INTEGRATION USING PARTIAL FRACTIONS

Very Short Answer Questions

Evaluate the following integrals.

1. $\int \frac{(x-1)dx}{(x-2)(x-3)}$

Sol.

$$\frac{(x-1)}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} \quad x-1 = A(x-3)+B(x-2)$$

Put x= 3, then B = 2 and put x=2 then A= -1

$$\therefore \frac{(x-1)}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}$$

$$\therefore \int \frac{(x-1)dx}{(x-2)(x-3)} = \int \left(\frac{-1}{x-2} + \frac{2}{x-3}\right)dx$$

$$= 2\log(x-3) - \log(x-2) + c$$

2. $\int \frac{x^2}{(x+1)(x+2)^2}dx$

Sol. let

$$\frac{x^2}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Rightarrow x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

Put x = –2 in (1)

$(-2)^2 = A(0) + B(0) + C(-2+1) \Rightarrow C = -4$

Put x = –1 in (1)

$(-1)^2 = A(-1+2)^2 + B(0) + C(0) \Rightarrow A = 1$

Equating coefficients of x^2 in (1)

$1 = A + B \Rightarrow B = 1 - A = 1 - 1 = 0$

$$\therefore \frac{x^2}{(x+1)(x+2)^2} = \frac{1}{x+1} + \frac{0}{x+2} + \frac{-4}{(x+2)^2}$$

$$\therefore \int \frac{x^2}{(x+1)(x+2)^2} dx = \int \frac{1}{x+1} dx - 4\int \frac{1}{(x+2)^2} dx$$

$$= \log|x+1| - 4\left(\frac{-1}{x+2}\right)$$

$$= \log|x+1| + \frac{4}{x+2} + C$$

3. $\int \frac{x+3}{(x-1)(x^2+1)} dx$

Sol. Let $\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\Rightarrow (x+3) = A(x^2+1) + (Bx+C)(x-1) \ldots(1)$

Put x = 1 in (1)

Then $4 = A(1 + 1) + 0 \Rightarrow A = 2$

Put x = 0 in (1)

$3 = A(1) + C(-1)$

$\Rightarrow A - C = 3 \Rightarrow C = A - 3 = 2 - 3 = 1$

Equating coefficient of x^2 in (1)

$0 = A + B \Rightarrow B = -A = -2$

$$\therefore \frac{x+3}{(x-1)(x^2+1)} = \frac{+2}{(x-1)} + \frac{-2x-1}{x^2+1}$$

$$\int \frac{x+3}{(x-1)(x^2+1)} dx$$

$$= 2\int \frac{1}{x-1} dx - \int \frac{2x}{x^2+1} dx - \int \frac{1}{x^2+1} dx$$

$$= 2\log|x-1| - \log|x^2+1| - \tan^{-1}(x) + C$$

4. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Sol. From partial fractions

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(b^2-a^2)}\left(\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right)$$

$$\therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{(b^2-a^2)}\left[\int \frac{1}{(x^2+a^2)}dx - \int \frac{1}{(x^2+b^2)}dx\right]$$

$$= \frac{1}{(b^2-a^2)}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right] + C$$

5. $\int \frac{dx}{e^x+e^{2x}}$

Sol. $\frac{1}{e^x+e^{2x}} = \frac{1}{e^x(1+e^x)} = \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)$

$$\therefore \int \frac{1}{e^x+e^{2x}}dx = \int \frac{1}{e^x}dx - \int \frac{1}{1+e^x}dx$$

$$= \int e^{-x}dx - \int \frac{e^x}{e^x(1+e^x)}dx$$

$$= \int e^{-x}dx + \int e^x\left\{\frac{1}{e^x} - \frac{1}{1+e^x}\right\}dx$$

$$= \int e^{-x}dx - \int 1dx + \int \frac{e^x}{1+e^x}dx$$

$$= \frac{e^x}{(-1)} - x + \log|1+e^x| + C$$

$$= -e^x + \log(1+e^x) - \log(e^x) + C \; [\because x = \log e^x]$$

$$= -e^{-x} + \log\left(\frac{1+e^x}{e^x}\right) + C$$

6. $\int \frac{dx}{(x+1)(x+2)}$

Sol. $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

7. $\int \frac{1}{e^x - 1} dx$

Sol. $\int \frac{1}{e^x-1} dx = \frac{e^x - (e^x - 1)}{e^x - 1} dx = \int \frac{e^x dx}{e^x - 1} - \int dx$

$$= \log(e^x - 1) - x = \log(e^x - 1) - \log e^x + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

8. $\int \frac{1}{(1-x)(4+x^2)} dx$

Sol. Let $\frac{1}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$

$\Rightarrow 1 = A(4 + x^2) + (Bx + C)(1 - x)$ …(1)

Put x = 1 in (1)

$1 = A(4 + 1) \Rightarrow A = 1/5$

Put x = 0 in (1)

$1 = A(4) + C(1) \Rightarrow C = 1 - 4A = 1/5$

Equating coefficients of x^2 in (1)

$0 = A - B \Rightarrow B = A = 1/5$

$$\therefore \frac{1}{(1-x)(4+x^2)} = \left(\frac{1}{5}\right)\frac{1}{1-x} + \frac{\left(\frac{1}{5}x+\frac{1}{5}\right)}{4+x^2}$$

$$= \frac{1}{5}\left(\frac{1}{1-x}\right) + \frac{1}{5}\frac{x}{4+x^2} + \frac{1}{5}\frac{1}{4+x^2}$$

$$\therefore \int \frac{1}{(1-x)(4+x^2)}dx = \frac{1}{5}\int\frac{1}{1-x}dx + \frac{1}{5}\int\frac{x}{4+x^2}dx + \frac{1}{5}\int\frac{1}{4+x^2}dx$$

$$= -\frac{1}{5}\int\frac{1}{1-x}dx + \frac{1}{10}\int\frac{2x}{4+x^2}dx + \frac{1}{5}\int\frac{1}{4+x^2}dx$$

$$= -\frac{1}{5}\log|1-x| + \frac{1}{10}\log|4+x^2| + \frac{1}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{5}\log|1-x| + \frac{1}{10}\log|4+x^2| + \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right) + C$$

9. $\int \frac{2x+3}{x^3+x^2-2x}dx$

Sol. $\frac{2x+3}{x(x^2+x-2)} = \frac{2x+3}{x(x+2)(x-1)}$

Let $\frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$

$\Rightarrow 2x+3 = A(x+2)(x-1) + Bx(x-1)$
$+C(x)(x+2) \quad ...(1)$

Put x = 0 in (1), then

3 = A(2)(–1) + B(0) + C(0) ⇒ A = –3/2

Put x = 1 in (1), then

2 + 3 = A(0) + B(0) + C(1)(3) ⇒ C = 5/3

Put x = –2 in (1), then

2(–2) + 3 = A(0) + B(–2)(–2–1) + C(0)

⇒ –1 = 6B ⇒ B = –1/6

$$\therefore \frac{2x+3}{x^3+x^2-2x} = \frac{2x+3}{x(x+2)(x-1)}$$

$$= \frac{-\frac{3}{2}}{x} + \frac{-\frac{1}{6}}{x+2} + \frac{\frac{5}{3}}{x-1}$$

Now $\int \frac{2x+3}{x^3+x^2-2x} dx =$

$$= -\frac{3}{2}\int \frac{1}{x}dx - \frac{1}{6}\int \frac{1}{x+2}dx + \frac{5}{3}\int \frac{1}{x-1}dx$$

$$= -\frac{3}{2}\log|x| - \frac{1}{6}\log|x+2| + \frac{5}{3}\log|x-1| + C$$

Short Answer Questions

Evaluate the following integrals.

1. $\int \frac{dx}{6x^2-5x+1}$

Sol.

$6x^2 - 5x + 1 = (3x - 1)(2x - 1)$

Let $\frac{1}{6x^2-5x+1} = \frac{A}{3x-1} + \frac{B}{2x-1}$

$\Rightarrow 1 = A(2x-1) + B(3x-1)$

Put x = 1/3, $1 = A\left(\frac{2}{3}-1\right) \Rightarrow A = -3$

Put x = $\frac{1}{2} \Rightarrow 1 = B\left(\frac{3}{2}-1\right) \Rightarrow B = 2$

$$\therefore \frac{1}{6x^2-5x+1} = \frac{-3}{3x-1} + \frac{2}{2x-1}$$

$$\int \frac{1}{6x^2-5x+1}dx = -3\int \frac{dx}{3x-1} + 2\int \frac{dx}{2x-1}$$

$$= -3\frac{\log|3x-1|}{3} + 2\frac{\log|2x-1|}{2}$$

$$= \log\left|\frac{2x-1}{3x-1}\right| + C$$

2. $\int \frac{dx}{x(x+1)(x+2)}$

Sol. $\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

$\Rightarrow 1 = A(x+1)(x+2) + B(x)(x+2) + C(x)(x+1)$

Put x = 0

1 = A(1)(2) + B(0) + C(0) $\Rightarrow$ A = 1/2

Put x = –1

1 = A(0) + B(–1)(–1+2) + C(0) $\Rightarrow$ B = –1

Put x = –2

1 = A(0) + B(0) + C(–2)(–2+1) $\Rightarrow$ C = 1/2

$$\frac{1}{x(x+1)(x+2)} = \frac{1/2}{x} - \frac{1}{x+1} + \frac{1/2}{x+2}$$

$$\int \frac{1}{x(x+1)(x+2)}dx = \frac{1}{2}\int\frac{1}{x}dx - \int\frac{1}{x+1}dx + \frac{1}{2}\int\frac{1}{x+2}dx$$

$$= \frac{1}{2}\log|x| - \log|x+1| + \frac{1}{2}\log|x+2| + C$$

3. $\int \frac{3x-2}{(x-1)(x+2)(x-3)}dx$

Sol. $\frac{3x-2}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

3x – 2 = A(x + 2)(x – 3) + B(x – 1)(x – 3) + C(x – 1)(x + 2)

Put x = 1

3(1) –2=A(1+2)(1–3)+B(0) + C(0) $\Rightarrow$ A = $-\frac{1}{6}$

Put x = 3

$3(3) - 2 = A(0)+B(0)+C(3-1)(3+2) \Rightarrow C = \frac{7}{10}$

Put x = –2

$3(-2) - 2 = A(0) + B(-2-1)(-2-3) + C(0) - 8$

$= 15B \Rightarrow B = \frac{-8}{15}$

$$\therefore \frac{3x-2}{(x-1)(x+2)(x-3)}$$

$$= \frac{-1}{6}\cdot\frac{1}{x-1} - \frac{8}{15}\cdot\frac{1}{x+2} + \frac{7}{10}\cdot\frac{1}{x-3}$$

$$\int \frac{3x-2}{(x-1)(x+2)(x-3)}dx = -\frac{1}{6}\log|x-1|$$

$$-\frac{8}{15}\log|x+2| + \frac{7}{10}\log|x-3| + C$$

4. $\int \frac{7x-4}{(x-1)^2(x+2)}dx$

Sol. $\frac{7x-4}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$\Rightarrow 7x - 4 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$ …(1)

Put x = 1 in (1)

$7 - 4 = A(0) + B(1+2) + C(0) \Rightarrow B = 1$

Put x = –2 in (1)

$7(-2) - 4 = A(0) + B(0) + C(-2-1)^2 \Rightarrow C = -2$

Equating coefficients of x^2 in (1)

$0 = A + C \Rightarrow A = -C \Rightarrow A = 2$

$$\therefore \frac{7x-4}{(x-1)^2(x+2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{2}{x+2}$$

$$\therefore \int \frac{7x-4}{(x-1)^2(x+2)}dx = 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} - 2\int \frac{dx}{x+2}$$

$$= 2\log|x-1| - \left(\frac{1}{x-1}\right) - \log|x+2| + C$$

Long Answer Questions

Evaluate the following integrals.

1. $\int \frac{1}{(x-a)(x-b)(x-c)}$ **dx**

Sol. $\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

$$= \frac{A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$$

$\Rightarrow 1 = A(x-b)(x-c) + B(x-a)(x-c)$

$+ C(x-a)(x-b)$ ...(1)

Put x = a, we get

$1 = A(a-b)(a-c) \Rightarrow A = \frac{1}{(a-b)(a-c)}$

Put x = b, we get $B = \frac{1}{(b-a)(c-b)}$

Similarly $C = \frac{1}{(c-a)(c-b)}$

$\therefore \frac{1}{(x-a)(x-b)(x-c)} =$

$$= \frac{\frac{1}{(a-b)(a-c)}}{x-a} + \frac{\frac{1}{(b-a)(b-c)}}{x-b} + \frac{\frac{1}{(c-a)(c-b)}}{x-c}$$

$\therefore \int \frac{1}{(x-a)(x-b)(x-c)}dx$

$$= \frac{1}{(a-b)(a-c)}\int \frac{1}{x-a}dx + \frac{1}{(b-a)(b-c)}$$

$$\int \frac{1}{x-b}dx + \frac{1}{(c-a)(c-b)}\int \frac{1}{x-c}dx$$

2. $\int \frac{2x+3}{(x+3)(x^2+4)}dx$

Sol. Let $\frac{2x+3}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$

$2x + 3 = A(x^2 + 4) + (Bx + C)(x + 3)$

$x = -3 \Rightarrow -3 = A(9 + 4) = 13A \Rightarrow A = -\frac{3}{13}$

Equating the coefficients of x^2

$0 = A + B \Rightarrow B = -A = \frac{3}{13}$

Equating the constants

$3 = 4A + 3C$

$3C = 3 - 4A = 3 + \frac{12}{13} = \frac{39+12}{13} = \frac{51}{13} \Rightarrow C = \frac{17}{13}$

$$\frac{2x+3}{(x+3)(x^2+4)} = \frac{-3}{13}\cdot\frac{1}{x+3} + \frac{3x+17}{13(x^2+4)}$$

$$\int \frac{2x+3}{(x+3)(x^2+4)}dx$$

$$= \frac{-3}{13}\int \frac{dx}{x+3} + \frac{3}{26}\int \frac{2xdx}{x^2+4} + \frac{17}{13}\int \frac{dx}{x^2+4}$$

$$= \frac{-3}{13}\log|x+3| + \frac{3}{26}\log|x^2+4| + \frac{17}{26}\tan^{-1}\left(\frac{x}{2}\right) + C$$

3. $\int \frac{2x^2+x+1}{(x+3)(x-2)^2}dx$

Sol. $\frac{2x^2+x+1}{(x+3)(x-2)^2} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$2x^2 + x + 1 = A(x-2)^2 + B(x+3)(x-2) + C(x+3)$

$x = 2 \Rightarrow 8 + 2 + 1 = C(2+3) = 5C \Rightarrow C = \frac{11}{5}$

$x = -3 \Rightarrow 18-3+1 = A(-5)^2 = 25A \Rightarrow A = \frac{16}{25}$

Equating the coefficients of x^2

$2 = A + B \Rightarrow B = 2 - A = 2 - \frac{16}{25} = \frac{34}{25}$

$\int \frac{2x^2+x+1}{(x+3)(x-2)^2} = \frac{16}{25}\int \frac{dx}{x+3} + \frac{34}{25}\int \frac{dx}{x-2} + \frac{11}{5}\int \frac{1}{(x-2)^2}dx$

$= \frac{16}{25}\log|x+3| + \frac{34}{25}\log|x-2| - \frac{11}{5(x-2)} + C$

4. $\int \frac{dx}{x^3+1}$

Sol. $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$

Let $\frac{1}{x^3+1} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow 1 = A(x^2 - x + 1) + (Bx + C)(x + 1) \ldots(1)$

Put $x = -1$ in (1)

$1 = A(1 + 1 + 1) + (-B + C)(0)$

$\Rightarrow 3A = 1 \Rightarrow A = 1/3$

Put $x = 0$ in (1)

$1 = A(1) + C(1) \Rightarrow C = 1 - A = 1 - \frac{1}{3} = \frac{2}{3}$

Equating the coefficient of x^2

$0 = A + B \Rightarrow B = -A = -1/3$

$$\frac{1}{x^3+1} = \frac{1}{3(x+1)} + \frac{-x+2}{3|x^2-x+1|}$$

$$\int \frac{1}{x^3+1}dx = \int\left(\frac{1}{3(x+1)} + \frac{-x+2}{3\left(x^2-x+1\right)}\right)dx$$

In the second integral of rhs,

Let $-x+2 = A\frac{d}{dx}\left(x^2-x+1\right)+B = A(2x-1)+B$

Comparing the coefficients,

2A = -1 and B-A =2

A = - 1/2, B =3/2

$$-x+2 = \frac{-1}{2}(2x-1)+\frac{3}{2}$$

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\int\frac{dx}{(x+1)} + \int\frac{-\frac{1}{2}(2x-1)+\frac{3}{2}}{x^2-x+1}dx$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

5. $\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$

Sol.

Put cosx = t ⇒ –sinx dx = dt

$$\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx = \int \frac{-t\,dt}{t^2 + 3t + 2}$$

$$= -\int \frac{t}{t^2 + 3t + 2} dt \qquad \ldots(1)$$

Let $\frac{t}{t^2 + 3t + 2} = \frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

⇒ t = A(t + 2) + B(t + 1) …(2)

Put t = –1 in (2)

–1 = A(–1 + 2) ⇒ A = –1

Put t = –2 in (2)

–2 = B(–2 + 1) ⇒ B = 2

$$\therefore \frac{t}{t^2 + 3t + 2} = \frac{-1}{t+1} + \frac{2}{t+2} \qquad \ldots(3)$$

∴ From (1) and (3)

$$\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx = -\left[\int \frac{-1}{t+1} dt + 2\int \frac{1}{t+2} dt\right]$$

$$= \int \frac{1}{t+1} dt - 2\int \frac{1}{t+2} dt$$

$$= \log|t+1| - 2\log|t+2| + C$$

$$= \log|1+\cos x| - 2\log|2+\cos x| + C$$

$$= \log|1+\cos x| - \log(2+\cos x)^2 + C$$

$$= \log\left|\frac{1+\cos x}{(2+\cos x)^2}\right| + C$$

6. $\int \frac{1}{a^2 - x^2} dx$ for $x \in I = (-a, a)$.

Sol: $\frac{1}{a^2 - x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}$

Then $1 = A(a + x) + B(a - x)$

Put $x = a$, $1 = 2a\,A \Rightarrow A = \frac{1}{2a}$

And if $x = -a$, then $2a\,B = 1 \Rightarrow B = \frac{1}{2a}$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\int \frac{dx}{a-x} + \frac{1}{2a}\int \frac{dx}{a+x}$$

$$= -\frac{1}{2a}\log|a-x| + \frac{1}{2a}\log|x+a| + c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a+x}\right| + c.$$

Reduction Formulae

Theorem: If $I_n = \int x^n e^{ax}\, dx$, then $I_n = \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}$ where n is a positive integer.

Proof:

$$I_n = \int x^n e^{ax}\, dx = x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} n\, x^{n-1}\, dx$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a}\int e^{ax} x^{n-1} dx = \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}$$

Theorem: If $I_n = \int \sin^n x\, dx$ then $I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$, where n is a positive integer.

Theorem: If $I_n = \int \cos^n x\, dx$ then $I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$.

Proof:

$$I_n = \int \cos^{n-1} x \cos x\, dx$$

$$= \cos^{n-1} x \sin x - \int \sin x (n-1) \cos^{n-2} x(-\sin x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x(1-\cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n (1+n-1) = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$\therefore I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Theorem: If $I_n = \int \tan^n x\, dx$ then $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$.

Proof:

$$I_n = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} dx = \frac{\tan^{n-1} dx}{n-1} - I_{n-2}$$

Theorem: If $I_n = \int \cot^n x\, dx$ then $I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$.

Proof:

$$I_n = \int \cot^{n-2} x \cot^2 x\, dx = \int \cot^{n-2} x(\csc^2 x - 1) dx$$

$$= \int \cot^{n-2} x \csc^2 x\, dx - \int \cot^{n-2} x\, dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

Theorem: If $I_n = \int \sec^n x\,dx$ then $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

Proof:

$$I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$$

$$= \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Theorem: If $I_n = \int \csc^n x\,dx$ then $I_n = \dfrac{-\csc^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

Proof:

$$I_n = \int \csc^{n-2} x \csc^2 x\,dx$$

$$= -\csc^{n-2} x \cot x - \int -\cot x (n-2) \csc^{n-3} x (-\csc x \cot x)\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^{n-2} x \cot^2 x\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^{n-2} x (\csc^2 x - 1)dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2} x\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = -\csc^{n-2} x \cot x + (n-2)I_{n-2}$$

$$I_n = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Theorem: If $I_n = \int (\log x)^n dx$ then $I_n = x(\log x)^n - nI_{n-1}$.

Proof:

$$I_n = \int (\log x)^n dx = x(\log x)^n - \int x\, n(\log x)^{n-1} \frac{1}{x} dx$$

$$x(\log x)^n - n\int (\log x)^{n-1} dx = x(\log x)^n - nI_{n-1}.$$

Theorem: If $I_{m,n} = \int \sin^m x \cos^n x\, dx$ then

i) $I_{m,n} = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \dfrac{n-1}{m+n} I_{m,n-2}$

ii) $I_{m,n} = -\dfrac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \dfrac{m-1}{m+n} I_{m-2,n}$

Proof:

i) $I_{m,n} = \int \sin^m x \cos^n x\, dx = \int (\sin^m x \cos^{n-1} x) \cos x\, dx$

$$= \sin^m x \cos^{n-1} x \sin x - \int \left[\sin^m x (n-1)\cos^{n-2} x(-\sin x) + \cos^{n-1} x\, m \sin^{m-1} x \cos x\right] \sin x\, dx$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} x \sin^2 x dx - m\int \sin^m x \cos^n x\, dx$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} x(1-\cos^2 x)dx - m\, I_{m,n}$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} x\, dx - (n-1)\int \sin^m x \cos^n x\, dx - m\, I_{m,n}$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1)I_{m,n-2} - (n-1)I_{m,n} - mI_{m,n}$$

$$\Rightarrow (m+n)I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1)I_{m,n-2}$$

$$\Rightarrow I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

ii) $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$= \int \sin^{m-1} x \cos^n x \sin x\, dx$$

$$= \sin^{m-1} x \cos^n x(-\cos x) - \int [\sin^{m-1} x\, n \cos^{n-1} x(-\sin x) - \cos^n x(m-1)\sin^{m-2} x \cos x](-\cos x)dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - n\int \sin^m x \cos^n x\, dx + (m-1)\int \sin^{m-2} x \cos^n x \cos^2 x\, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1)\int \sin^{m-2} x \cos^n x(1-\sin^2 x)dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1)\int \sin^{m-2} x \cos^n x\, dx - (m-1)\int \sin^m x \cos^n x\, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1)I_{m-2,n} - (m-1)I_{m,n}$$

$$\Rightarrow (m+n)I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1)I_{m-2,n}$$

$$\Rightarrow I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}.$$

Very Short Answer Questions

Evaluate the following integrals.

1. $\int e^x (1+x^2)dx$

Sol. $\int e^x (1+x^2)dx = \int e^x dx + \int x^2 e^x dx$

$$= e^x + (x^2 \cdot e^x - 2\int x \cdot e^x dx)$$
$$= e^x + x^2 \cdot e^x - 2(x \cdot e^x - \int e^x dx)$$

$$= e^x + x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C$$
$$= e^x (x^2 - 2x + 3) + C$$

2. $\int x^2 e^{-3x} dx$

Sol. $\int x^2 e^{-3x} dx = \frac{x^2 e^{-3x}}{-3} + \frac{1}{3}\int e^{-3x} \cdot 2x\, dx$

$$= -\frac{x^2 e^{-3x}}{3} + \frac{2}{3}\left(\frac{xe^{-3x}}{-3} + \frac{1}{3}\int e^{-3x} dx\right)$$

$$= -\frac{x^2 e^{-3x}}{3} - \frac{2}{9} x \cdot e^{-3x} - \frac{2}{27} e^{-3x} + C$$

$$= \frac{-e^{-3x}}{27}\left(9x^2 + 6x + 2\right) + C$$

3. $\int x^3 e^{ax} dx$

Sol. $\int x^3 e^{ax} dx = \frac{x^3 e^{ax}}{a} - \frac{1}{a}\int e^{ax}(3x^2\,dx)$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax} dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left(\frac{x^2 e^{ax}}{a} - \frac{1}{a}\int e^{ax} 2x\,dx\right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}\cdot x^2 \cdot e^{ax} + \frac{6}{a^2}\int x \cdot e^{ax} dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}\left(\frac{x e^{ax}}{a} - \frac{1}{a}\int e^{ax}\cdot dx\right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x \cdot e^{ax}}{a^3} - \frac{6}{a^4} e^{ax} + C$$

$$= \frac{e^{ax}}{a^4}\left[a^3x^3 - 3a^2x^2 + 6ax - 6\right] + C$$

Short Answer Questions

1. Show that

$$\int x^n \cdot e^{-x} dx = -x^n e^{-x} + n\int x^{n-1} \cdot e^{-x} dx$$

Sol. $\int x^n \cdot e^{-x} dx = \frac{x^n e^{-x}}{(-1)} + \int e^{-x} \cdot nx^{n-1} dx$

$$= -x^n e^{-x} + n\int x^{n-1} \cdot e^{-x} dx$$

2. If $I_n = \int \cos^n x\, dx$ **, then show that** $I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$ **.**

Sol. $I_n = \int \cos^n x\, dx = = \int \cos^{n-1} x \cos x\, dx$

$$= \cos^{n-1} x \sin x - \int \sin x (n-1)\cos^{n-2} x(-\sin x) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n (1+n-1) = \cos^{n-1} x \cdot \sin x + (n-1) I_n$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Long Answer Questions

1. **Obtain reduction formula for** $I_n = \int \cot^n x\,dx$**, n being a positive integer, n ≥ 2 and deduce the value of** $\int \cot^4 x\,dx$**.**

Sol. $I_n = \int \cot^n x\,dx = \int \cot^{n-2} x \cdot \cot^2 x\,dx$

$$= \int \cot^{n-2} x \cdot (\csc^2 x - 1)\,dx$$

$$= \int \cot^{n-2} x \cdot \csc^2 x dx - I_{n-2}$$

$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$n = 4 \Rightarrow I_4 = -\frac{\cot^3 x}{3} - I_2$$

$$n = 2 \Rightarrow I_2 = -\cot x - I_0 \text{ where } I_0 = \int dx = x$$

$$I_2 = -\cot x - x$$

$$I_4 = -\frac{\cot^3 x}{3} - (-\cot x - x) + C$$

$$= -\frac{\cot^3 x}{3} + \cot x + x + C$$

2. **Obtain the reduction formula for** $I_n = \int \csc^n x\,dx$**, n being a positive integer, n ≥ 2 and deduce the value of** $\int \operatorname{cosec}^5 x\,dx$**.**

Sol. $I_n = \int \csc^n x\,dx = \int \csc^{n-2} x \cdot \csc^2 x\,dx$

$$= \csc^{n-2} x(-\cot x) + \int \cot x (n-2)\csc^{n-3} x(\cot x)dx$$

$$= -\csc^{n-2} x \cot x + (n-2)\int \csc^{n-2} x(\csc^2 x - 1)dx$$

$$= -\csc^{n-2} x \cot x + (n-2)I_{n-2} - (n-2)I_n$$

$$I_n(1 + n - 2) = -\csc^{n-2} x \cdot \cot x + (n-2)I_{n-2}$$

$$I_n = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$n = 5 \Rightarrow I_5 = -\frac{\csc^3 x \cdot \cot x}{4} + \frac{3}{4} I_3$$

$$I_3 = -\frac{\csc x \cdot \cot x}{2} + \frac{1}{2} I_1$$

$$I_1 = \int \csc x \, dx = \log\left|\tan\frac{x}{2}\right|$$

$$I_3 = -\frac{\csc x \cdot \cot x}{2} + \frac{1}{2}\log\left|\tan\frac{x}{2}\right|$$

$$I_5 = -\frac{\csc^3 x \cdot \cot x}{4} - \frac{3}{8}\csc x \cot x + \frac{3}{8}\log\left|\tan\frac{x}{2}\right| + C$$

3. If $I_{m,n} = \int \sin^m x \cos^n x \, dx$**, then show that** $I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2}, n$ **for a positive integer n and an integer m ≥ 2.**

Sol. $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \sin^{m-1} x \cdot (\cos x)^n \sin x \, dx$$

$$= \int \sin^{m-1}(x)(\cos x)^n (-\sin x) dx$$

$$= -\left[\sin^{m-1}(x)\int (\cos x)^n (-\sin x) dx - \int \left\{\frac{d}{dx}\sin^{m-1}(x) \cdot \int \cos^n(x)(-\sin x) dx\right\} dx\right]$$

$$= -\left[\sin^{m-1}(x)\frac{\cos^{n+1}(x)}{n+1} - \int \left\{(m-1)\sin^{m-2}(x)\cos x \frac{\cos^{n+1} x}{n+1}\right\} dx\right]$$

$$= -\sin^{m-1}(x)\frac{\cos^{n+1}(x)}{n+1} + \frac{m-1}{n+1}\int \{\sin^{m-2}(x)\cos^n x \cos^2 x\} dx$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)\int \{\sin^{m-2}(x)\cos^n x - \sin^m(x)\cos^n(x)\} dx$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right)\left[\int \sin^{m-2}(x)\cos^n x dx - \int \sin^m(x)\cos^n(x) dx\right]$$

$$= -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n}$$

$$\therefore I_{m,n} + \left(\frac{m-1}{n+1}\right) I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$\therefore \left(\frac{m+n}{n+1}\right) I_{m,n} = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{n+1} + \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$\therefore I_{m,n} = \frac{1}{m+n}(\sin^{m-1}(x)\cos^{n+1}(x)) + \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

4. Evaluate $\int \sin^5 x \cos^4 x dx$ **.**

Sol.Reduction formula

$$I_{m,n} \frac{-\sin^{m-1} x \cdot \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

$$I_{5,4} = -\frac{\sin^4 x \cos^5 x}{9} + \frac{4}{9} I_{3,4}$$

$$I_{3,4} = -\frac{\sin^2 x \cos^5 x}{7} + \frac{2}{7} I_{1,4}$$

$$I_{1,4} = \int \sin x \cos^4 x \, dx = -\frac{\cos^5 x}{5}$$

$$I_{3,4} = -\frac{\sin^2 x \cos^5 x}{7} - \frac{2}{35}\cos^5 x$$

$$I_{5,4} = -\frac{\sin^4 x \cos^5 x}{9} + \frac{4}{9}\left(-\frac{\sin^2 x \cos^5 x}{7} - \frac{2}{35}\cos^5 x\right) + C$$

$$= -\frac{\sin^4 x \cos^5 x}{9} - \frac{4}{63}\sin^2 x \cdot \cos^5 x - \frac{8}{315}\cos^5 x + C$$

5. Find reduction formula for $\int x^n e^{nx} dx$, n being a positive integer and hence evaluate $\int x^3 e^{5x} dx$.

Sol: Let $I_n = \int x^n e^{ax} dx$ using integration by parts

$$= x^n \frac{e^{ax}}{a} - \int n \cdot x^{n-1} \frac{e^{ax}}{a} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$ is the reduction formula.

Now $I_3 = \int x^3 e^{5x} dx$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5} I_2$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}\left[\frac{x^2 e^{5x}}{5} - \frac{2}{5} I_1\right]$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \int x^3 e^{5x} dx$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25}\left(\frac{x e^{5x}}{5} - \int \frac{e^{5x}}{5} dx\right)$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x^2 e^{5x} - \frac{6}{625} e^{5x} + c = \frac{x^3 e^{5x}}{5} - \frac{3}{5^2} x^2 e^{5x} + \frac{6}{5^3} x^2 e^{5x} - \frac{6}{5^4} e^{5x} + c.$$

6. Obtain reduction formula for $\int \sin^n x\, dx$ for an integer $n \geq 2$ and hence obtain $\int \sin^4 x\, dx$.

Sol: Let $I_n = \int \sin^n x\, dx$

$= \int \sin^{n-1} x \sin x\, dx$ using integration by parts.

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n = \frac{1}{n}[(n-1)I_{n-2} - \sin^{n-1} x \cos x]$$

$= \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$ is the reduction formula for $\int \sin^n x\, dx$.

Now $\int \sin^4 x\, dx = \dfrac{-\sin^3 x \cos x}{4} + \dfrac{3}{4} I_2$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left[\frac{-\sin x \cos x}{2} + \frac{1}{2} I_0\right]$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x\, dx + \frac{3}{8}\int dx + c$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x\, dx + \frac{3}{8} x + c.$$

7. Obtain reduction formula for $\int \sin^m x \cos^n x\, dx$ for a positive integer m and integer n ≥ 2.

Sol: Let $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$= \int \sin^m x \cos^{n-1} x \cos x\, dx$$

Using integration by parts for $u = \cos^{n-1}x$ and $v = \sin^m x$.

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x(-\sin x)\frac{\sin^{m+1} x}{m+1} dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \left(\frac{n-1}{m+1}\right)\int \sin^{m+2} x \cos^{n-2} x dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \left(\frac{n-1}{m+1}\right)\int \sin^m x \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x - \frac{n-1}{m+1}\int \sin^m x \cos^n x\, dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\therefore I_{m,n}\left(1 + \frac{n-1}{m+1}\right) = \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,n-2}$$

$$\Rightarrow I_{m,n}\left(\frac{m+n}{n+1}\right)=\frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x+\frac{n-1}{m+1}I_{m,n-2}$$

$$\Rightarrow I_{m,n}=\frac{m+n}{n+1}\left[\frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x+\frac{n-1}{m+1}I_{m,n-2}\right]$$

$$=\frac{1}{m+1}[\cos^{n-1}x\sin^{m+1}x+(n-1)I_{m,n-2}]$$

is the reduction formula for

$$I_{m,n}=\int\sin^m x\cos^n x\,dx.$$

8. Obtain reduction formula for $\int\tan^n x\,dx$ for an integer $n\geq 2$ and hence find $\int\tan^6 x\,dx$.

Sol: Let $I_n=\int\tan^n x\,dx$

$$=\int\tan^{n-2}x\tan^2 x\,dx$$

$$=\int\tan^{n-2}x(\sec^2 x-1)dx$$

$$=\int\tan^{n-2}x\sec^2 x\,dx-\int\tan^{n-2}x\,dx$$

$=\dfrac{\tan^{n-1}x}{n-1}-I_{n-2}$ is the reduction formula for $\int\tan^n x dx$.

Now $I_6=\int\tan^6 x\,dx$

$$=\frac{\tan^5 x}{5}-I_4$$

$$=\frac{\tan^5 x}{5}-\left[\frac{\tan x^3}{3}-I_2\right]$$

$$=\frac{\tan^5 x}{5}-\frac{\tan x^3}{3}+\frac{\tan x}{1}-I_0$$

$$=\frac{\tan^5 x}{5}-\frac{\tan x^3}{3}+\tan x-x+c.$$

9. Obtain reduction formula for $\int \sec^n x\,dx$ for $n \geq 2$ and hence evaluate $\int \sec^5 x\,dx$.

Sol: Let $I_n = \int \sec^n x\,dx$

$$= \int \sec^{n-2} x \sec^2 x\,dx$$

$$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \cdot \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)I_n - (n-2)I_{n-2}$$

$$= \sec^{n-2} x \tan x - (n-2)[I_n - I_{n-2}]$$

$$\Rightarrow I_n(1+n-2) = \sec^{n-2} \tan x + (n-2)I_{n-2}$$

$$\Rightarrow I_n = \frac{1}{n-1}\left[\sec^{n-2} x \tan x + (n-2)I_{n-2}\right]$$

$$= \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}I_{n-2}$$

Now $\int \sec^5 x\,dx = I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}I_3$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left[\frac{1}{2}\sec x \tan x + \frac{1}{2}I_1\right]$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\int \sec x\,dx$$

$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\log|\sec x + \tan x| + c$$

10. If $I_n = \int (\log x)^n dx$, then show that $I_n = x(\log x)^n - nI_{n-1}$, and hence find $\int (\log x)^4 dx$.

Sol. $I_n = \int (\log x)^n dx$

$$= (\log x)^n x - \int x \cdot n \cdot (\log x)^{n-1}\frac{1}{x}dx$$

$$= x(\log x)^n - n\int (\log x)^{n-1} dx$$

$$= x(\log x)^n - n \cdot I_{n-1}$$

$I_4 = x(\log x)^4 - 4\cdot I_3$

$I_3 = (x\log x)^3 - 3\cdot I_2$

$I_2 = (x\log x)^2 - 2\cdot I_1$

$I_1 = x\log x - I_0$ where $I_0 = \int dx = x$

$I_1 = x \log x - x$

$I_2 = (x\log x)^2 - 2x\log x + 2x$

$I_3 = (x)(\log x)^3 - 3x(\log x)^2 - 2x\log x + 2x$

$= x(\log x)^3 - 3x(\log x)^2 + 6x(\log x) - 6x$

$I_4 = x(\log x)^4 - 4(x)(\log x)^3 - 3x(\log x)^2 + 6x(\log x) - 6x] + C$

$= x[(\log x)^4 - 4(\log x)^3 + 12(\log x)^2 - 24(\log x) + 24] + C$

Problems for Practice

Very Short Answer Questions

1. Find $\int 2x^7 dx$ **on R.**

Ans. $\frac{x^8}{4} + C$

2. Evaluate $\int \cot^2 x\, dx$ **on I ⊂ R \ {nπ : n ∈ Z}.**

Ans. $-\cot x - x + C$

3. Evaluate $\int \frac{x^6 - 1}{1 + x^2} dx$ for $x \in R$.

Ans. $\frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1} x + C$

4. Find $\int (1-x)(4-3x)(3+2x)dx, x \in R$ **.**

Ans. $\frac{3}{2}x^4 - \frac{5}{3}x^3 - \frac{13}{2}x^2 + 12x + C$

5. Evaluate $\int \left(x + \frac{1}{x}\right)^3 dx, x > 0$**.**

Ans. $\frac{x^4}{4} + \frac{3x^2}{2} + 3\log x - \frac{1}{2x^2} + C$

6. Find $\int \sqrt{1+\sin 2x}\, dx$ **on R.**

Ans. cos x – sin x + C

7. Evaluate $\int \frac{2x^3 - 3x + 5}{2x^2} dx$ **for x > 0 and check the result by differentiation.**

Ans. $\frac{x^2}{2} - \frac{3}{2}\log x - \frac{5}{2x} + C$

8. Evaluate $\int \frac{x^5}{1+x^{12}} dx$ **on R.**

Ans. $\frac{1}{6}\tan^{-1} x^6 + C$

9. Evaluate $\int \cos^3 x \sin x\, dx$ **on R.**

Ans. $\frac{-\cos^4 x}{4} + C$

10. Find $\int \left(1 - \frac{1}{x^2}\right) e^{\left(x + \frac{1}{x}\right)} dx$ **on I where I = (0, ∞)**

Ans. $e^{\left(x + \frac{1}{x}\right)} + C$

11. Evaluate $\int \frac{1}{\sqrt{\sin^{-1} x}\sqrt{1-x^2}} dx$ **on I = (0, 1).**

Ans. $2\sqrt{\sin^{-1} x} + C$

12. Evaluate

$$\int \frac{\sin^4 x}{\cos^6 x} dx, x \in I \subset R \setminus \left\{\frac{(2n+1)\pi}{2} : n \in Z\right\}$$

Ans. $\frac{1}{5}\tan^5 x + C$

13. Evaluate $\int \sin^2 x \, dx$ **on R.**

Ans. $\frac{1}{2}\sin 2x + C$

14. Find $\int \frac{x^2}{\sqrt{x+5}} dx$ **on (–5, ∞)**

Ans. $\frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50\sqrt{x+5} + C$

15. $\int \frac{(x^3 - x)^{1/3}}{x^4} dx$ **on (0, ∞)**

Ans. $\frac{3}{8}\left(1 - \frac{1}{x^2}\right)^{4/3} + C$

16. Find $\int \frac{x}{\sqrt{1-x}} dx, \ x \in I = (0,1)$ **.**

Ans. $\frac{2}{3}(1-x)^{3/2} - 2\sqrt{1-x} + C$

17. Evaluate $\int \frac{dx}{(x+5)\sqrt{x+4}}$ **on (–4, ∞)**

Ans. $2\tan^{-1}(\sqrt{x+4})+C$

18. Evaluate $\int \frac{dx}{\sqrt{4-9x^2}}$ on $I=\left(-\frac{2}{3},\frac{2}{3}\right)$

Ans. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$

19. Evaluate $\int \frac{1}{a^2-x^2}dx$ for $x\in I=(-a,a)$.

Ans. $\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

20. Evaluate $\int \frac{1}{1+4x^2}dx$ **on R.**

Ans. $\frac{1}{2}\tan^{-1}2x+C$

21. Find $\int \frac{1}{\sqrt{4-x^2}}dx$ **on (–2, 2).**

Ans. $\sin^{-1}\left(\frac{x}{2}\right)+C$

22. Evaluate $\int \sqrt{4x^2+9}\,dx$ **on R.**

Ans. $\frac{x\sqrt{4x^2+9}}{2}+\frac{9}{4}\sinh^{-1}\left(\frac{2x}{3}\right)+C$

23. Evaluate $\int \sqrt{9x^2 - 25}\, dx$ on $\left(\frac{5}{3}, \infty\right)$**.**

Ans. $\frac{x\sqrt{9x^2-25}}{2} - \frac{25}{6}\cosh^{-1}\left(\frac{3x}{5}\right) + C$

24. Evaluate $\int \sqrt{16 - 25x^2}\, dx$ on $\left(\frac{-4}{5}, \frac{4}{5}\right)$**.**

Ans. $\frac{x\sqrt{16-25x^2}}{2} + \frac{8}{5}\sin^{-1}\left(\frac{5x}{4}\right) + C$

25. Evaluate $\int x \sin^{-1} x\, dx$ **on (–1, 1).**

Ans. $\frac{x^2 \sin^{-1} x}{2} - \frac{1}{4}(\sin^{-1} x - x\sqrt{1-x^2}) + C$

26. Evaluate $\int x^2 \cos x\, dx$

Ans. $(x^2 - 2)\sin x + 2x\cos x + C$

27. Evaluate $\int e^x \sin x\, dx$ **on R.**

Ans. $\frac{1}{2}e^x(\sin x - \cos x) + C$

33. Evaluate $\frac{dx}{4x^2 - 4x - 7}$

Ans. $\frac{1}{8\sqrt{2}}\log\left|\frac{2x-1-2\sqrt{2}}{2x-1+2\sqrt{2}}\right| + C$

34. Find $\int \frac{dx}{5-2x^2+4x}$.

Ans. $\frac{1}{2\sqrt{14}}\log\left|\frac{x-1+\sqrt{\frac{7}{2}}}{x-1-\sqrt{\frac{7}{2}}}\right|+C$

35. Evaluate $\int \frac{dx}{x^2+x+1}$

Ans. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

36. Evaluate $\int \frac{dx}{\sqrt{x^2+2x+10}}$

Ans. $\sinh^{-1}\left(\frac{x+1}{3}\right)+C$

37. Evaluate $\int \frac{dx}{\sqrt{1+x-x^2}}$

Ans. $\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C$

38. Evaluate $\int \sqrt{3+8x-3x^2}\,dx$

Ans. $\frac{(3x-4)\sqrt{3+8x-3x^2}}{6}+\frac{25}{6\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)+C$

39. Evaluate $\int \frac{x+1}{x^2+3x+12}dx$

Ans. $\frac{1}{2}\log|x^2+3x+12|-\frac{1}{\sqrt{39}}\tan^{-1}\left(\frac{2x+3}{\sqrt{39}}\right)+C$

40. Evaluate $\int (3x-2)\sqrt{2x^2-x+1}dx$

Ans. $\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}}-\frac{35}{64\sqrt{2}}\sinh^{-1}\left(\frac{4x-1}{\sqrt{7}}\right)+C$

41. Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}}dx$

Ans. $2\sqrt{x^2-2x+10}+7\sinh^{-1}\left(\frac{x-1}{3}\right)+C$

42. Evaluate $\int \frac{dx}{5+4\cos x}$

Ans. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+C$

43. Find $\int \frac{dx}{3\cos x+4\sin x+6}$

Ans. $\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3\tan(x/2)+4}{\sqrt{11}}\right)+C$

44. Evaluate $\int \frac{\cos x+3\sin x+7}{\cos x+\sin x+1}dx$

Ans. $-\log|\cos x+\sin x+1|+2x+5\log\left|1+\tan\frac{x}{2}\right|+C$

45. Find $\int \frac{x^3-2x+3}{x^2+x-2}dx$

Ans. $\frac{(x-1)^2}{2}+\log|c(x+2)^{1/3}(x-1)^{2/3}+C$

46. Find $\frac{dx}{x^2-81}$

Ans. $\frac{1}{18}\log\left|\frac{x-9}{x+9}\right|+C$

47. Find $\int \frac{2x^2-5x+1}{x^2(x^2-1)}dx$

Ans. $\frac{1}{x}+\log\left|\frac{x^5}{(x^2-1)(x+1)^3}\right|+C$

48. Find $\int \frac{3x-5}{x(x^2+2x+4)}dx$

Ans. $-\frac{5}{4}\log|x|+\frac{5}{8}\log|x^2+2x+4|+\frac{17}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$

49. $\int \frac{2x+1}{x(x^2+4)^2}dx$

Ans. $\frac{1}{16}\log|x|-\frac{1}{32}\log(x^2+4)+\frac{1}{8(x^2+4)}+\frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{4}\left(\frac{x}{4+x^2}\right)+C$

50. Evaluate $x^3\cdot e^{5x}dx$ **.**

Ans.

$\frac{x^3\cdot e^{5x}}{5}-\frac{3}{25}x^2\cdot e^{5x}+\frac{6}{125}x\cdot e^{5x}-\frac{6}{625}e^{5x}+C$

51. Evaluate $\int \sin^4 x dx$

Ans. $-\frac{\sin^3 x\cdot\cos x}{4}-\frac{3}{8}\sin x\cdot\cos x+\frac{3}{8}+C$

52. $\int \sec^5 x\,dx$

Ans. $\frac{\sec^3 x\cdot\tan x}{4}+\frac{3}{8}\sec x\tan x+\frac{3}{8}\log|\sec x+\tan x|+C$