal Chord: A chord of the parabola passing through the focus is called a focal chord.

Latus Rectum: A focal chord of a parabola perpendicular to the principal axis of the parabola is called latus rectum. If the latus rectum meets the parabola in L and L′, then LL′ is called length of the latus rectum.

Theorem: The length of the latus rectum of the parabola $y^2 = 4ax$ is 4a.

Proof:

Let LL′ be the length of the latus rectum of the parabola $y^2 = 4ax$.



Let SL = l, then L = (a, l).

Since L is a point on the parabola $y^2 = 4ax$, therefore $l^2 = 4a(a)$

$\Rightarrow l^2 = 4a^2 \Rightarrow l = 2a \Rightarrow SL = 2a$

$\therefore LL' = 2SL = 4a$.

Focal Distance: If P is a point on the parabola with focus S, then SP is called focal distance of P.

Theorem: The focal distance of $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $\mathbf{x_1 + a}$.

Notation: We use the following notation in this chapter

$S \equiv y^2 - 4ax$

$S_1 \equiv yy_1 - 2a(x + x_1)$

$S_{11} = S(x_1, y_1) \equiv y_1^2 - 4ax_1$

$S_{12} \equiv y_1y_2 - 2a(x_1 + x_2)$

Note:

Let $P(x_1, y_1)$ be a point and $S \equiv y^2 - 4ax = 0$ be a parabola. Then

i) P lies on the parabola $\Leftrightarrow S_{11} = 0$

ii) P lies inside the parabola $\Leftrightarrow S_{11} < 0$

iii) P lies outside the parabola $\Leftrightarrow S_{11} > 0$.

Theorem: The equation of the chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ on the parabola $S = 0$ is $S_1 + S_2 = S_{12}$.

Theorem: The equation of the tangent to the parabola $S = 0$ at $P(x_1, y_1)$ is $S_1 = 0$.

Normal:

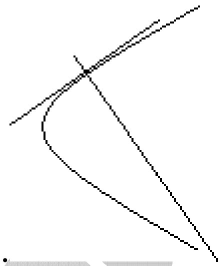
Let $S = 0$ be a parabola and P be a point on the parabola $S = 0$. The line passing through P and perpendicular to the tangent of $S = 0$ at P is called the normal to the parabola $S = 0$ at P.

Theorem: The equation of the normal to the parabola $y^2 = 4ax$ at $P(x_1, y_1)$ is

$y_1(x - x_1) + 2a(y - y_1) = 0$.

Proof:

The equation of the tangent to $S = 0$ at P is $S_1 = 0$



$\Rightarrow yy_1 - 2a(x + x_1) = 0$.

$\Rightarrow yy_1 - 2ax - 2ax_1 = 0$

The equation of the normal to $S = 0$ at P is :

$$y_1(x - x_1) + 2a(y - y_1) = 0$$

Theorem: The condition that the line y = mx + c may be a tangent to the parabola $y^2 = 4ax$ is c = a/m.

Proof:

Equation of the parabola is $y^2 = 4ax$ ----- (1)

Equation of the line is y = mx + c … (2)

Solving (1) and (2),

$$(mx + c)^2 = 4ax \Rightarrow m^2x^2 + c^2 + 2mcx = 4ax$$

$\Rightarrow m^2x^2 + 2(mc - 2a)x + c^2 = 0$ Which is a quadratic equation in x. Therefore it has two roots.

If (2) is a tangent to the parabola, then the roots of the above equation are equal.

$\Rightarrow$ its disc eminent is zero

$$\Rightarrow 4(mc - 2a)^2 - 4m^2c^2 = 0$$
$$\Rightarrow m^2c^2 + 4a^2 - 4amc - m^2c^2 = 0$$
$$\Rightarrow a^2 - amc = 0$$
$$\Rightarrow a = mc$$
$$\Rightarrow c = \frac{a}{m}$$

II Method:

Given parabola is $y^2 = 4ax$.

Equation of the tangent is y = mx + c ----- (1)

Let $P(x_1, y_1)$ be the point of contact.

The equation of the tangent at P is

$yy_1 - 2a(x + x_1) = 0 \Rightarrow yy_1 = 2ax + 2ax_1$ … (2)

Now (1) and (2) represent the same line.

$$\therefore \frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{c} \Rightarrow x_1 = \frac{c}{m}, y_1 = \frac{2a}{m}$$

P lies on the line $y = mx + c \Rightarrow y_1 = mx_1 + c$

$$\Rightarrow \frac{2a}{m} = m\left(\frac{c}{m}\right) + c \Rightarrow \frac{2a}{m} = 2c \Rightarrow c = \frac{a}{m}$$

Note: The equation of a tangent to the parabola

$y^2 = 4ax$ can be taken as $y = mx + a/m$. And the point of contact is $(a/m^2, 2a/m)$.

Corollary: The condition that the line $lx + my + n = 0$ to touché the parabola $y^2 = 4ax$ is $am^2 = ln$.

Proof:

Equation of the parabola is $y^2 = 4ax$ ----- (1)

Equation of the line is $lx + my + n = 0$

$$\Rightarrow\ y = -\frac{l}{m}x - \frac{n}{m}$$

But this line is a tangent to the parabola, therefore

$$C = a/m \ \Rightarrow -\frac{n}{m} = \frac{a}{-l/m} \Rightarrow \frac{n}{m} = \frac{am}{l} \Rightarrow am^2 = ln$$

Hence the condition that the line $lx + my + n = 0$ to touché the parabola $y^2 = 4ax$ is $am^2 = ln$.

Note: The point of contact of $lx + my + n = 0$ with $y^2 = 4ax$ is $(n/l, -2am/l)$.

Theorem: The condition that the line $lx + my + n = 0$ to touch the parabola $x^2 = 4ax$ is $al^2 = mn$.

Proof:

Given line is $lx + my + n = 0$ … (1)

Let $P(x_1, y_1)$ be the point of contact of (1) with the parabola $x^2 = 4ay$.

The equation of the tangent at P to the parabola is $xx_1 = 2a(y + y_1)$

$\Rightarrow x_1x - 2ay - 2ay_1 = 0$ …(2)

Now (1) and (2) represent the same line.

$$\therefore\ \frac{x_1}{l} = -\frac{2a}{m} = -\frac{2ay_1}{n} \Rightarrow x_1 = -\frac{2al}{m}, y_1 = \frac{n}{m}$$

P lies on the line $lx + my + n = 0$

$\Rightarrow lx_1 + my_1 + n = 0 \Rightarrow l\left(\frac{-2al}{m}\right) + m\left(\frac{n}{m}\right) + n = 0$

$\Rightarrow -2al^2 + mn + mn = 0 \Rightarrow al^2 = mn$,

Theorem: Two tangents can be drawn to a parabola from an external point.

Note:

1. If m_1, m_2 are the slopes of the tangents through P, then m_1, m_2 become the roots of equation (1). Hence $m_1 + m_2 = y_1/x_1$, $m_1m_2 = a/x_1$.

2: If P is a point on the parabola S = 0 then the roots of equation (1) coincide and hence only one tangent can be drawn to the parabola through P.

3: If P is an internal point to the parabola
S = 0 then the roots of (1) are imaginary and hence no tangent can be drawn to the parabola through P.

Theorem: The equation in the chord of contact of $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.

Theorem: The equation of the chord of the parabola S = 0 having $P(x_1, y_1)$ as its midpoint is $S_1 = S_{11}$.

Pair of Tangents:

Theorem: The equation to the pair of tangents to the parabola S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

Parametric Equations of the Parabola:

A point (x, y) on the parabola $y^2 = 4ax$ can be represented as $x = at^2$, $y = 2at$ in a single parameter t. Theses equations are called parametric equations of the parabola $y^2 = 4ax$. The point $(at^2, 2at)$ is simply denoted by t.

Theorem: The equation of the tangent at $(at^2, 2at)$ to the parabola is $y^2 = 4ax$ is $yt = x + at^2$.

Proof:

Equation of the parabola is $y^2 = 4ax$.

Equation of the tangent at $(at^2, 2at)$ is $S_1 = 0$.

$$\Rightarrow (2at)y - 2a(x + at^2) = 0$$

$$\Rightarrow 2aty = 2a(x + at^2) \Rightarrow yt = x + at^2.$$

Theorem: The equation of the normal to the parabola $y^2 = 4ax$ at the point t is $y + xt = 2at + at^3$.

Proof:

Equation of the parabola is $y^2 = 4ax$.

The equation of the tangent at t is:

$$yt = x + at^2 \Rightarrow x - yt + at^2 = 0$$

The equation of the normal at $(at^2, 2at)$ is

$$t(x - at^2) + 1(y - 2at) = 0$$

$$\Rightarrow xt - at^3 + y - 2at = 0 \Rightarrow y + xt = 2at + at^3$$

Theorem: The equation of the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ is $y(t_1+t_2) = 2x + 2at_1t_2$.

Proof:

Equation of the parabola is $y^2 = 4ax$.

Given points on the parabola are

$$P(at_1^2, 2at_1),\ Q(at_2^2, 2at_2).$$

Slope of $\overleftrightarrow{PQ}$ is

$$\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2}$$

The equation of $\overleftrightarrow{PQ}$ is $y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$.

$\Rightarrow (y - 2at_1)(t_1 + t_2) = 2(x - at_1^2)$

$\Rightarrow y(t_1 + t_2) - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$

$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$.

Note: If the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ is a focal chord then $t_1t_2 = -1$.

Proof:

Equation of the parabola is $y^2 = 4ax$

Focus S = (a, o)

The equation of the chord is $y(t_1 + t_2) = 2x + 2at_1t_2$

If this is a focal chord then it passes through the focus (a, 0).

$\therefore$ $0 = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$.

Theorem: The point of intersection of the tangents to the parabola $y^2 = 4ax$ at the points t_1 and t_2 is $(at_1t_2, a[t_1 + t_2])$.

Proof:

Equation of the parabola is $y^2 = 4ax$

The equation of the tangent at t_1 is $yt_1 = x + at_1^2$ … (1)

The equation of the tangent at t_2 is

$yt_2 = x + at_2^2$ …………… (2)

(1) – (2) $\Rightarrow y(t_1 - t_2) = a\left(t_1^2 - t_2^2\right) \Rightarrow y = a(t_1 + t_2)$

(1) $\Rightarrow a(t_1 + t_2)t_1 = x + at_1^2$

$\Rightarrow at_1^2 + at_1t_2 = x + at_1^2 \Rightarrow x = at_1t_2$

$\therefore$ Point of intersection = $(at_1t_2, a[t_1 + t_2])$.

Theorem: Three normals can be drawn form a point (x_1, y_1) to the parabola $y^2 = 4ax$.

Corollary: If the normal at t_1 and t_2 to the parabola $y^2 = 4ax$ meet on the parabola, then $t_1t_2 = 2$.

Proof:

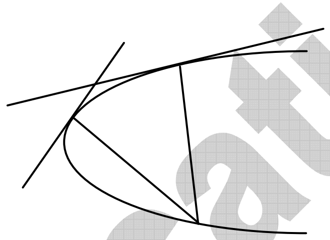
Let the normals at t_1 and t_2 meet at t_3 on the parabola.

The equation of the normal at t_1 is:

$$y + xt_1 = 2at_1 + at_1^3 \qquad \ldots(1)$$

Equation of the chord joining t_1 and t_3 is:

$$y(t_1 + t_3) = 2x + 2at_1t_3 \quad \ldots(2)$$



(1) and (2) represent the same line

$$\therefore \quad \frac{t_1 + t_3}{1} = \frac{-2}{t_1} \Rightarrow t_3 = -t_1 - \frac{2}{t_1}$$

Similarly $t_3 = -t_2 - \frac{2}{t_2}$

$$\therefore \quad -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2} \Rightarrow t_1 - t_2 = \frac{2}{t_2} - \frac{2}{t_1}$$

$$\Rightarrow t_1 - t_2 = \frac{2(t_1 - t_2)}{t_1t_2} \Rightarrow t_1t_2 = 2$$

Very Short Answer Questions

1. **Find the vertex and focus of $4y^2 + 12x - 20y + 67 = 0$.**

Sol. Given parábola

$4y^2 + 12x - 20y + 67 = 0$

$4y^2 - 20y = -12x - 67$

$$y^2 - 5y = -3x - \frac{67}{4}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 - \frac{25}{4} = -3x - \frac{67}{4}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4} = -3\left(x + \frac{7}{2}\right)$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3\left[x - \left(-\frac{7}{2}\right)\right]$$

$$\therefore h = -\frac{7}{2},\ k = \frac{5}{2},\ a = \frac{3}{4}$$

Vertex A is $\left(-\frac{7}{2}, \frac{5}{2}\right)$

Focus is s (h-a, k) = $\left(-\frac{7}{2} - \frac{3}{4}, \frac{5}{2}\right) = \left(\frac{-17}{4}, \frac{5}{2}\right)$

2. **Find the vertex and focus of $x^2 - 6x - 6y + 6 = 0$.**

Sol. Given parabola is

$x^2 - 6x - 6y + 6 = 0$

$x^2 - 6x = 6y - 6$

$(x - 3)^2 - 9 = 6y - 6$

$(x - 3)^2 = 6y + 3$

$$(x - 3)^2 = 6\left(y + \frac{1}{2}\right) = 6\left[y - \left(\frac{-1}{2}\right)\right]$$

$\therefore$ h = 3, k = $\frac{-1}{2}$, a = $\frac{6}{4} = \frac{3}{2}$

Vertex = (h, k) = $\left(3, \frac{-1}{2}\right)$

Focus = (h, k+a) = $\left(3, -\frac{1}{2} + \frac{3}{2}\right) = (3, 1)$

3. Find the equations of axis and directrix of the parabola $y^2 + 6y - 2x + 5 = 0$.

Sol. Given parabola is $y^2 + 6y = 2x - 5$

$[y - (-3)]^2 - 9 = 2x - 5$

$[y - (-3)]^2 = 2x - 5 + 9$

$[y - (-3)]^2 = 2x + 4$

$[y - (-3)]^2 = 2[x - (-2)]$

Comparing with $(y - k)^2 = 4a(x - h)$ we get,

(h, k) = (–2, –3), a = $\frac{1}{2}$

Equation of the axis y – k = 0 i.e. y + 3 = 0

Equation of the directrix x – h + a = 0

i.e. $x - (-2) + \frac{1}{2} = 0$

2x + 5 = 0.

4. Find the equation of axis and directrix of the parabola $4x^2 + 12x - 20y + 67 = 0$.

Sol. Given parabola $4x^2 + 12x - 20y + 67 = 0$

$4x^2 + 12x = 20y - 67$

$x^2 + 3x = 5y - \frac{67}{4}$

$\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} = 5y - \frac{67}{4}$

$$\left(x+\frac{3}{2}\right)^2 = 5y-\frac{58}{4} = 5y-\frac{29}{2}$$

$$\left[x-\left(-\frac{3}{2}\right)\right]^2 = 5\left[y-\frac{29}{10}\right]$$

Comparing with $(x - h)^2 = 4a(y - k)$

$(h, k) = \left(-\frac{3}{2}, \frac{29}{10}\right)$, $a = \frac{5}{4}$

Equation of the axis $x - h = 0$

i.e. $x+\frac{3}{2}=0 \Rightarrow 2x + 3 = 0$

Equation of the directrix, $y - k + a = 0$

$$y-\frac{29}{10}+\frac{5}{4}=0 \Rightarrow 20y-33=0$$

5. Find the equation of the parabola whose focus is s(1, –7) and vertex is A(1, –2).

Sol.

Focus $s = (1, -7)$, vertex $A(1, -2)$

$h = 1, k = -2, a = -2 + 7 = 5$

Since x coordinates of S and A are equal, axis of the parabola is parallel to y-axis.

And the y coordinate of S is less than that of A, therefore the parabola is a down ward parabola.

Let equation of the parabola be

$(x - h)^2 = -4a(y - k)$

$(x - 1)^2 = -20(y + 2)$

$x^2 - 2x + 1 = -20y - 40$

$\Rightarrow x^2 - 2x + 20y + 41 = 0$

6. Find the equation of the parabola whose focus is S(3, 5) and vertex is A(1, 3).

Sol.

Focus S(3, 5) and vertex A(1, 3)`

Let Z (x, y) be the projection of S on directrix. The A is the midpoint of SZ.

$$\Rightarrow (1,3) = \left(\frac{3+x}{2}, \frac{5+y}{2}\right) \Rightarrow x = -1, y = 1$$

Z =(-1, 1)

Slope of directrix = -1/(slope of SA)

$$= \frac{-1}{\left(\frac{5-3}{3-1}\right)} = -1$$

Equation of directrix is y-1 = -1 (x+1)

i.e., x + y =0 ----(1)

Let P (x,y) be any point on the parabola. Then

SP = PM $\Rightarrow$ $SP^2 = PM^2$ where PM is the perpendicular from P to the directrix.

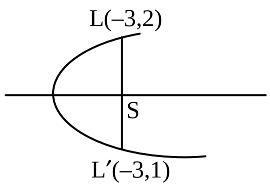
$$\Rightarrow (x-3)^2 + (y-5)^2 = \frac{(x+y)^2}{1+1}$$

$$\Rightarrow 2(x^2 - 6x + 9 + y^2 - 10y + 25) = (x + y)^2$$

$$\Rightarrow 2x^2 + 2y^2 - 12x - 20y + 68 = x^2 + 2xy + y^2$$

i.e. $x^2 - 2xy + y^2 - 12x - 20y + 68 = 0$.

7. **Find the equation of the parabola whose latus rectum is the line segment of joining the points (–3, 2) and (–3, 1).**



Sol. Ends of the latus rectum are L(–3, 2) and L′(–3, 1) .

Length of the latus rectum is $LL' = \sqrt{(-3+3)^2 + (2-1)^2} = \sqrt{0+1} = 1$ (= 4a)

$$\Rightarrow 4|a| = 1 \Rightarrow |a| = \frac{1}{4} \Rightarrow a = \pm\frac{1}{4}$$

S is the midpoint of LL′

$$\Rightarrow S = \left(-3, \frac{3}{2}\right)$$

Case I: a = –1/4

$$\Rightarrow A = \left[-3 + \frac{1}{4}, \frac{3}{2}\right]$$

Equation of the parabola is

$$\left(y - \frac{3}{2}\right)^2 = -\left(x + 3 - \frac{1}{4}\right)$$

$$\Rightarrow \frac{(2y-3)^2}{4} = \frac{-(4x+12-1)}{4}$$

$$\Rightarrow (2y-3)^2 = -(4x+11)$$

Case II: a = 1/4

$$\Rightarrow A = \left[-3 - \frac{1}{4}, \frac{3}{2}\right]$$

Equation of the parabola is

$$\left(y-\frac{3}{2}\right)^2=\left(x+3+\frac{1}{4}\right)$$

$$\Rightarrow \frac{(2y-3)^2}{4}=\frac{(4x+12+1)}{4}$$

$$\Rightarrow (2y-3)^2=4x+13$$

8. Find the position (interior or exterior or on) of the following points with respect to the parabola $y^2 = 6x$.

(i) (6, –6), (ii) (0, 1), (iii) (2, 3)

Sol. Equation of the parabola is $y^2 = 6x$

$\Rightarrow S \equiv y^2 - 6x$

i) $S_{11} = (-6)^2 - 6.6 = 36 - 36 = 0$

$\therefore$ (6, –6) lies on the parabola.

ii) (0, 1)

$S_{11} = 1^2 - 6.0 = 1 > 0$

$\therefore$ (0, 1) lies outside the parabola.

iii)(2, 3)

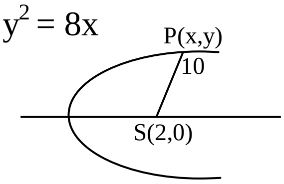
$S_{11} = 3^2 - 6.2 = 9 - 12 = -3 < 0$

$\therefore$ (2, 3) lies inside the parabola.

9. Find the coordinates of the point on the parabola $y^2 = 8x$ whose focal distance is 10.

Sol. Equation of the parabola is $y^2 = 8x$

$4a = 8 \Rightarrow a = 2$



$\Rightarrow S = (2, 0)$

Let P(x, y) be a point on the parabola

Given SP = 10

$\Rightarrow |x+a| = 10 \Rightarrow x+2 = \pm 10$

$\Rightarrow x = 8$ or -12

II Method:

Given SP = 10 $\Rightarrow$ $SP^2 = 100$

$(x - 2)^2 + y^2 = 100$

But $y^2 = 8x$

$\Rightarrow (x - 2)^2 + 8x = 100$

$\Rightarrow x^2 - 4x + 4 + 8x - 100 = 0$

$\Rightarrow x^2 + 4x - 96 = 0 \Rightarrow (x + 12)(x - 8) = 0$

$x + 12 = 0 \Rightarrow x = -12$

$x - 8 = 0 \Rightarrow x = 8$

Case I: x = 8

$y^2 = 8x = 8 \times 8 = 64$

$y = \pm 8$

Coordinates of the required points are (8, 8) and (8, –8)

Case II: x = –12

$y^2 = 8 \times -12 = -96 < 0$

y is not real.

10. If (1/2, 2) is one extremity of a focal chord of the parabola $y^2 = 8x$. Find the coordinates of the other extremity.

Sol.

Given parabola $y^2 = 8x$

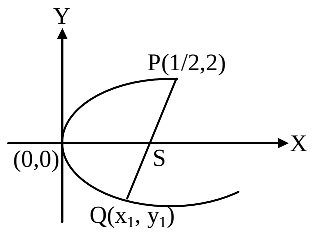
focus $S = (2, 0)$

One end of the focal chord is P $\left(\frac{1}{2}, 2\right)$,

Let $Q = (x_1, y_1)$ be the other end of the focal chord.

Q is a point on the parabola, $y_1^2 = 8x_1 \Rightarrow x_1 = \frac{y_1^2}{8}$

$$\Rightarrow Q = \left(\frac{y_1^2}{8}, y_1\right)$$



Slope of SP = $\frac{0-2}{2-\frac{1}{2}} = \frac{-4}{3}$

Slope of SQ = $\frac{y_1 - 0}{\frac{y_1^2}{8} - 2} = \frac{8y_1}{y_1^2 - 16} = \frac{-4}{3}$

PSQ is a focal chord $\Rightarrow$ the points P, S, Q are collinear.

Therefore, Slope of SP = Slope of SQ

$24y_1 = -4y_1^2 + 64$

$\Rightarrow 4y_1^2 + 24y_1 - 64 = 0$

$\Rightarrow y_1^2 + 6y_1 - 16 = 0$

$\Rightarrow (y_1 + 8)(y_1 - 2) = 0$

$y_1 = 2, -8\ ;\ x_1 = \frac{1}{2}, 8$

Therefore (8, –8) other extremity.

(If $x_1 = \frac{1}{2}$ then $y_1 = 2$ which is the given point .)

11. 1. Find equation of the tangent and normal to the parabola $y^2 = 6x$ at the positive end of the latus rectum.

Sol. Equation of parabola $y^2 = 6x$

$4a = 6 \Rightarrow a = 3/2$

Positive end of the Latus rectum is (a, 2a) = $\left(\frac{3}{2}, 3\right)$

Equation of tangent $yy_1 = 2a(x + x_1)$

$yy_1 = 3(x + x_1)$

$3y = 3\left(x + \frac{3}{2}\right)$

$2y - 2x - 3 = 0$ is the equation of tangent

Slope of tangent is 1

Slope of normal is –1

Equation of normal is $y - 3 = -1\left(x - \frac{3}{2}\right)$

$2x + 2y - 9 = 0$

12. Find the equation of the tangent and normal to the parabola $x^2 - 4x - 8y + 12 = 0$ at (4, 3/2).

Sol.

Equation of the parabola is

$x^2 - 4x - 8y + 12 = 0$

$\Rightarrow (x - 2)^2 - 4 = 8y - 12$

$\Rightarrow (x - 2)^2 = 8y - 8$

$\Rightarrow (x - 2)^2 = 8(y - 1)$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Equation of tangents at (x_1, y_1) is

$$(x-2)(x_1-2)=2a(y-1+y_1-1)$$
$$(x-2)(4-2)=2a\left(y-1+\frac{3}{2}-1\right)$$
$$2(x-2)=4\left(\frac{2y-1}{2}\right)$$
$$x-2y-1=0$$

Equation of normal will be

$y - y_1 = m(x - x_1)$

m-slope of normal

Slope of tangent is: 1/2

Slope of normal is: –2

$$y-\frac{3}{2}=-2(x-4)$$
$$2y-3=-4x+16$$
$$4x+2y-19=0$$

13. Find the value of k if the line 2y = 5x + k is a tangent to the parabola y^2 = 6x.

Sol.

Equation of the parabola is $y^2 = 6x$

Given line is 2y = 5x + k

$$\Rightarrow y=\left(\frac{5}{2}\right)x+\left(\frac{k}{2}\right)$$

Therefore $m = \frac{5}{2}$, $c = \frac{k}{2}$

$y=\left(\frac{5}{2}\right)x+\left(\frac{k}{2}\right)$ is a tangent to $y^2 = 6x$

$$\Rightarrow c=\frac{a}{m}\Rightarrow\frac{k}{2}=\frac{3/2}{5/2}\Rightarrow k=\frac{6}{5}$$

14. Find the equation of the normal to the parabola $y^2 = 4x$ which is parallel to $y - 2x + 5 = 0$.

Sol. Given the parabola is $y^2 = 4x$

$\therefore a = 1$

Given line $y - 2x + 5 = 0$

Slope $m = 2$

The normal is parallel to the line $y - 2x + 5 = 0$

Slope of the normal = 2

Equation of the normal at 't' is $y + tx = 2at + at^3$

$\therefore$ Slope $= -t = 2$ ($\Rightarrow t = -2$)

Equation of the normal is $y - 2x = 2 \cdot 1(-2) + 1(-2)^3 = -4 - 8 = -12$

$$2x - y - 12 = 0.$$

15. Show that the line $2x - y + 2 = 0$ is a tangent to the parabola $y^2 = 16x$. Find the point of contact also.

Sol. Given parabola is $y^2 = 16x$

$\Rightarrow 4a = 16 \Rightarrow a = 4$

Given line is $2x - y + 2 = 0$

$y = 2x + 2$

$\Rightarrow m = 2, c = 2$

$\frac{a}{m} = \frac{4}{2} = 2 = c$

Therefore given line is a tangent to the parabola.

$\therefore$ Point of contact =

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{2^2}, \frac{2(4)}{2}\right) = (1, 4)$$

16. Find the equation of tangent to the parabola $y^2 = 16x$ inclined at an angle 60° with its axis and also find the point of contact.

Sol.

Given parabola $y^2 = 16x$

Inclination of the tangent is

$\theta = 60° \Rightarrow m = \tan 60° = \sqrt{3}$

Therefore equation of the tangent is $y = mx + \frac{a}{m}$

$$\Rightarrow y = \sqrt{3}x + \frac{4}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}y = 3x + 4$$

Point of contact = $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$

17. A double ordinate of the curve $y^2 = 4ax$ is of length 8a. Prove that the line from the vertex to its ends are at right angles.

Sol. Let P = $(at^2, 2at)$ and P′ = $(at^2, -2at)$ be the ends of double ordinate PP′. Then

$$8a = PP' = \sqrt{0 + (4at)^2} = 4at \Rightarrow t = 2$$

$\therefore$ P = (4a, 4a), P′ = (4a, –4a)

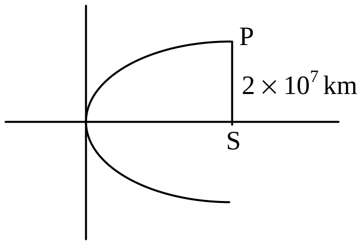
Slope of $\overline{AP} \times$ slope of $\overline{AP'}$

$$= \left(\frac{4a}{4a}\right)\left(-\frac{4a}{4a}\right) = -1$$

$$\therefore \angle PAP' = \frac{\pi}{2}$$

18. A comet moves in a parabolic orbit with the sun as the focus. When the comet is 2×10^7 km from the sun, the line from the sun to it makes an angle $\pi/2$ with the axis of the orbit. Find how near the comet comes to the sun.

Sol:



Let the equation of the parabolic orbit be $y^2 = 4ax$

Let S be the position of the sun (focus) on the axis of parabola. Let P be the position of comet when it is at a distance of 2×10^7 Km from the Sun S.

$\therefore SP = 2\times10^7$

$\Rightarrow 2a = 2\times10^7\,\text{Km}$

$\Rightarrow a = 10^7\,\text{Km}$

The distance of the comet from the Sun S is minimum when it is at the vertex.

$\therefore$Nearest distance of the comet from the sun S is SA = a = 10^7 Km.

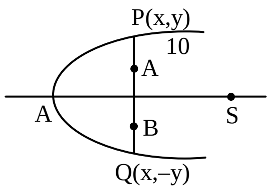
Short Answer Questions

1. Find the locus of the points of trisection of double ordinate of a parabola $y^2 = 4ax$ (a > 0).

Sol.

Given parabola is $y^2 = 4ax$

Let P(x, y) and Q(x, –y) be the ends of the double ordinate.



Let A, B be the points of trisection of the double ordinate.

A divides PQ in the ratio 1: 2.

$$\Rightarrow A = \left(x, \frac{-y+2y}{3}\right) = \left(x, \frac{y}{3}\right)$$

B divides PQ in the ratio 2 : 1

Coordinates of B are $\left(x, \frac{y-2y}{3}\right) = \left(x, -\frac{y}{3}\right)$

Let (x_1, y_1) be the coordinates of the points of trisection.

Then $x_1 = x$ and

$$y_1 = \pm\frac{y}{3} \Rightarrow y_1^2 = \frac{y^2}{9} \Rightarrow y^2 = 9y_1^2$$

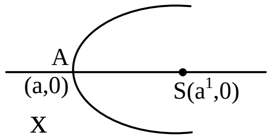
$$4ax_1 = 9y_1^2$$

Locus of (x_1, y_1) is $9y^2 = 4ax$.

2. Find the equation of the parabola whose vertex and focus are on the positive x-axis at a distance of a and a′ from origin respectively.

Sol. Vertex A (a, 0) and focus S (a′, 0)

AS = a′ – a



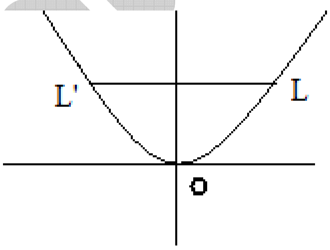
Latus rectum = 4 (a′ – a)

Equation of the parabola is $y^2 = 4(a' - a)(x - a)$

3. If L and L′ are the ends of the latus rectum of the parabola $x^2 = 6y$. Find the equations of OL and OL′ where O is the origin. Also find the angle between them.

Sol. Given parabola is $x^2 = 6y$

Curve is symmetric about Y-axis



$4a = 6 \Rightarrow a = \frac{3}{2}$

$L = (2a,\ a) = \left(3, \frac{3}{2}\right)$ and $L' = (-2a,\ a) = \left(-3, \frac{3}{2}\right)$

Now equation of OL is $x = 2y$

And equation of OL′ is $x = -2y$

Let θ be the angle between the lines, then

$$\tan\theta = \left|\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{4}}\right| = \frac{4}{3} \Rightarrow \theta = \mathrm{Tan}^{-1}\left(\frac{4}{3}\right)$$

4. Find the equation of the parabola whose axis is parallel to x-axis and which passes through these points. A(–2, 1), B(1, 2), C(–1, 3)

Sol.

Given that axis of the parabola is parallel to X-axis,

Let the equation of the parabola be $x = ay^2 + by + c$

It is Passing through (–2, 1), (1, 2) , (–1, 3)

$(-2, 1) \Rightarrow -2 = a + b + c$ … (i)

$(1, 2) \Rightarrow 1 = 4a + 2b + c$ … (ii)

$(-1, 3) \Rightarrow -1 = 9a + 3b + c$ … (iii)

(ii) – (iii) $2 = -5a - b$

(ii) – (i) $3 = 3a + b$

$5 = -2a$

$$a = -\frac{5}{2}, b = \frac{21}{2}, c = -10$$

$$x = -\frac{5}{2}y^2 + \frac{21}{2}y - 10$$

$$5y^2 + 2x - 21y + 20 = 0$$

5. Find the equation of the parabola whose axis is parallel to Y-axis and which passes through the points (4, 5), (–2, 11), (–4, 21).

Sol.

Given that axis of the parabola is parallel to X-axis,

Let the equation of the parabola be $y = ax^2 + bx + c$

It is Passing through (4, 5), (–2, 11), (–4, 21)

$(4, 5), \Rightarrow 5 = 16a + 4b + c$ … (i)

(–2, 11), $\Rightarrow$ 11 = 4a – 2b + c … (ii)

(–4, 21) $\Rightarrow$ 21 = 16a – 4b + c … (iii)

(ii) – (i) we get: 6 = –12 – 6b

(iii) – (ii) : 10 = 12a – 2b

Solving we get

b = –2, a = 1/2, c = 5

$$y = \frac{1}{2}x^2 - 2x + 5$$

$x^2 - 2y - 4x + 10 = 0$

6. Find the equations of tangents to the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line 2x – y + 5 = 0. Find the coordinates of the points of contact also.

Sol.

Given the parabola is $y^2 = 16x$

$\Rightarrow$ 4a = 16 $\Rightarrow$ a = 4

Equation of the tangent parallel to 2x – y + 5 = 0 is y = 2x + c

Condition for tangency is $c = \frac{a}{m} = \frac{4}{2} = 2$

Equation of the tangent is

y = 2x + 2 $\Rightarrow$ 2x – y + 2 = 0

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{4}, \frac{8}{2}\right) = (1, 4)$

Equation of the tangent perpendicular to 2x – y + 5 = 0 is x+2y+c = 0

$$\Rightarrow 2y = -x - c \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}c$$

If above line is a tangent the c = a/m

$$\Rightarrow -\frac{1}{2}c = \frac{4}{\left(-\frac{1}{2}\right)} \Rightarrow c = 16$$

Equation of the perpendicular tangent is

$$y = -\frac{1}{2}x - 8$$
$$2y = -x - 16$$
$$x + 2y + 16 = 0$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$$= \left(\frac{4}{(1/4)}, \frac{8}{(-1/2)}\right) = (16, -16).$$

7. If lx + my + n = 0 is a normal to the parabola y^2 = 4ax, then show that $al^3 + 2alm^2 + nm^2 = 0$.

Sol. Given parabola is $y^2 = 4ax$

Equation of the normal is $y + tx = 2at + at^3$

$tx + y - (2at + at^3) = 0$ .... …(1)

Equation of the given line is

$lx + my + n = 0$ .... …(2)

(1), (2) are representing the same line, therefore

$$\frac{t}{\ell} = \frac{1}{m} = \frac{-(2at + at^3)}{n}$$

$$\frac{t}{\ell} = \frac{1}{m} \Rightarrow t = \frac{\ell}{m}$$

$$\frac{1}{m} = -\frac{(2at + at^3)}{n}$$

$$\Rightarrow \frac{-n}{m} = 2a \cdot t + at^3$$

$$\Rightarrow 2a \cdot \frac{\ell}{m} + a \cdot \left(\frac{\ell}{m}\right)^3 = \frac{2a\ell}{m} + \frac{a\ell^3}{m^3}$$

$$\Rightarrow -nm^2 = 2al\,m^2 + al^3$$

$$\Rightarrow al^3 + 2alm^2 + nm^2 = 0$$

8. Show that the equation of common tangents to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ are $y = \pm(x + 2a)$.

Sol.

Given parabola $y^2 = 8ax$

$\Rightarrow y^2 = 4.2ax$

The equation of tangent to parabola is

$$y = mx + \frac{2a}{m}.$$

$$m^2x - my + 2a = 0 \quad \ldots (1)$$

If (1) is a tangent to the circle $x^2 + y^2 = 2a^2$, then the length of perpendicular from its centre (0, 0) to (1) is equal to the radius of the circle.

$$\left|\frac{2a}{\sqrt{m^2+m^4}}\right| = a\sqrt{2}$$

$$\Rightarrow 4 = 2(m^4 + m^2)$$

$$m^4 + m^2 - 2 = 0$$

$$(m^2 + 2)(m^2 - 1) = 0 \text{ or } m = \pm 1$$

Required tangents are:

$$y = (1)x + \frac{2a}{(1)},\ y = (-1)x + \frac{2a}{(-1)}$$

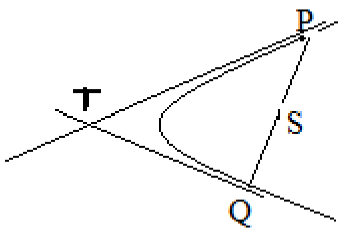
$$\Rightarrow y = \pm(x + 2a)$$

9. Prove that the tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

Sol. Let the parabola be $y^2 = 4ax$

Equation of the tangent at $P(t_1)$ is

$$t_1 y = x + at_1^2$$



Equation of the tangent at $Q(t_2)$ is

$$t_2 y = x + at_2^2$$

Solving, point of intersection is

$$T[at_1t_2, a(t_1 + t_2)]$$

Equation of the chord PQ is

$$(t_1 + t_2)y = 2x + 2at_1t_2$$

Since PQ is a focal chord, S (a,0) is a point on PQ.

Therefore, $0 = 2a + 2a\,t_1t_2$

$\Rightarrow \quad t_1t_2 = -1.$

Therefore point of intersection of the tangents is $[-a, a(t_1 + t_2)]$.

The x coordinate of this point is a constant. And that is x = -a which is the equation of the directrix of the parabola.

Hence tangents are intersecting on the directrix.

10. Find the condition for the line y = mx + c to be a tangent to $x^2 = 4ay$.

Sol.

Equation of the parabola is $x^2 = 4ay$.----(1)

Equation of the line is $y = mx + c$ ---- (2)

Solving above equations,

$x^2 = 4a(mx + c) \Rightarrow x^2 - 4amx - 4ac = 0$ which is a quadratic in x.

If the given line is a tangent to the parabola, the roots of above equation are real and equal.

$\Rightarrow b^2 - 4ac = 0 \Rightarrow 16a^2m^2 + 16ac = 0$

$\Rightarrow am^2 + c = 0 \Rightarrow c = -am^2$ is the required condition.

11. Three normals are drawn (k, 0) to the parabola $y^2 = 8x$ one of the normal is the axis and the remaining two normals are perpendicular to each other, then find the value of k.

Sol.

Equation of parabola is $y^2 = 8x$

Equation of the normal to the parabola is

$y + xt = 2at + at^3$ which is a cubic equation in t. Therefore it has 3 roots. Say t_1, t_2, t_3. Where $-t_1, -t_2, -t_3$ are the slopes of the normals.

This normal is passing through (k, 0)

$\therefore kt = 2at + at^3$

$at^3 + (2a - k)t = 0$

$at^2 + (2a - k) = 0$

Given one normal is axis i.e., x axis and the remaining two are perpendicular. Thererfore

$m_1 = 0 = t_1$, and $m_2m_3 = -1$

$(-t_2)(-t_3) = -1$, $t_2t_3 = -1$

$$\frac{2a-k}{a} = -1 \Rightarrow 2a - k = -a$$

$$\Rightarrow k = 2a + a = 3a$$

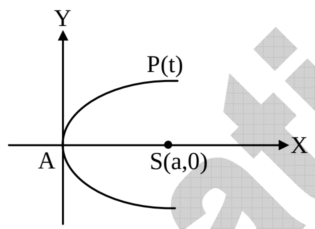
Equation of the parabola is $y^2 = 8x$

$4a = 8 \Rightarrow a = 2$

$k = 3a = 3 \times 2 = 6.$

12. Prove that the point on the parabola $y^2 = 4ax$ $(a > 0)$ nearest to the focus is vertex.

Sol. Let $P(at^2, 2at)$ be the point on the parabola.



$y^2 = 4ax$, which is nearest to the focus $S(a, 0)$ then

$sp^2 = (at^2 - a)^2 + (2at - 0)^2$

$f(t) = a^2 2(t^2 - 1)(2t) + 4a^2(2t)$

$= 4a^2t(t^2 - 1 + 2) = 4a^2t(t^2 + 1)$

For minimum value of $f(t) = 0 \Rightarrow t = 0$

$f''(L) = 4a^2 (3t^2 + 1)$

$f'(0) = 4a^2 > 0$

$\therefore$ At $t = 0$, $f(t)$ is minimum

Then $P = (0, 0)$

$\therefore$ The point on the parabola $y^2 = 4ax$, which is nearest to the focus is its vertex $A(0, 0)$.

13. Find the equation of the tangent and normal to the parabola $y^2 = 8x$ at (2, 4).

Sol. Equation of the tangent at (x_1, y_1) to $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

Equation of tangent at (2, 4) to $y^2 = 8x$ be

$4y = 4(x + 2)$ $[4a = 8 \Rightarrow a = 2]$

$y = x + 2$

Equation of normal at (x_1, y_1) is

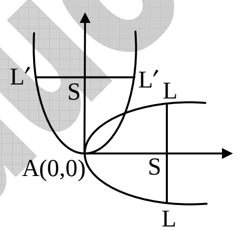
$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$$(y - 4) = \frac{-4}{4}(x - 2)$$

$$y - 4 = -x + 2 \Rightarrow x + y = 6$$

14. Two parabolas have the same vertex and equal length of latus rectum such that their axes are at right angles. Prove that the common tangent touches each at the end of a latus rectum.

Sol.



Equations of the parabolas can be taken as

$y^2 = 4ax$ and $x^2 = 4ay$

Equation of the tangent at $(2at, at^2)$ to

$x^2 = 4ay$ is

$2atx = 2a(y + at^2)$

$y = tx - at^2$

This is a tangent to $y^2 = 4ax$

$\therefore$ The condition is $c = a/m$

$$-at^2 = \frac{a}{t} \Rightarrow t^3 = -1 \Rightarrow t = -1$$

Equation of the tangent is y = –x – a

x + y + a = 0

Equation of the tangent at L′(a, –2a) is

y(–2a) = 2a(x + a)

x + y + a = 0

∴ Common tangent to the parabolas touches the parabola $y^2 = 4ax$ at L(a, –2a).

Equation of the tangent at L(–2a, a) to

$x^2 = 4ay$

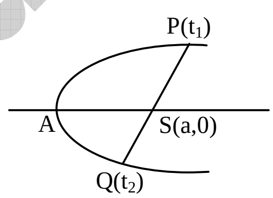
x(–2a) = 2a(y + a)

x + y + a = 0

Common tangent to the parabolas touch the parabola at L′(–2a, a).

15. Show that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.

Sol.



$P(t_1)$, $Q(t_2)$ are the ends of a focal chord.

Slope of PS = Slope of PQ

$$\frac{2at_1}{a(t_1^2-1)} = \frac{2a(t_1-t_2)}{a(t_1^2-t_2^2)}$$

$$\frac{t_1}{t_1^2-1}=\frac{1}{t_1+t_2}$$

$$t_1+t_2=\frac{t_1^2-1}{t_1}=t_1-\frac{1}{t_1}$$

$$t_2=-\frac{1}{t_1} \qquad \ldots (1)$$

Equation of the tangent at $P(t_1)$ is

$$t_1y=x+at_1^2$$

Slope of the tangent at $P=\frac{1}{t_1}$ … (2)

Equation of the normal at $Q(t_2)$ is

$$y+xt_2=2at_2+at_2^3$$

Slope of the normal at $Q=-t_2$ … (3)

From (1), (2), (3) we get

Slope of the tangent at P =slope of normal at Q

Slope of the tangent at P is parallel to the normal at Q.

16. The sum of the ordinates of two points $y^2 = 4ax$ is equal to the sum of the ordinates of two other points on the same curve. Show that the chord joining the first two points is parallel to the chord joining the other two points.

Sol: Given equation of the parabola is $y^2 = 4ax$ … (1)

Let $A=(at_1^2,2at_1), B=(at_2^2,2at_2)$, $C=(at_3^2,2at_3)$ and $D=(at_4^2,2at_4)$ be the four points on the parabola (1).

Given that $2at_1+2at_2=2at_3+2at_4$

$$\Rightarrow t_1+t_2=t_3+t_4 \qquad \ldots (2)$$

The equation of the chord $\overline{AB}$ of the parabola $y^2 = 4ax$ is $y(t_1+t_2)=2x+2at_1t_2$.

$$\Rightarrow 2x-(t_1+t_2)y+2at_1t_2=0 \qquad \ldots (3)$$

Let m_1 be the slope of the line (3) then $m_1 = \frac{2}{t_1 + t_2}$.

The equation of the chord $\overline{CD}$ of the parabola (1) is $y(t_3 + t_4) = 2x + 2at_3t_4$ and slope of this chord $m_2 = \frac{2}{t_3 + t_4}$.

$\therefore$ From (2) we have $t_1 + t_2 = t_3 + t_4$

Hence $m_1 = m_2$

$\therefore$ Chord $\overline{AB}$ is parallel to the chord $\overline{CD}$.

17. If Q is the foot of the perpendicular from a point p on the parabola $y^2 = 8(x - 3)$ to its directrix. S is the focus of the parabola and if SPQ is an equilateral triangle, then find the length of the side of the triangle.

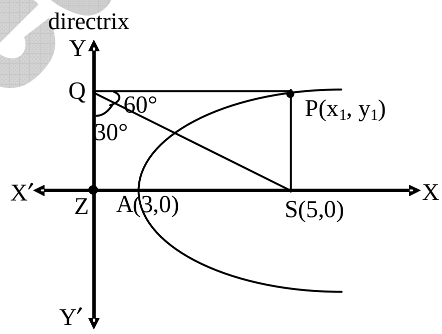
Sol: Given equation of parabola is $(y - 0)^2 = 8(x - 3)$

Which is of the form $(y - k)^2 = 4a(x - h)$

Where $4a = 8 \Rightarrow a = 2$

$\therefore$ Vertex = (h, k) = (3, 0)

and focus = (h+a, k) = (3+2, 0) = (5, 0)



Since PQS is an equilateral triangle.

$\angle SQP = 60° \Rightarrow \angle SQZ = 30°$

Also in Δ SZQ, we have $\sin 30° = \frac{SZ}{SQ}$

$$\therefore SQ = \frac{SZ}{\sin 30°} = 2(SZ) = 2(4) = 8$$

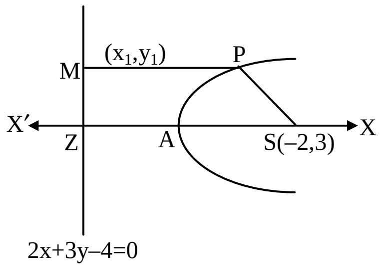
$$(\because SA = \sqrt{(5-3)^2} = 2)$$

Hence length of each side of the triangle is 8.

Long Answer Questions

1. Find the equation of the parabola whose focus is (–2, 3) and directrix is the line 2x + 3y – 4 = 0. Also find the length of the latus rectum and the equation of the axis of the parabola.

Sol.



Focus S(–2, 3)

Equation of the directrix is 2x + 3y – 4 = 0.

Let $P(x_1, y_1)$ be any point on the parabola.

$SP^2 = (x_1 + 2)^2 + (y_1 - 3)^2$

Let PM be the perpendicular from P to the directrix.

From Def. of parabola $SP = PM \Rightarrow SP^2 = PM^2$

$$(x_1 + 2)^2 + (y_1 - 3)^2 = \frac{(2x_1 + 3y_1 - 4)^2}{13}$$

$$13(x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9) = (2x_1 + 3y_1 - 4)^2$$
$$13x_1^2 + 13y_1^2 + 52x_1 - 78y_1 + 169 =$$
$$4x_1^2 + 9y_1^2 + 16 + 12x_1y_1 - 16x_1 - 24y_1$$

$9x_1^2 - 12x_1y_1 + 4y_1^2 + 68x_1 - 54y_1 + 153 = 0$

Locus of $P(x_1, y_1)$ is

$9x^2 - 12xy + 4y^2 + 68x - 54y + 153 = 0$

Length of the latus rectum = 4a

2a =Perpendicular distance from S on directrix = $\dfrac{|2(-2)+3\cdot 3-4|}{\sqrt{4+9}} = \dfrac{1}{\sqrt{13}}$

Length of the latus rectum = $4a = \dfrac{2}{\sqrt{3}}$

The axis is perpendicular to the directrix

Equation of the directrix can be taken as

$3x - 2y + k = 0$

This line passes through S(–2, 3)

$-6 - 6 + k = 0 \Rightarrow k = 12$

Equation of the axis is : $3x - 2y + 12 = 0$

2. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ sq.units where y_1, y_2, y_3 are the ordinates of its vertices.

Sol.

Given parabola $y^2 = 4ax$

Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$, $R(at_3^2, 2at_3)$ be the vertices of ΔPQR.

Area of ΔPQR =

$$= \frac{1}{2}\begin{vmatrix} at_1^2 - at_2^2 & at_2^2 - at_3^2 \\ 2at_1 - 2at_2 & 2at_2 - 2at_3 \end{vmatrix} = \frac{1}{2}\left|2a^2\left(t_1^2 - t_2^2\right)(t_2 - t_3) - 2a^2\left(t_2^2 - t_3^2\right)(t_1 - t_2)\right|$$

$$= a^2\left|(t_1 - t_2)(t_2 - t_3)\left(t_1 + t_2 - t_2 - t_3\right)\right|$$

$$= a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

$$= \frac{a^3}{a} |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

$$= \frac{1}{8a} |(2at_1 - 2at_2)(2at_2 - 2at_3)(2at_3 - 2at_1)|$$

$$= \frac{1}{8a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$$

Where $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ are the vertices of ΔPQR.

3. Find the coordinates of the vertex and focus, equation of the directrix and axis of the following parabolas.

i) $y^2 + 4x + 4y - 3 = 0$

ii) $x^2 - 2x + 4y - 3 = 0$

Sol.i) Given parábola is $y^2 + 4x + 4y - 3 = 0$

$\Rightarrow y^2 + 4y = -4x + 3$

$\Rightarrow (y + 2)^2 - 4 = -4x + 3$

$\Rightarrow (y + 2)^2 = -4x + 7$

$\Rightarrow [y - (-2)]^2 = -4\left[x - \frac{7}{4}\right]$

$h = \frac{7}{4}, k = -2, a = 1$

Vertex $A(h, k) = \left(\frac{7}{4}, -2\right)$

Focus $(h-a, k) = \left(\frac{7}{4} - 1, -2\right) = \left(\frac{3}{4}, -2\right)$

Equation of the directrix: $x - h - a = 0$

$x - \frac{7}{4} - 1 = 0 \Rightarrow 4x - 11 = 0$

Equation of the axis is: $y - k = 0 \Rightarrow y + 2 = 0$

ii) Given parábola is $x^2 - 2x + 4y - 3 = 0$

$\Rightarrow x^2 - 2x = -4y + 3$

$\Rightarrow (x-1)^2 - 1 = -4y + 3$

$\Rightarrow (x-1)^2 = -4y + 4$

$\Rightarrow (x-1)^2 = -4[y-1]$

$h = 1, k = 1, a = 1$

Vertex $A(h, k) = (1, 1)$

Focus $(h, k - a) = (1, 1 - 1) = (1, 0)$

Equation of the directrix: $y - k - a = 0$

$y - 1 - 1 = 0 \Rightarrow y - 2 = 0$

Equation of the axis is, $x - h = 0 \Rightarrow x - 1 = 0$.

4. If the normal at the point t_1 on the parabola $y^2 = 4ax$ meets it again at point t_2 then prove that $t_1t_2 + t_1^2 + 2 = 0$.

Sol.

Equation of the parabola is $y^2 = 4ax$

Equation of normal at $t_1 = (at_1^2, 2at_1)$ is

$y + xt_1 = 2at_1 + at_1^3$.

This normal meets the parabola again at $(at_2^2, 2at_2)$.

Therefore,

$2at_2 + at_2^2t_1 = 2at_1 + at_1^3$

$\Rightarrow 2(t_2 - t_1) = t_1(t_1^2 - t_2^2)$

$\Rightarrow 2 = -t_1(t_1 - t_2)$

$\Rightarrow t_1t_2 + t_1^2 + 2 = 0$

5. From an external point P tangents are drawn to the parabola $y^2 = 4ax$ and these tangents make angles θ_1, θ_2 with its axis such that $\cot\theta_1 + \cot\theta_2$ is a constant 'a' show that P lies on a horizontal line.

Sol.

Equation of the parabola is $y^2 = 4ax$

Equation of any tangent to the parabola is

$$y = mx + \frac{a}{m}$$

This tangent passes through $P(x_1, y_1)$

$$y_1 = mx_1 + \frac{a}{m}$$

$$\Rightarrow my_1 = m^2x_1 + a \Rightarrow m^2x_1 - my_1 + a = 0$$

Let m_1, m_2 be the roots of the equation

$m_1 + m_2 = \frac{y_1}{x_1}$, $m_1m_2 = \frac{a}{x_1}$ Where m_1 and m_2 are the slopes of the tangents.

Given $\cot\theta_1 + \cot\theta_2 = a$

$$\frac{1}{m_1} + \frac{1}{m_2} = a \Rightarrow \frac{m_1 + m_2}{m_1m_2} = a$$

$$\Rightarrow m_1 + m_2 = a \cdot m_1m_2$$

$$\frac{y_1}{x_1} = a \cdot \frac{a}{x_1} \Rightarrow y_1 = a^2$$

Locus of $P(x_1, y_1)$ is $y = a^2$ which is a horizontal line.

6. Show that the common tangent to the circle $2x^2 + 2y^2 = a^2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola $y^2 = -4ax$.

Sol.

Given parabola is $y^2 = 4ax$

Let $y = mx + \frac{a}{m}$ be the tangent. But this is also the tangent to $2x^2 + 2y^2 = a^2$

$\Rightarrow$ Perpendicular distance from centre (0, 0) = radius

$$\Rightarrow \left|\frac{a/m}{\sqrt{m^2+1}}\right| = \frac{a}{\sqrt{2}} \Rightarrow \frac{a^2/m^2}{m^2+1} = \frac{a^2}{2}$$

$$\Rightarrow \frac{2a^2}{m^2} = a^2(m^2+1)$$

$$\Rightarrow 2 = m^4 + m^2 \Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2-1)(m^2+2) = 0 \quad (\because m^2+2 \neq 0)$$

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

Therefore, equations of the tangents are

$y = -x - a$ and $y = x + a$.

The point of intersection of these two tangents is (- a, 0) which is the focus of the parabola $y^2 = -4ax$.

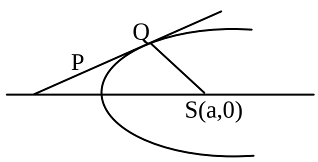
7. Show that the foot of the perpendicular from focus to the tangent of the parabola $y^2 = 4ax$ lies on the tangent at the vertex.

Sol. Equation of any tangent to the parabola is:

$$y = mx + \frac{a}{m}$$

$Q(x_1, y_1)$ is the foot of the perpendicular

$$\therefore \; y_1 = mx_1 + \frac{a}{m} \qquad \ldots(1)$$



Slope of SQ = $\frac{y_1}{x_1 - a}$

SQ and PQ are perpendicular

$$\therefore \; m \times \frac{y_1}{x_1 - a} = -1$$

$$m = \frac{-(x_1 - a)}{y_1} = \frac{a - x_1}{y_1}$$

Substituting in (1) we get

$$y_1 = \frac{x_1(a - x_1)}{y_1} + \frac{a y_1}{a - x_1}$$

$$\Rightarrow y_1 = \frac{x_1(a - x_1)^2 + a y_1^2}{y_1(a - x_1)}$$

$$\Rightarrow y_1^2(a - x_1) = x_1(a - x_1)^2 + a y_1^2$$

$$\Rightarrow a y_1^2 - x_1 y_1^2 = x_1(a^2 + x_1^2 - 2a x_1) + a y_1^2$$

$$\Rightarrow x_1[x_1^2 - 2a x_1 + a^2 + y_1^2] = 0$$

$$\Rightarrow x_1[(x_1 - a)^2 + y_1^2] = 0$$

$$\Rightarrow x_1 = 0$$

Locus of $Q(x_1, y_1)$ is $x = 0$ i.e. the tangent at the vertex of the parabola.

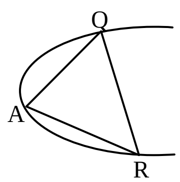
8. If a normal chord at a point t on the parabola $y^2 = 4ax$, subtends a right angle at vertex, then prove that $t = \pm\sqrt{2}$ **.**

Sol. Equation of the parabola is $y^2 = 4ax$ … (1)

Equation of the normal at t is:

$$tx + y = 2at + at^3$$

$$\frac{tx + y}{2at + at^3} = 1 \qquad \dots (2)$$



Homogenising (1) with the help of (2) combined equation of AQ, AR is

$$y^2 = \frac{4ax(tx+y)}{a(2t+t^3)}$$

$$y^2(2t+t^3) = 4tx^2 + 4xy$$

$$4tx^2 + 4xy - (2t+t^3)y^2 = 0$$

AQ, AR are perpendicular

Coefficient of x^2 + coefficient of $y^2 = 0$

$4t - 2t - t^3 = 0$

$2t - t^3 = 0$

$-t(t^2 - 2) = 0$

$t^2 - 2 = 0 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$

9. (i) If the coordinates of the ends of a focal chord of the parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then prove that $x_1x_2 = a^2$, $y_1y_2 = -4a^2$.

(ii) For a focal chord PQ of the parabola $y^2 = 4ax$, if SO = l and SQ = l' then prove that $\frac{1}{l} + \frac{1}{l'} = \frac{1}{a}$.

Sol. i) Let $P(x_1,y_1) = (at_1^2, 2at_1)$ and $Q(x_2, y_2) = (at_2^2, 2at_2)$ be two end points of a focal chord.

P, S, Q are collinear.

Slope of $\overline{PS}$ = Slope of $\overline{QS}$

$$\frac{2at_1}{at_1^2 - a} = \frac{2at_2}{at_2^2 - a}$$

$$t_1t_2^2 - t_1 = t_2t_1^2 - t_2$$

$$t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$$

$$1 + t_1t_2 = 0 \Rightarrow t_1t_2 = -1$$

From (1)

$$x_1x_2 = at_1^2at_2^2 = a^2(t_2t_1)^2 = a^2$$

$$y_1y_2 = 2at_1 2at_2 = 4a^2(t_2t_1) = -4a^2$$

ii) Let P(at_1^2, $2at_1$) and Q(at_2^2, $2at_2$) be the extremities of a focal chord of the parabola, then $t_1t_2 = -1$ (from(1))

$$l = SP = \sqrt{(at_1^2 - a)^2 + (2at_1 - 0)^2}$$

$$= a\sqrt{(t_1^2 - 1)^2 + 4t_1^2} = a(1 + t_1^2)$$

$$l' = SQ = \sqrt{(at_2^2 - a)^2 + (2at_2 - 0)^2}$$

$$= a\sqrt{(t_2^2 - 1)^2 + 4t_2^2} = a(1 + t_2^2)$$

$$\therefore \ (l - a)(l' - a) = a^2 t_1^2 t_2^2 = a^2 (t_1 t_2)^2 = a^2 \qquad [\because t_1 t_2 = -1]$$

$$ll' - a(l + l') = 0 \Rightarrow \frac{1}{l} + \frac{1}{l'} = \frac{1}{a}$$

10. Show that the common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is

$xa^{1/3} + yb^{1/3} + a^{2/3}b^{2/3} = 0$.

Sol. The equations of the parabolas are

$y^2 = 4ax$ … (1) and

$x^2 = 4by$ …(2)

Equation of any tangent to (1) is of the form

$$y = mx + \frac{a}{m} \quad \text{… (3)}$$

If the line (3) is a tangent to (2) also, we must get only one point of intersection of (2) and (3).

Substituting the value of y from (3) in (2), we get $x^2 = 4b\left(mx + \frac{a}{m}\right)$ is

$mx^2 - 4bm^2x - 4ab = 0$ should have equal roots therefore its discriminant must be zero. Hence

$16b^2m^4 - 4m(-4ab) = 0$

$16b\,(bm^4 + am) = 0$

$m(bm^3 + a) = 0$, but $m \neq 0$

$\therefore$ $m = -a^{1/3}b^{1/3}$ substituting in (3) the equation of the common tangent becomes

$$y = -\left(\frac{a}{b}\right)^{1/3} x + \frac{a}{\left(-\frac{a}{b}\right)^{1/3}} \text{ or}$$

$$a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0.$$

11. Prove that the area of the triangle formed by the tangents at (x_1, y_1), (x_2, y_2) and (x_3, y_3) to the parabola $y^2 = 4ax$ $(a > 0)$ is

$\frac{1}{16a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ **sq.units.**

Sol. Let $D(x_1, y_1) = (at_1^2, 2at_1)$

$E(x_2, y_2) = (at_2^2, 2at_2)$ and

$F(x_3, y_3) = (at_3^2, 2at_3)$

Be three points on the parabola.

$y^2 = 4ax$ $(a > 0)$

The equation of the tangents at D, E and F are

$t_1y = x + at_1^2$ ...(1)

$t_2y = x + at_2^2$ ...(2)

$t_3y = x + at_3^2$ ...(3)

(1) – (2) $\Rightarrow (t_1 - t_2)y = a(t_1 - t_2)(t_1 + t_2)$

$\Rightarrow y = a(t_1 + t_2)$ substituting in (1) we get,

$x = at_1t_2$

$\therefore$ The point of intersection of the tangents at D and E is say $P[at_1t_2, a(t_1+t_2)]$

Similarly the points of intersection of tangent at E, F and at F, D are $Q[at_2t_3, a(t_2+t_3)]$ and $R[at_3t_1, a(t_3+t_1)]$ respectively.

Area of ΔPQR

$$= \text{Absolute value of } \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_2+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_1t_3 & a(t_1+t_3) & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1t_2 & t_2+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1(t_2-t_3) & t_2-t_3 & 0 \\ t_3(t_2-t_1) & t_2-t_1 & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

= Absolute value of

$$\frac{a^2}{2}(t_2-t_3)(t_2-t_1)\begin{vmatrix} t_1 & 1 & 0 \\ t_3 & 1 & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \frac{a^2}{2}|(t_2-t_3)(t_2-t_1)(t_1-t_3)|$$

$$= \frac{1}{16a}|2a(t_1-t_2)2a(t_2-t_3)2a(t_3-t_1)|$$

$$= \frac{1}{16a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)| \text{ sq. units.}$$

12. Prove that the normal chord at the point other than origin whose ordinate is equal to its abscissa subtends a right angle at the focus.

Sol. Let the equation of the parabola be $y^2 = 4ax$ and $P(at^2, 2at)$ be any point …(1)

On the parabola for which the abscissa is equal to the ordinate.

i.e. $at^2 = 2at \Rightarrow t = 0$ or $t = 2$. But $t \neq 0$.

Hence the point (4a, 4a) at which the normal is

$y + 2x = 2a(2) + a(2)^3$

$y = (12a - 2x)$ …………. (2)

Substituting the value of

$y = (12a - 2x)$ in (1) we get

$(12a - 2x)^2 = 4ax$

$x^2 - 13ax + 36a^2 = (x - 4a)(x - 9a) = 0$

$\Rightarrow x = 4a, 9a$

Corresponding values of y are 4a and –6a.

Hence the other points of intersection of that normal at P(4a, 4a) to the given parabola is Q(9a, –6a), we have S(a, 0).

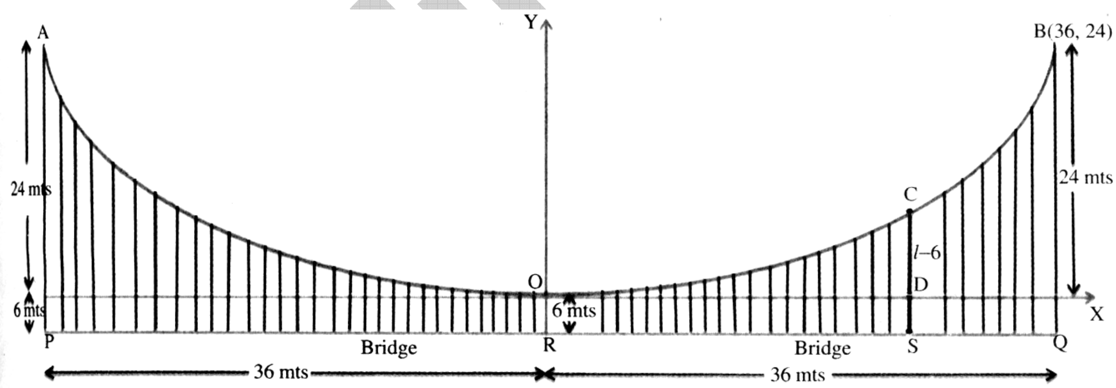
Slope of the $\overline{SP} = m_1 = \dfrac{4a-0}{4a-a} = \dfrac{4}{3}$

Slope of the $\overline{SQ} = m_2 = \dfrac{-6a-0}{9a-a} = -\dfrac{3}{4}$

Clearly $m_1m_2 = -1$, so that $\overline{SP} \perp \overline{SQ}$.

14. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 72 mt. long is supported by a vertical wires attached to the cable. The longest being 30 mts. and the shortest being 6 mts. Find the length of the supporting wire attached to the road-way 18 mts. from the middle.

Sol.



Let AOB be the cable [O is its lowest point and A, B are the highest points]. Let PRQ be the suspension bridge suspended with PR = RQ = 36 mts.

PA = QB = 30 mts (longest wire) OR = 6 mts (shortest wire)

Therefore, PR = RQ = 36 m. We take O as origin of coordinates at O, X-axis along the tangent at O and Y-axis along $\overline{RO}$. So the equation of the cash is $x^2 = 4ay$ for some $a > 0$.

$\therefore$ B = (36, 24) and B is a point on $x^2 = 4ay$.

We have $(36)^2 = 4a \times 24$.

$$\Rightarrow 4a = \frac{36\times 36}{24} = 54\,\text{mts}$$

If RS = 18 m and SC is the vertical through S meeting the cable at C and the X-axis at D.

Then SC is the length of the required wire.

Let SC = l mts, then DC = $(l - 6)$ m.

$\therefore$ C = $(18, l - 6)$ which lies on $x^2 = 4ay$

$\Rightarrow (18)^2 = 4a(l - 6)$

$\Rightarrow (18)^2 = 54(l - 6)$

$$\Rightarrow (l - 6) = \frac{18\times 18}{54} = 6$$

$\Rightarrow l = 12.$

15. Prove that the two parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect (other than the origin) at an angle of $\text{Tan}^{-1}\left[\dfrac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}\right]$.

Sol. Take a > 0 and b > 0.

And $y^2 = 4ax$ …(1)

And $x^2 = 4by$ …(2)

Are the given parabolas.

Solving (1) and (2) we get the point of intersection other than the origin.

$$y^4 = 16a^2x^2$$
$$= 16a^2(4by)$$
$$= 64a^2by$$

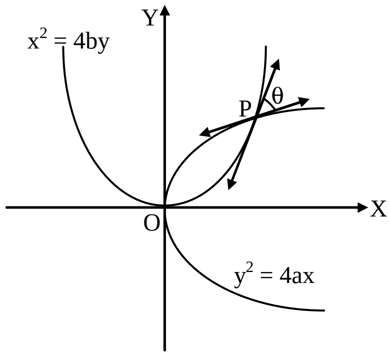
$$\therefore y(y^3 - 64a^2b) = 0$$

$$\Rightarrow y^3 - 64a^2b = 0$$

$$\Rightarrow y = (64a^2b)^{1/3} = 4a^{2/3}b^{1/3}$$

Also from $y^2 = 4ax$, we have

$$x = \frac{y^2}{4a} = \frac{16a^{4/3}b^{2/3}}{4a} = 4a^{1/3}b^{2/3}$$



$\therefore$ P = $(4a^{1/3}\,b^{2/3}, 4a^{2/3}b^{1/3})$

Differentiating $y^2 = 4ax$ w.r.t. ‘x’ we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

$\therefore$ m_1 = slope of the tangent at P

$$= \frac{2a}{4a^{2/3}b^{1/3}} = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}$$

Similarly differentiating $x^2 = 4by$ w.r.t. x

$$2x = 4b\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{4b} = \frac{x}{2b}$$

And m_2 = slope of the tangent at P to

$x^2 = 4by$

$$= \frac{4a^{1/3}b^{2/3}}{2b} = 2\left(\frac{a}{b}\right)^{1/3}$$

If θ is the acute angle between the two tangents to the curves at P then

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\left(\frac{a}{b}\right)^{1/3}\left(\frac{1}{2} - 2\right)}{1 + \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}(2)\left(\frac{a}{b}\right)^{1/3}}\right|$$

$$= \left|\frac{-\frac{3}{2}\left(\frac{a}{b}\right)^{1/3}}{1 + \left(\frac{a}{b}\right)^{2/3}}\right|$$

$$= \left|\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}\right|$$

$$\Rightarrow \theta = \mathrm{Tan}^{-1}\left[\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}\right]$$

16. Prove that the orthocenter of the triangle formed by any three tangents to a parabola lies on the directrix of the parabola.

Sol. Let $y^2 = 4ax$ be the parabola and, $A = (at_1^2, 2at_1), B = (at_2^2, 2at_2)$, $C = (at_3^2, 2at_3)$ be any three points on it.

If P, Q, R are the points of intersection of tangents at A and B, B and C, C and A then

$P = [at_1t_2, a(t_1 + t_2)], Q = [at_2t_3, a(t_2 + t_3)]$, $R = [at_3t_1, a(t_3 + t_1)]$.

Consider the ΔPQR

Then equation of $\overline{QR}$ (Tangent at C) is $x - yt_3 + at_3^2 = 0$.

$\therefore$ Altitude through P of ΔPQR is

$$t_3x + y = at_1t_2t_3 + a(t_1 + t_2) \quad \ldots (1)$$

[$\therefore$ Slope = $\frac{1}{t_3}$ and equation is

$$y - a(t_1 + t_2) = -t_3[x - at_1t_2]$$
$$\Rightarrow y + xt_3 = at_1t_2t_3 + a(t_1 + t_2)]$$

Similarly, the altitude through Q is

$$t_1x + y = at_1t_2t_3 + a(t_2 + t_3) \quad \ldots (2)$$

Solving (1) and (2), we get

$(t_3 - t_1)x = a(t_1 - t_3)$

i.e., x = –a.

Hence, the orthocenter of the triangle PQR with x coordinate as –a, lies on the directrix of the parabola.

PROBLEMS FOR PRACTICE

1. Find the coordinates of the vertex and focus, and the equations of the directrix and axes of the following parabolas.

i) $y^2 = 16x$ ii) $x^2 = -4y$

iii) $3x^2 - 9x + 5y - 2 = 0$ iv) $y^2 - x + 4y + 5 = 0$

2. Find the equation of the parabola whose vertex is (3, –2) and focus is (3, 1).

Ans. $(x - 3)^2 = 12(y + 2)$

3. Find the coordinates of the points on the parabola $y^2 = 2x$ whose focal distance is 5/2.

Ans. (2, 2) and (2, –2)

4. Find the equation of the parabola passing through the points (–1, 2), (1, –1) and (2, 1) and having its axis parallel to the x-axis.

Ans. $7y^2 - 3y + 6x - 16 = 0$

5. If Q is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x - 3)$ to its directrix. S is the focus of the parabola and if SPQ is an equilateral triangle then find the length of side of the triangle.

Ans. 8

6. Find the condition for the straight line $lx + my + n = 0$ to be a tangent to the parabola $y^2 = 4ax$ and find the coordinates of the point of contact.

Ans. $\left(\frac{n}{l}, \frac{-2am}{l}\right)$

7. **Show that the straight line $7x + 6y = 13$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$ and find the point of contact.**

Ans. (1, 1)

8. **From an external point P, tangent are drawn to the parabola $y^2 = 4ax$ and these tangent make angles θ_1, θ_2 with its axis, such that $\tan\theta_1 + \tan\theta_2$ is constant b. Then show that P lies on the line $y = bx$.**

9. **Prove that the poles of normal chord of the parabola $y^2 = 4ax$ lie on the curve $(x + 2a)y^2 + 4a^3 = 0$.**