INTEGRATION BY PARTS

Theorem: If f(x) and g(x) are two integrable functions then

$$\int f(x)\cdot g(x)dx = f(x)\int g(x)dx - \int f'(x)\left[\int g(x)dx\right]dx.$$

Proof:

$$\frac{d}{dx}\left[f(x)\cdot\int g(x)dx\right] = f(x)\frac{d}{dx}\left[\int g(x)dx\right] + \int g(x)dx\cdot\frac{d}{dx}[f(x)]$$

$$= f(x)g(x) + \left[\int g(x)dx\right]f'(x)$$

$$\therefore \int\left[f(x)g(x) + f'(x)\int g(x)dx\right]dx = f(x)\int g(x)dx$$

$$\Rightarrow \int f(x)g(x)dx + \int f'(x)\left[\int g(x)dx\right]dx = f(x)\int g(x)dx$$

$$\therefore \int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\left[\int g(x)dx\right]dx$$

Note 1: If u and v are two functions of x then $\int u\,dv = uv - \int v\,du$.

Note 2: If u and v are two functions of x; u', u'', u''' …… denote the successive derivatives of u and v_1, v_2, v_3, v_4, v_5 … the successive integrals of v then the extension of integration by pairs is $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$

Note 3: In integration by parts, the first function will be taken as the following order.

Inverse functions, Logarithmic functions, Algebraic functions, Trigonometric functions and Exponential functions. (To remember this a phrase ILATE).

Theorem: $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

Proof: Let I = $\int e^{ax}\cos bx\,dx = \cos bx\int e^{ax}dx - \int\left[d(\cos bx)\int e^{ax}dx\right]dx$

$$= \cos bx\frac{e^{ax}}{a} - \int(-b\sin bx)\frac{e^{ax}}{a}dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\sin bx \frac{e^{ax}}{a} - \int b\cos bx \frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I$$

$$\Rightarrow I\left(1+\frac{b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left[a\cos bx + b\sin bx\right]$$

$$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left[a\cos bx + b\sin bx\right]$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$$

Theorem: $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

Proof : Let I = $\int e^{ax}\sin bx\,dx = \sin bx\int e^{ax}dx - \int\left[d(\sin bx)\int e^{ax}dx\right]dx$

$$= \sin bx\frac{e^{ax}}{a} - \int b\cos bx\frac{e^{ax}}{a}dx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left[\cos bx\frac{e^{ax}}{a} - \int(-b\sin bx)\frac{e^{ax}}{a}dx\right]$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I$$

$$\Rightarrow I\left(1+\frac{a^2}{b^2}\right) = \frac{1}{a^2}e^{ax}\left[a\sin bx - b\cos bx\right]$$

$$\Rightarrow I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}}{a^2}\left[a\sin bx - b\cos bx\right]$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c$$

Theorem: $\int e^x [f(x)+f'(x)]dx = e^x f(x)+c$

Proof:

$$\int e^x [f(x)+f'(x)]dx = \int e^x f(x)dx + \int e^x f'(x)dx$$

$$= f(x)\int e^x dx - \int\left[d[f(x)]\int e^x dx\right]dx + \int e^x f'(x)dx$$

$$= f(x)e^x - \int f'(x)e^x dx + \int e^x f'(x)dx = e^x f(x)+c$$

Note: $\int e^{-x}[f(x)-f'(x)]dx = -e^{-x}f(x)+c$

Very Short Answer Questions

Evaluate the following integrals.

1. $\int x\sec^2 x\,dx$ **on** $I \subset R \setminus \left\{\frac{(2n+1)\pi}{2} : \text{ n is an integer}\right\}$

Sol. $\int x\sec^2 x\,dx = x(\tan x) - \int \tan x\,dx$

$$= x\tan x - \log|\sec x| + C$$

2. $\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$**,** $x \in R$**.**

Sol.

Let $f(x)=\tan^{-1}x$ so that $f'(x)=\frac{1}{1+x^2}$

$$\therefore \int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx = e^x\tan^{-1}x + C \left(\because \int e^x[f(x)+f'(x)]dx = e^x\cdot f(x)+C\right)$$

3. $\int \frac{\log x}{x^2}dx$ **on** $(0,\infty)$**.**

Sol. $\int \frac{\log x}{x^2}dx = (\log x)\left(-\frac{1}{x}\right) + \int \frac{1}{x}\cdot\frac{1}{x}dx$

$$= -\frac{1}{x}\log x - \frac{1}{x} + C$$

4. $\int (\log x)^2 dx$ **on (0, ∞).**

Sol. $\int (\log x)^2 dx = (\log x)^2 x - \int x \cdot 2\log x \cdot \frac{1}{x} dx$

$= x(\log x)^2 - 2\int \log x\, dx$

$= x(\log x)^2 - 2\left(x\log x - \int x \frac{1}{x} dx \right)$

$= x(\log x)^2 - 2x \cdot \log x + x + c$

5. $\int e^x (\sec x + \sec x \tan x)dx$ **on** $I \subset R \setminus \left\{ (2n+1)\frac{\pi}{2} : n \in Z \right\}$

Sol. $\int e^x (\sec x + \sec x \tan x)dx = e^x \cdot \sec x + C$

$\left(\because \int e^x [f(x) + f'(x)]dx = e^x f(x) + C \right)$

6. $\int e^x \cos x\, dx$ **on R.**

Sol. $I = \int e^x \cos x\, dx = e^x \sin x - \int \sin x \cdot e^x dx$

$= e^x \cdot \sin x + e^x \cdot \cos x - \int e^x \cdot \cos x\, dx$

$= e^x (\sin x + \cos x) - I$

$2I = e^x (\sin x + \cos x)$

$I = \frac{e^x}{2} (\sin x + \cos x) + C$

7. $\int e^x (\sin x + \cos x)dx$ **on R.**

Sol. $\int e^x (\sin x + \cos x)dx$

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$\therefore \int e^x (\sin x + \cos x)dx = e^x \cdot \sin x + C$

8. $\int (\tan x + \log \sec x)e^x dx$ **on** $\left(\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right) n \in Z$

Sol. let $f = \log|\sec x| \Rightarrow f' = \frac{1}{\sec x} \cdot \sec x \cdot \tan x \cdot$

$= \tan x$

$\int (\tan x + \log \sec x)e^x dx = e^x \cdot \log|\sec x| + C \left(\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + C\right)$

Short Answer Questions

Evaluate the following integrals.

1. $\int x^n \log x\, dx$ **on (0, ∞), n is a real number and n ≠ –1.**

Sol. $\int x^n \log x\, dx = (\log x)\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx$

$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\int x^n dx$

$= \frac{x^{n+1}(\log x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$

$= \frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1] + C$

2. $\int \log(1+x^2)dx$ **on R.**

Sol. $\int \log(1+x^2)dx = \int 1.\log(1+x^2)dx =$

$= \log(1+x^2) \cdot x - \int x \frac{1}{1+x^2} 2x\, dx$

$= x\log(1+x^2) - 2\int \frac{1+x^2-1}{1+x^2}dx$

$$= x\log(1+x^2) - 2\int dx + 2\int \frac{dx}{1+x^2}$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C$$

3. $\int \sqrt{x}\log x\,dx$ **on (0, ∞).**

Sol. $\int \sqrt{x}\log x\,dx =$

$$= \log x \cdot \frac{2}{3}x^{3/2} - \frac{2}{3}\int x^{3/2}\cdot\frac{1}{x}dx$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{2}{3}\frac{x^{3/2}}{3/2} + C$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{4}{9}x^{3/2} + C$$

4. $\int e^{\sqrt{x}}dx$ **on (0, ∞).**

Sol. let $\sqrt{x} = t \Rightarrow x = t^2,\ dx = 2t\,dt$

$$\int e^{\sqrt{x}}dx = 2\int t\,e^t dt = 2\left[te^t - \int e^t dt\right]$$

$$= 2(te^t - e^t) + C$$

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

5. $\int x^2\cos x\,dx$ **on R.**

Sol. $\int x^2\cos x\,dx = x^2(\sin x) - \int \sin x(2x\,dx)$

$$= x^2\sin x + 2\int x(-\sin x)dx$$

$$= x^2\cdot\sin x + 2[x\cos x - \int\cos x\,dx]$$

$$= x^2\sin x + 2x\cos x - 2\sin x + c$$

6. $\int x\sin^2 x\,dx$ **on R.**

Sol. $\int x\sin^2 x\,dx = \frac{1}{2}\int x(1-\cos x)dx$

$$= \frac{1}{2}\left[\int x dx - \int x\cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left\{x\cdot\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right\}\right]$$

$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{4}\int \sin 2x\,dx$$

$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$$

7. $\int x\cos^2 x\,dx$ **on R.**

Sol. $\int x\cos^2 x\,dx = \frac{1}{2}\int x(1+\cos 2x)dx$

$$= \frac{1}{2}\left[\int x\,dx + \int x\cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \left\{x\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right\}\right]$$

$$= \frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{4}\int \sin 2x\,dx$$

$$= \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + C$$

8. $\int \cos\sqrt{x}\,dx$ **on R.**

Sol. $x = t^2 \Rightarrow dx = 2t\,dt$

$$I = 2\int t\cdot\cos t\,dt = 2(t\sin t - \int \sin t\,dt)$$

$$= 2(t\sin t + \cos t) + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

9. $\int x\sec^2 2x\,dx$ **on** $I \subset R \setminus \left\{(2n\pi+1)\frac{\pi}{4} : n \in Z\right\}$

Sol. $\int x\sec^2 2x\,dx = x\frac{\tan 2x}{2} - \frac{1}{2}\int \tan 2x\,dx$

$$= x\frac{\tan 2x}{2} - \frac{1}{2}\cdot\frac{1}{2}\log|\sec 2x| + C$$

$$= x\frac{\tan 2x}{2} - \frac{1}{4}\log|\sec 2x| + C$$

10. $\int x\cot^2 x\,dx$ **on** $I \subset R \setminus \{n\pi : n \in Z\}$**.**

Sol. $\int x\cot^2 x\,dx = \int x(\csc^2 x - 1)dx$

$$= \int x\csc^2 x\,dx - \int x\,dx$$

$$= x(-\cot x) + \int \cot x\,dx - \frac{x^2}{2}$$

$$= -x\cot x + \log|\sin x| - \frac{x^2}{2} + C$$

11. $\int e^x(\tan x + \sec^2 x)dx$ **on** $I \subset R \setminus \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Sol. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x\,dx$

$$I = \int e^x[f(x) + f'(x)]dx = e^x f(x) + C$$

$$= e^x \tan x + C$$

12. $\int e^x\left(\frac{1 + x\log x}{x}\right)dx$ **on (0, ∞).**

Sol. $\int e^x\left(\frac{1 + x\log x}{x}\right)dx = \int e^x\left(\log x + \frac{1}{x}\right)dx$

$$= e^x \log x + C$$

13. $\int \frac{dx}{(x^2+a^2)^2}, (a>0)$ on R.

Sol: Take substitution x = a tan θ

So that dx = a sec²θ dθ

$$\therefore \int \frac{dx}{(x^2+a^2)^2} = \int \frac{a\sec^2\theta d\theta}{(a^2\tan^2\theta+a^2)^2}$$

$$= \int \frac{a\sec^2\theta d\theta}{a^4(1+\tan^2\theta)^2} = \frac{1}{a^3}\int \frac{\sec^2\theta d\theta}{\sec^4\theta}$$

$$= \frac{1}{a^3}\int \cos^2\theta d\theta$$

$$= \frac{1}{a^3}\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2a^3}\left[\int 1\cdot d\theta + \int \cos 2\theta\, d\theta\right]$$

$$= \frac{1}{2a^3}\left[\theta + \frac{1}{2}\sin 2\theta\right]$$

$$= \frac{1}{2a^3}\left[\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left[2\tan^{-1}\left(\frac{x}{a}\right)\right]\right] + c$$

$$= \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{4a^3}\sin\left[2\tan^{-1}\left(\frac{x}{a}\right)\right] + c.$$

14. $\int e^x \log(e^{2x}+5e^x+6)dx$ on r.

Sol: $e^{2x}+5e^x+6 = (e^x)^2+5e^x+6$

$$= (e^x)^2+3e^x+2e^x+6$$

$$= e^x(e^x+3)+2(e^x+3)$$

$$= (e^x+3)(e^x+2)$$

$$\int e^x \log(e^{2x}+5e^x+6)dx$$

$$= \int e^x \log[(e^x+2)(e^x+3)]dx$$

$$= \int e^x \log(e^x+2)dx + \int e^x \log(e^x+3)dx \quad (\because \log ab = \log a + \log b)$$

Let $e^x = t$ then $e^x\,dx = dt$

$$\therefore \int e^x \log(e^{2x}+5e^x+6)dx$$

$$=\int \log(t+2)dt+\int \log(t+3)dt$$

$$=\log(t+2)t-\int \frac{t}{t+2}dt+\log(t+3)\cdot t-\int \frac{t}{t+3}dt$$

(using integration by parts on two integrals)

$$=t\cdot\log(t+2)-\int\left(\frac{(t+2)-2}{t+2}\right)dt+t\cdot\log(t+3)-\int\left(\frac{(t+3)-3}{t+3}\right)dt$$

$$=t\cdot\log(t+2)-\int dt+2\int\frac{dt}{t+2}+t\log(t+3)-\int dt+3\int\frac{dt}{t+3}$$

$$=t\log(t+2)-t+2\log(t+2)+t\log(t+3)-t+3\log(t+3)$$

$$=2\log|(t+2)|+3\log|(t+3)|-2t+t[\log(t+2)(t+3)]$$

$$=t[\log(t^2+5t+6)]-2t+2\log|t+2|+3\log|t+3|+c$$

$$=e^x[\log(e^{2x}+5e^x+6)]-2e^x+2\log|e^x+2|+3\log|e^x+3|+c.$$

15. $\int \cos(\log x)dx$ **on (0, ∞).**

Sol: Let $I=\int \cos(\log x)dx=\int \cos(\log x)1\cdot dx$

Take u = cos(log x) and v = 1 and using integration by parts successively.

$$I=\cos(\log x)x-\int -\sin(\log x)\frac{1}{x}x\cdot dx$$

$$=x\cos(\log x)+\int \sin(\log x)dx$$

$$=x\cos(\log x)+\sin(\log x)\cdot x-\int \cos(\log x)\frac{1}{x}\cdot x\cdot dx$$

$$=x\cos(\log x)+x\cdot\sin(\log x)-\int \cos(\log x)dx$$

$$=x[\cos(\log x)+\sin(\log x)]-1$$

$$\therefore 2I=x[\cos(\log x)+\sin(\log x)]$$

$$\Rightarrow I=\frac{x}{2}[\cos(\log x)+\sin(\log x)]+c$$

$$\therefore \int \cos(\log x)dx=$$

$$\frac{x}{2}[\cos(\log x)+\sin(\log x)]+c$$

16. $\int e^{x} \frac{x+2}{(x+3)^2} dx$ **on I ⊂ R \ {–3}**

Sol. $\int e^{x} \frac{x+2}{(x+3)^2} dx$

Hint: $\int e^{x} [f(x)+f'(x)] dx = e^{x} - f(x) + C$

$$= \int e^{x} \left\{ \frac{x+3-1}{(x+3)^2} \right\} dx$$

$$= \int e^{x} \left\{ \frac{1}{x+3} + \frac{(-1)}{(x+3)^2} \right\} dx = e^{x} \left(\frac{1}{x+3} \right) + C$$

17. $\int \frac{xe^{x}}{(x+1)^2} dx$ **on I ⊂ R \ {–1}**

Sol. $\int \frac{xe^{x}}{(x+1)^2} dx = \int \left[\frac{x+1-1}{(x+1)^2} \right] e^{x} dx$

$$= \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2} \right] e^{x} dx$$

$$= \int \left[\left(\frac{1}{x+1} \right) + \frac{(-1)}{(x+1)^2} \right] e^{x} dx$$

Hint: $\int e^{x} [f(x)+f'(x)] dx = e^{x} - f(x) + C$

$$= \left(\frac{1}{x+1} \right) e^{x} + C = \frac{e^{x}}{x+1} + C$$

Long Answer Questions

Evaluate the following integrals.

1. $\int x\tan^{-1}x\,dx,\ x\in R$

Sol. $\int x\tan^{-1}x\,dx = (\tan^{-1}x)\frac{x^2}{2} - \frac{1}{2}\int x^2\cdot\frac{1}{1+x^2}dx$

$= \frac{x^2(\tan^{-1}x)}{2} - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$

$= \frac{x^2(\tan^{-1}x)}{2} - \frac{1}{2}(x-\tan^{-1}x)+C$

$= \frac{x^2(\tan^{-1}x)}{2} - \frac{x}{2} + \frac{\tan^{-1}x}{2}+C$

$= \frac{(x^2+1)}{2}\tan^{-1}x - \frac{x}{2}+C$

2. $\int x^2\tan^{-1}x\,dx,\ x\in R$.

Sol. $\int x^2\tan^{-1}x\,dx = (\tan^{-1}x)\frac{x^3}{3} - \frac{1}{3}\int x^3\frac{1}{1+x^2}dx$

$= \frac{x^3(\tan^{-1}x)}{3} - \frac{1}{3}\int\frac{x(x^2+1)-x}{1+x^2}dx$

$= \frac{x^3(\tan^{-1}x)}{3} - \frac{1}{3}\int x\,dx + \frac{1}{3}\int\frac{x\,dx}{1+x^2}$

$= \frac{x^3(\tan^{-1}x)}{3} - \frac{x^2}{6} + \frac{1}{6}\log|1+x^2|+C$

3. $\int\frac{\tan^{-1}x}{x^2}dx, x\in I\subset R\setminus\{0\}$

Sol. $\int\frac{\tan^{-1}x}{x^2}dx = \int\tan^{-1}x\frac{1}{x^2} = (\tan^{-1}x)\left(-\frac{1}{x}\right)+\int\frac{1}{x}\frac{1}{1+x^2}dx$

$= -\frac{\tan^{-1}x}{x} + \frac{1}{2}\int\frac{2x\,dx}{x^2(1+x^2)}$

$= -\frac{\tan^{-1}x}{x} + \frac{1}{2}\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)(2x\,dx)$

$$= \frac{\tan^{-1} x}{x} + \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$$

$$= -\frac{\tan^{-1} x}{x} + \log|x| - \frac{1}{2}\log|1+x^2| + C$$

4. $\int x\cos^{-1} x\,dx, x \in (-1,1)$

Sol. $\int x\cos^{-1} x$

$$= \cos^{-1}\int x\,dx - \int\left[\frac{d}{dx}[\cos^{-1} x]\int x\,dx\right]dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \int \frac{-1}{\sqrt{1-x^2}}\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\cos^{-1} x + \frac{1}{2}\int \frac{x^2}{1-x^2}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\int \sqrt{1-x^2}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{2}\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] + \frac{1}{2}\sin^{-1} x + C$$

$$= \frac{x^2}{2}\cos^{-1} x - \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x + C$$

5. $\int x^2\sin^{-1} x\,dx, x \in (-1,1)$

Sol. $\int x^2\sin^{-1} x\,dx$

$$= (\sin^{-1} x)\frac{x^3}{3} - \frac{1}{3}\int x^3\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{3}\int \frac{x[1-(1-x^2)]}{\sqrt{1-x^2}}dx$$

$$=\frac{x^3}{3}\sin^{-1}x-\frac{1}{3}\int\frac{xdx}{\sqrt{1-x^2}}+\frac{1}{3}\int x\sqrt{1-x^2}dx$$

$$=\frac{x^3}{3}\sin^{-1}x+\frac{1}{3}\sqrt{1-x^2}+\frac{1}{3}\frac{(1-x^2)^{3/2}}{(3/2)(-2)}+C$$

$$=\frac{x^3}{3}\sin^{-1}x+\frac{\sqrt{1-x^2}}{3}-\frac{1}{9}(1-x^2)^{3/2}+C$$

6. $\int x\log(1+x)dx, x\in(-1,\infty)$

Sol. $\int x\log(1+x)dx$

$$=\log(1+x)\left(\frac{x^2}{2}\right)-\frac{1}{2}\int\frac{x^2}{1+x}dx$$

$$=\frac{x^2}{2}\log(1+x)-\frac{1}{2}\int\frac{1-(1-x^2)}{1+x}dx$$

$$=\frac{x^2}{2}\log(1+x)-\frac{1}{2}\int\frac{dx}{1+x}+\frac{1}{2}\int(1-x)dx$$

$$=\frac{x^2}{2}\log(1+x)-\frac{1}{2}\log(1+x)+\frac{1}{2}\left(x-\frac{x^2}{2}\right)+C$$

$$=\frac{(x^2-1)}{2}\log(1+x)+\frac{x}{2}-\frac{x^2}{4}+C$$

7. $\int\sin\sqrt{x}\,dx$ **on (0, ∞).**

Sol. put $x=t^2\Rightarrow dx=2t\,dt$

$$=2\left[t(-\cos t)+\int\cos t\,dt\right]$$

$$=-2t\cos t+2\sin t$$

$$=-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$$

8. $\int e^{ax}\sin(bx+c)dx$ **, (a, b, c ∈ R, b ≠ 0) on R.**

Sol.

Let $I = \int e^{ax}\sin(bx+c)dx$

$$= e^{ax}\left(-\frac{\cos(bx+c)}{b}\right) + \frac{1}{b}\int \cos(bx+c)e^{ax}a\,dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b}\int e^{ax}\cos(bx+c)dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b}\left(e^{ax}\cdot\sin\frac{bx+c}{b}\right) - \frac{1}{b}\int \sin(bx+c)e^{ax}\cdot a\cdot dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}I$$

$$\left(1+\frac{a^2}{b^2}\right)I = -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c) \quad \frac{a^2+b^2}{b^2}I = \frac{e^{ax}}{b^2}[a\sin(bx+c) - b(\cos(bx+c)]$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b(\cos(bx+c)] + C_1$$

9. $\int a^x \cos 2x\,dx$ **on R(a > 0 and a ≠ 1).**

Sol. $\int a^x \cos 2x\,dx$

$$= a^x\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\cdot a^x \log a\,dx$$

$$= \frac{a^x\cdot\sin 2x}{2} + \frac{\log a}{2}\int a^x(-\sin 2x)dx$$

$$= \frac{a^x\sin 2x}{2} + \frac{\log a}{2}\left(a^x\cdot\cos\frac{2x}{2} - \frac{1}{2}\int \cos 2x\cdot a^x\log a\,dx\right)$$

$$= \frac{a^x\sin 2x}{2} + \frac{a^x\log a\cos 2x}{4} - \frac{(\log a)^2}{4}I$$

$$\left(1+\frac{(\log a)^2}{4}\right)I = \frac{a^x[2\sin 2x + (\log a)\cos 2x]}{4}$$

$$\frac{4+(\log a)^2}{4}I = \frac{a^x[2\sin 2x + (\cos 2x)\log a]}{4}$$

$$\therefore I = \frac{2\cdot a^x \cdot \sin 2x + (a^x \cdot \log a)\cos 2x}{(\log a)^2 + 4} + c$$

10. $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$ **on** $I \subset R \setminus \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$.

Sol. Put x = tan t ⇒ dx = $\sec^2$ t dt

Then $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$

$$= \int \tan^{-1}\left(\frac{3\tan t - \tan^3 t}{1 - 3\tan^2 t}\right)\sec^2 t\,dt$$

$$= \int \tan^{-1}(\tan 3t)\sec^2 t\,dt = 3\int t\sec^2 t\,dt$$

$$= 3\left[t\int \sec^2 t\,dt - \int\left\{\frac{d}{dt}(t)\int \sec^2 t\,dt\right\}dt\right]$$

$$= 3[t(\tan t) - \int (1)\tan t\,dt]$$

$$= 3(t\tan t - \log|\sec t|) + C$$

$$= 3\left(x\cdot\tan^{-1}x - \log\sqrt{1+x^2}\right) + C$$

$$= 3x\left[\tan^{-1}x - \frac{3}{2}\log(1+x^2)\right] + C$$

$$= 3x\tan^{-1}(x) - \frac{3}{2}\log(1+x^2) + C$$

11. $\int \sinh^{-1}x\,dx$ **on R.**

Sol. $\int \sinh^{-1}x\,dx = \int 1.\sinh^{-1}x\,dx$

$$= x.\sinh^{-1}x - \int \frac{1}{\sqrt{1+x^2}}.xdx$$

$$= x.\sinh^{-1}x - \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx$$

$$= x.\sinh^{-1}x - \frac{1}{2}2.\sqrt{1+x^2} + c$$

$$= x.\sinh^{-1}x - \sqrt{1+x^2} + c$$

12. $\int \cosh^{-1} x dx$ **on [1, ∞].**

Sol. $\int \cosh^{-1} x dx = \int 1.\cosh^{-1} x dx$

Apply integration of parts.

Ans. $x\cosh^{-1} x - \sqrt{x^2-1} + C$

13. $\int \tanh^{-1} x\, dx$ **on (–1, 1).**

Sol. $\int \tanh^{-1} x\, dx = \int 1.\tanh^{-1} x\, dx$

$$= \int 1.\tanh^{-1} x\, dx$$

$$= x.\tanh^{-1} x - \int \frac{1}{1-x^2} x dx$$

$$= x.\tanh^{-1} x + \frac{1}{2}\int \frac{-2x}{1-x^2} dx$$

$$= x.\tanh^{-1} x + \frac{1}{2}\log\left(1-x^2\right) + c$$

14. Find $\int e^{ax}\cos(bx+c)dx$ **on R where a, b, c are real numbers and b ≠ 0.**

Sol. Let A = $\int e^{ax}\cos(bx+c)dx$

Then from the formula for integration by parts

$$A = e^{ax}\left[\frac{\sin(bx+c)}{b}\right] - \int ae^{ax}\left[\frac{\sin(bx+c)}{b}\right]dx$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) - \frac{a}{b}\int e^{ax}\sin(bx+c)dx$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) - \frac{a}{b}\left[e^{ax}\left\{\frac{-\cos(bx+c)}{b}\right\} - \int ae^{ax}\left\{-\frac{\cos(bx+c)}{b}\right\}dx\right] + C_1$$

$$= \frac{1}{b}e^{ax}\sin(bx+c) + \frac{a}{b^2}e^{ax}\cos(bx+c) - \frac{a^2}{b^2}A + C_2$$

$$\left(1+\frac{a^2}{b^2}\right)A = \frac{a}{b^2}e^{ax}\cos(bx+c) + \frac{1}{b}e^{ax}\sin(bx+c) + C_2$$

$$(a^2+b^2)A = ae^{ax}\cos(bx+c) + be^{ax}\sin(bx+c) + C_3$$

Hence $A = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + K$

Where $k = \frac{c_3}{a^2+b^2}$ a constant

By taking c = 0, we get

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + K$$

15. Evaluate $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$ **on (–1, 1).**

Sol. Put x = cosθ, θ∈ (0, π)dx = –sin θ dθ

$$\frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\theta/2}{2\cos^2\theta/2} = \tan^2\frac{\theta}{2}$$

$$\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx = \int \tan^{-1}\sqrt{\tan^2\frac{\theta}{2}}(-\sin\theta)d\theta$$

$$= -\int \tan^{-1}\left(\tan\frac{\theta}{2}\right)(\sin\theta)d\theta$$

$$= -\frac{1}{2}\int \theta\cdot\sin\theta\,d\theta$$

$$= -\frac{1}{2}\left[\theta(-\cos\theta) - \int(-\cos\theta)d\theta\right] + C$$

$$= \frac{1}{2}(\theta\cos\theta - \sin\theta) + C$$

$$= \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$$

16. Evaluate $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$ **on I ⊂ R \ {2nπ : n ∈ Z}.**

Sol. $\frac{1-\sin x}{1-\cos x} = \frac{1-\sin x}{2\sin^2 x/2}$

$$= \frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} = \frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\csc^2\frac{x}{2} - \cot\frac{x}{2}$$

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int e^x\left(\frac{1}{2}\csc^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$$

$$= \int e^x\left[f(x) + f'(x)\right]dx \text{ where } f(x) = -\cot\frac{x}{2}$$

$$= e^x f(x) + C = -e^x \cot\frac{x}{2} + C$$

17. Evaluate $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$ on $I \subset R \setminus (-1, 1)$.

Sol. Let x = tan θ ⇒ dx = $\sec^2\theta\, d\theta$

$$\frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}(\tan 2\theta) = 2\theta + n\pi$$

Where n = 0 if |x| < 1

= –1 if x > 1

= 1 if x < –1

We have $d\theta = \frac{1}{1+x^2}dx$ and

$1+x^2 = 1 + \tan^2\theta = \sec^2\theta$

$$\therefore \int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

$$= \int\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)(1+x^2)\frac{1}{1+x^2}dx$$

$$= \int(2\theta + n\pi)\int \sec^2\theta d\theta$$

$$= 2\int \theta\sec^2\theta d\theta + n\pi\int \sec^2\theta d\theta + c$$

$$= 2\left(\theta\tan\theta - \int \tan\theta d\theta\right) + n\pi\tan\theta + c$$

$$= 2(\theta\tan\theta + \log|\cos\theta| + n\pi\tan\theta + c$$

$$= (2\theta + n\pi)\tan\theta + 2\log\cos\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\cos^2\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\sec^2\theta + c$$

$$= x\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \log(1+x^2) + c$$

18. Find $\int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx$ on (–1, 1) where m is a real number. (Here for Y ∈ R, exp.(y) stands for e^y).

Sol. Let t = $\sin^{-1}$ x, then

$$x = \sin t, dt = \frac{1}{\sqrt{1-x^2}}dx, \text{ for } x \in (-1, 1)$$

$$\text{Hence } \int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx = \int e^{mt}\sin^2 t\,dt$$

$$= \int e^{mt}\left(\frac{1-\cos 2t}{2}\right)dt$$

$$= \frac{1}{2}\int e^{mt}dt - \frac{1}{2}\int e^{mt}\cdot\cos 2t\,dt + c \qquad \ldots(1)$$

Case (i): m = 0

$$\text{From (1) } \int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\int dt - \frac{1}{2}\int \cos 2t\,dt + C$$

$$= \frac{t}{2} - \frac{\sin 2t}{4} + C$$

$$= \frac{\sin^{-1}x}{2} - \frac{1}{4}\sin(2\sin^{-1}x) + C$$

Case (ii): m ≠ 0

$$\text{From (1) } \int x^2 \frac{\exp(m\sin^{-1}x)}{\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\frac{e^{mt}}{m} - \frac{1}{2}\frac{e^{mt}}{m^2+4}(m\cos 2t + 2\sin 2t) + C_1$$

$$= \frac{e^{mt}}{2}\left(\frac{1}{m} - \frac{1}{m^2+4}(m\cos 2t + 2\sin 2t)\right) + C_1$$

$$= \frac{e^{m\sin^{-1}x}}{2}\left(\frac{1}{m} - \frac{1}{m^2+4}(m\cos(2\sin^{-1}x) + -2\sin(2\sin^{-1}x)\right) + C_1$$

Integration of Some Special Types of Functions

Type I: If the integral is of the form $\int \frac{px+q}{ax^2+bx+c}dx$ then take px + q = $A\frac{d}{dx}(ax^2+bx+c)+B$.

Type II: $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$. Take px + q = $A\frac{d}{dx}(ax^2+bx+c)+B$.

Type III: $\int (px+q)\sqrt{ax^2+bx+c}\,dx$. Take px + q = $A\frac{d}{dx}(ax^2+bx+c)+B$.

Type IV: $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}dx$. To evaluate this put px + q = $\frac{1}{t}$.

Type V: $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$. To evaluate this, put x = $\frac{1}{t}$.

Type VI : $\int \frac{px+q}{\sqrt{ax+b}}dx$ or $\int \frac{\sqrt{ax+b}}{px+q}dx$ or $\int (px+q)\sqrt{ax+b}\,dx$ or $\int \frac{1}{(px+q)\sqrt{ax+b}}dx$. Put ax + b = t^2.

Then $dx = \frac{1}{a}2t\,dt$.

Integration Of Functions Which Are Rational in sin x and cos x.

Type I: If the integral is of the form $\int \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$ or $\int \frac{dx}{a\cos^2 x+b\sin x\cos x+c\sin^2 x}$ then multiply both numerator and denominator with $\sec^2 x$ and take tan x = t.

Type II: If the integral is of the form $\int \frac{dx}{a+b\cos x}$ or $\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a\cos x+b\sin x+c}$, take

$\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=dt \Rightarrow (1+\tan^2 x/2)dx = \quad 2dt \quad \Rightarrow dx=\frac{2dt}{1+t^2}$. $\quad \sin x=\frac{2\tan x/2}{1+\tan^2 x/2}=\frac{2t}{1+t^2}$,

$\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}=\frac{1-t^2}{1+t^2}$.

Type III : If the integral is of the form $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx$, take a cos x + b sin x = $A\frac{d}{dx}$(c cos x + d sin x). By equating the coefficients of cos x, sin x we get the values of A and B. Then the given integral becomes A log|ccos x + d sin x| + Bx + k.

Very Short Answer Questions

Evaluate the following integrals.

1. $\int \frac{dx}{\sqrt{2x - 3x^2 + 1}}$

Sol. $\int \frac{dx}{\sqrt{2x - 3x^2 + 1}}$

$$= \int \frac{dx}{\sqrt{3\left(\frac{2x}{3} - x^2 + \frac{1}{3}\right)}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{x - \frac{1}{3}}{\frac{2}{3}}\right) + C = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right) + C$$

2. $\int \frac{\sin\theta}{\sqrt{2 - \cos^2\theta}}d\theta$

Sol. $\int \frac{\sin\theta}{\sqrt{2 - \cos^2\theta}}d\theta$

$$\text{Put } \cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$$

$$= \int -\frac{dt}{\sqrt{2 - t^2}} = -\int \frac{dt}{\sqrt{(\sqrt{2})^2 - t^2}}$$

$$= -\sin^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = -\sin^{-1}\left(\frac{\cos\theta}{\sqrt{2}}\right) + C$$

3. $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

Sol. $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

put $\sin x = t \Rightarrow \cos x dx = dt$

$$= \int \frac{dt}{t^2 + 4t + 5} = \int \frac{dt}{(t+2)^2 + 1}$$

$$= \tan^{-1}(t+2) + C = \tan^{-1}(\sin x + 2) + C$$

4. $\int \frac{dx}{1 + \cos^2 x}$

Sol. $\int \frac{dx}{1+\cos^2 x} = \int \frac{\sec^2 dx}{\sec^2 x + 1} = \int \frac{\sec^2 xdx}{\tan^2 x + 2}$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

5. $\int \frac{dx}{2\sin^2 x + 3\cos^2 x}$

Sol. $\int \frac{dx}{2\sin^2 x + 3\cos^2 x} = \int \frac{\sec^2 xdx}{2\tan^2 x + 3}$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{dt}{2t^2 + 3} = \frac{1}{2} \int \frac{dt}{t^2 + \left(\sqrt{\frac{3}{2}}\right)^2}$$

$$= \frac{2}{2\sqrt{\frac{3}{2}}} \tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right) + C$$

6. $\int \frac{1}{1+\tan x} dx$

Sol. $\int \frac{1}{1+\tan x} dx$

$$= \int \frac{1}{1+\frac{\sin x}{\cos x}} dx = \int \frac{\cos x\, dx}{\sin x + \cos x} = \frac{1}{2}\int \frac{2\cos x\, dx}{\sin x + \cos x}$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$$

7. $\int \frac{1}{1-\cot x} dx$

Sol. $\int \frac{1}{1-\cot x} dx = \int \frac{1}{1-\frac{\cos x}{\sin x}} dx = \int \frac{\sin x dx}{\sin x - \cos x}$

$$= \frac{1}{2}\int \frac{(\sin x - \cos x) + (\cos x + \sin x)}{\sin x - \cos x} dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x - \cos x} dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$$

Short Answer Questions

Evaluate the following integrals.

1. $\int \sqrt{1+3x-x^2}\,dx$

Sol. $\int \sqrt{1+3x-x^2}\,dx = \int \sqrt{1-(x^2-3x)}\,dx$

$= \int \sqrt{1-(x-\frac{3}{2})^2 - \frac{9}{2}}\ dx$

$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right) - \left(x-\frac{3}{2}\right)^2}$

$= \frac{\left(x-\frac{3}{2}\right)\sqrt{1+3x-x^2}}{2} + \frac{13}{8}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$

$= \frac{(2x-3)\sqrt{1+3x-x^2}}{2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$

2. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx$

Sol. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx$

let 9cosx-sinx $= A\ \frac{d}{dx}(4\sin x + 5\cos x) + B(4\sin x + 5\cos x)$

9cosx-sinx $= A\ (4\cos x - 5\sin x) + B(4\sin x + 5\cos x)$

Comparing the coefficients of sin and cos , we get

9 = 4A+5B and -5 = -5A+4B

Solving these equations , A =1 and B=1.

$\therefore$ 9cosx-sinx $=1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)$

$=1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)$

$$\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}dx = \int \frac{(4\sin x + 5\cos x) + (4\cos x - 5\sin x)}{4\sin x + 5\cos x}dx$$

$$= \int dx + \int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx$$

$$= x + \log|4\sin x + 5\cos x| + C$$

3. $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}dx$

Sol. Let 2 cos x + 3 sin x = A(4 cosx + 5 sin x) + B(–4 sin x + 5 cos x)

Equating the coefficient of sinx and cosx, we get 4A + 5B = 2, 5A – 4B = 3.

$$\begin{array}{cccc} & A & B & 1 \\ +5 & -2 & 4 & +5 \\ -4 & -3 & 5 & -4 \end{array}$$

$$\frac{A}{-15-8} = \frac{B}{-10+12} = \frac{1}{-16-25}$$

$$A = \frac{23}{41}, B = -\frac{2}{41}$$

$$\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}dx =$$

$$= \frac{23}{41}\int dx - \frac{2}{41}\int \frac{-4\sin x + 5\cos x}{4\cos x + 5\sin x}dx$$

$$= \frac{23}{41}x - \frac{2}{41}\log|4\cos x + 5\sin x| + C$$

4. $\int \frac{dx}{1 + \sin x + \cos x}$

Sol. $\int \frac{dx}{1 + \sin x + \cos x}$

$$= \int \frac{dx}{\left[1 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{1 + \tan^2 \frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2 \frac{x}{2}}$$

$$= \int \frac{\sec^2 \frac{x}{2}}{2 + 2\tan\frac{x}{2}} \text{ put } \tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2} dx = dt$$

$$= 2\int \frac{dt}{2+2t} = \int \frac{dt}{1+t} \log|1+t| + C$$

$$= \log\left|1 + \tan\frac{x}{2}\right| + C$$

5. $\int \frac{dx}{3x^2 + x + 1}$

Sol. $\int \frac{dx}{3x^2 + x + 1} = \int \frac{dx}{3\left(x^2 + \frac{1}{3}x + \frac{1}{3}\right)}$

$$= \frac{1}{3}\int \frac{dx}{\left(x + \frac{1}{6}\right)^2 + \frac{1}{3} - \frac{1}{36}} = \frac{1}{3}\int \frac{dx}{\left(x + \frac{1}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2}$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{\sqrt{11}}{6}} \tan^{-1}\left(\frac{x + (1/6)}{(\sqrt{11}/6)}\right) + C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right) + C$$

6. $\int \frac{dx}{\sqrt{5 - 2x^2 + 4x}}$

Sol. $\int \frac{dx}{\sqrt{5 - 2x^2 + 4x}}$

$5 - 2x^2 + 4x$

$$= -2\left[x^2 - 2x - \frac{5}{2}\right] = -2\left[(x-1)^2 - 1 - \frac{5}{2}\right]$$

$$= -2\left[(x-1)^2 - \frac{7}{2}\right] = 2\left[\frac{7}{2} - (x-1)^2\right]$$

Now $\int \frac{1}{\sqrt{5-2x^2+4x}}dx$

$$= \int \frac{1}{\sqrt{2}\left\{\frac{7}{2} - (x-1)^2\right\}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{7}{2}\right)^2 - (x-1)^2}}dx$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{(x-1)}{\sqrt{7/2}} + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{2}{7}}(x-1) + C$$

Long Answer Questions

Evaluate the following integrals.

1. $\int \frac{x+1}{\sqrt{x^2-x+1}}dx$

Sol. take $x+1 = A\frac{d}{dx}\left(x^2-x+1\right)+B$

$x+1 = A\left(2x-1\right)+B$

Comparing the coefficients of like terms,

$2A = 1$ and $B-A = 1$

$\Rightarrow A=\frac{1}{2}$ and $b = \frac{3}{2}$

$\therefore x+1 = \frac{1}{2}\left(2x-1\right)+\frac{3}{2}$

$$\int \frac{x+1}{\sqrt{x^2-x+1}}dx = \int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{\sqrt{x^2-x+1}}dx$$

$$= \frac{1}{2}\int \frac{(2x-1)dx}{\sqrt{x^2-x+1}}+\frac{3}{2}\int \frac{dx}{\sqrt{x^2-x+1}}$$

$$= \sqrt{x^2-x+1}+\frac{3}{2}\int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2-x+1}+\frac{3}{2}\sinh^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \sqrt{x^2-x+1}+\frac{3}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

2. $\int (6x+5)\sqrt{6-2x^2+x}\,dx$

Sol.

let $6x+5=A\dfrac{d}{dx}\left(6-2x^2+x\right)+B$

$\Rightarrow 6x + 5 = A(1 - 4x) + B$

Equating the coefficients

$$6=-4A \Rightarrow A=\frac{-3}{2}$$

Equating the constants

$A + B = 5$

$B = 5 - A = 5+\frac{3}{2}=\frac{13}{2}$

$\int (6x+5)\sqrt{6-2x^2+x}\,dx$

$$=-\frac{3}{2}\int (1-4x)\sqrt{6-2x^2+x}\,dx+\frac{13}{2}\sqrt{6-2x^2+x}\,dx$$

$$=-\frac{3}{2}\frac{(6-2x^2+x)^{3/2}}{3/2}+\frac{13}{2}\sqrt{2}\int\sqrt{3-x^2+\frac{x}{2}}dx$$

$$=-(6-2x^2+x)^{3/2}+\frac{13}{\sqrt{2}}\int\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}dx$$

$$=-(6-2x^2+x)^{3/2}+\frac{13}{\sqrt{2}}$$

$$\left(\frac{\left(x-\frac{1}{4}\right)\sqrt{3-x^2+\frac{x}{2}}}{2}+\frac{49}{32}\sin^{-1}\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{7}{4}\right)}\right)+C$$

$$=-(6-2x^2+x)^{3/2}+\frac{13}{\sqrt{2}}\left[\frac{(4x-1)\sqrt{6-2x^2+x}}{16\times 2}+\frac{49}{32}\sin^{-1}\left(\frac{4x-1}{7}\right)\right]+C$$

$$=-(6-2x^2+x)^{3/2}+\frac{13}{16}(4x-1)\sqrt{6-2x^2+x}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$$

3. $\int \frac{dx}{4+5\sin x}$

Sol. $\int \frac{dx}{4+5\sin x} = \int \frac{dx}{4+5\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$$\text{put } \tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$$

$$\Rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+t^2}$$

$$\text{G.I.} = 2\int \frac{\frac{dt}{1+t^2}}{4+5\frac{2t}{1+t^2}} = 2\int \frac{dt}{4+4t^2+10t}$$

$$= \frac{1}{2}\int \frac{dt}{t^2+\frac{5t}{2}+1} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right| + C = \frac{1}{3}\log\left|\frac{2t+1}{2t+4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\left(\tan\frac{x}{2}\right)+2}\right| + C$$

4. $\int \frac{1}{2-3\cos 2x}dx$

Sol. $\int \frac{1}{2-3\cos 2x}dx = \int \frac{dx}{2-3\frac{1-\tan^2 x}{1+\tan^2 x}}$

put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$dx = \frac{dt}{1+t^2}$$

$$GI = \int \frac{\frac{dt}{1+t^2}}{2-3\frac{1-t^2}{1+t^2}} = \int \frac{dt}{2+2t^2-3+3t^2}$$

$$= \int \frac{dt}{5t^2-1} = \frac{1}{5}\int \frac{dt}{t^2-\left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\frac{(1/2)}{\sqrt{5}}\log\left|\frac{t-\frac{1}{\sqrt{5}}}{t+\frac{1}{\sqrt{5}}}\right| + C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right| + C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right| + C$$

5. $\int x\sqrt{1+x-x^2}\,dx$

Sol. Let x = A(1 – 2x) + B

Equating the coefficients of x

1 = –2A ⇒ A = –1/2

Equating the constants

0 = A + B ⇒B = –A = 1/2

$\int x\sqrt{1+x-x^2}\,dx =$

$$-\frac{1}{2}\int(1-2x)\sqrt{1+x-x^2}dx+\frac{1}{2}\int\sqrt{1+x-x^2}dx$$

$$=-\frac{1}{2}\frac{(1+x-x^2)^{3/2}}{3/2}+\frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}dx$$

$$=-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{2}\left(\frac{\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2}}{2}+\frac{25}{8}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)\right)$$

$$=-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{(2x-1)\sqrt{1+x-x^2}}{8}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C$$

6. $\int\frac{dx}{(1+x)\sqrt{3+2x-x^2}}$

Sol. $\int\frac{dx}{(1+x)\sqrt{3+2x-x^2}}=\int\frac{dx}{(1+x)\sqrt{(3-x)(1+x)}}$

Put $1+x=t^2 \Rightarrow dx=2t\,dt$

$$\text{G.I.}=\int\frac{2t\,dt}{t^2\sqrt{t^2(4-t^2)}}=\int\frac{2dt}{t^2\sqrt{4-t^2}}=\int\frac{2}{t^3}\frac{dt}{\sqrt{\frac{4}{t^2}-1}}$$

Put $\frac{4}{t^2}-1=y^2 \Rightarrow -\frac{8}{t^3}dt=2y\,dy$

$$\Rightarrow \frac{2}{t^3}dt=-\frac{y}{4}dy$$

$$\text{G.I.}=2\int-\frac{y}{4}\frac{dy}{\sqrt{y^2}}=-\frac{1}{2}\int dy=-\frac{1}{2}y+C$$

$$=-\frac{1}{2}\sqrt{\frac{4}{t^2}-1}+C$$

$$=-\frac{1}{2}\sqrt{\frac{4}{1+x}-1}+C-\frac{1}{2}\sqrt{\frac{3-x}{3+x}}+C$$

7. $\int \frac{dx}{4\cos x + 3\sin x}$

Sol. $\int \frac{dx}{4\cos x + 3\sin x} = \int \frac{dx}{4\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 3\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}$

$$I = \int \frac{\frac{2dt}{1+t^2}}{4\frac{(1-t^2)}{1+t^2} + \frac{3\cdot 2t}{1+t^2}} = 2\int \frac{dt}{4-4t^2+6t}$$

$$= -\frac{1}{2}\int \frac{dt}{t^2 - \frac{3}{2}t - 1} = -\frac{1}{2}\int \frac{dt}{\left(t-\frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2}$$

$$= -\frac{1}{2}\frac{1}{2\cdot\frac{5}{4}}\log\left|\frac{t-\frac{3}{4}-\frac{5}{4}}{t-\frac{3}{4}+\frac{5}{4}}\right| + C$$

$$= -\frac{1}{5}\log\left|\frac{t-2}{t+(1/2)}\right| + C = -\frac{1}{5}\log\left|\frac{2t-4}{2t+1}\right| + C$$

$$= -\frac{1}{5}\log\left|\frac{2\left(\tan\frac{x}{2}-2\right)}{2\tan\frac{x}{2}+1}\right| + C$$

8. $\int \frac{1}{\sin x + \sqrt{3}\cos x}dx$

Sol. Let $t = \tan\frac{x}{2}$ so that $dx = \frac{2dt}{1+t^2}$

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$I=\int\frac{2\frac{dt}{1+t^2}}{\frac{2t}{1+t^2}+\frac{\sqrt{3}(1-t^2)}{1+t^2}}=2\int\frac{dt}{\sqrt{3}(1-t^2)+2t}$$

$$=\frac{2}{\sqrt{3}}\int\frac{dt}{1-t^2+\frac{2}{\sqrt{3}}t}=\frac{2}{\sqrt{3}}\int\frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2-\left(t-\frac{1}{\sqrt{3}}\right)^2}$$

$$=\frac{2}{\sqrt{3}}\frac{1/4}{\sqrt{3}}\log\left|\frac{\frac{2}{\sqrt{3}}+t-\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}-t+\frac{1}{\sqrt{3}}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{t+\frac{1}{\sqrt{3}}}{\sqrt{3}-t}\right|+C=\frac{1}{2}\log\left|\frac{\sqrt{3}t+1}{\sqrt{3}(\sqrt{3}-t)}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{\sqrt{3}\tan\frac{x}{2}+1}{\sqrt{3}\left(\sqrt{3}-\tan\frac{x}{2}\right)}\right|+c$$

9. $\int\frac{dx}{5+4\cos 2x}$

Sol. $t=\tan x \Rightarrow dt=\sec^2 x\,dx,\ dx=\frac{dt}{1+t^2}$

$$I=\int\frac{\frac{dt}{1+t^2}}{\frac{5+4(1-t^2)}{1+t^2}}=\int\frac{dt}{5+5t^2+4-4t^2}$$

$$=\int\frac{dt}{t^2+9}=\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$$

$$=\frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right)+C$$

10. $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$

Sol.

$\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$

Let $2\sin x + 3\cos x + 4 = A(3\sin x + 4\cos x + 5) + B\frac{d}{dx}(3\sin x + 4\cos x + 5) + C$

2 sinx + 3 cos x + 4

= A(3sinx + 4cosx + 5) + 3(3cosx – 4sinx) + C

Equating the coefficients of

sin x and cos x,

we get 3A – 4B = 2

and 4A + 3B = 3

Solving these equations,

$A = \frac{18}{25}, B = \frac{1}{25}$

Equating the constants

4 = 5A + C

$C = 4 - 5A = 4 - 5 \cdot \frac{18}{25} = \frac{2}{5}$

$2\sin x + 3\cos x + 4 = \frac{18}{25}(3\sin x + 4\cos x + 5) + \frac{1}{25}(3\cos x - 4\sin x) + \frac{2}{5}$

$\therefore \int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$

$= \frac{18}{25}\int dx + \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x + 5}$

$+ \frac{2}{5}\int \frac{dx}{3\sin x + 4\cos x + 5}$

$= \frac{18}{25}x + \frac{1}{25}\log|3\sin x + 4\cos x + 5|$

$+ \frac{2}{5}\int \frac{dx}{3\sin x + 4\cos x + 5}$ ...(1)

Let $I = \int \frac{dx}{3\sin x + 4\cos x + 5}$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}$ $I = \int \frac{\frac{2dt}{1+t^2}}{\frac{3-2t}{1+t^2} + \frac{4(1+t^2)}{1+t^2} + 5}$

$$= 2\int \frac{dt}{6t+4-4t^2+5+5t^2} = 2\int \frac{dt}{t^2+6t+9}$$

$$= 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} = -\frac{2}{3+\tan\frac{x}{2}}$$

Substituting in (1)

$$I = \frac{18}{25}\cdot x + \frac{1}{25}\log|3\sin x + 4\cos x + 5|$$

$$-\frac{4}{5\left(3+\tan\frac{x}{2}\right)} + C$$

11. $\int \sqrt{\frac{5-x}{x-2}}\,dx$ **on (2, 5).**

Sol: $\int \sqrt{\frac{5-x}{x-2}}\,dx = \int \sqrt{\frac{(5-x)^2}{(x-2)(5-x)}}\,dx$

$$= \int \frac{5-x}{\sqrt{(x-2)(5-x)}}\,dx$$

$$= \int \frac{5-x}{\sqrt{5x\ -x^2-10+2x}}\,dx$$

$$= \int \frac{5-x}{\sqrt{7x-x^2-10}}\,dx$$

Let $5-x = A\frac{d}{dx}(7x-x^2-10)+B$

$= A(7-2x)+B$

Equating coefficient of x and constant terms on both sides we get

$-2A = -1 \Rightarrow A = 1/2$ and $7A + B = 5 \Rightarrow B = 5 - 7A = 5-\frac{7}{2} = \frac{3}{2}$.

$\therefore 5-x=\frac{1}{2}(7-2x)+\frac{3}{2}$

$\therefore \int \frac{5-x}{\sqrt{7x-x^2-10}}dx$

$=\frac{1}{2}\int \frac{(7-2x)}{\sqrt{7x-x^2-10}}dx+\frac{3}{2}\int \frac{dx}{\sqrt{7x-x^2-10}}$ …(1)

Consider the first integral on RHS and suppose $7x - x^2 - 10 = t \Rightarrow (7 - 2x)d = dt$

$\therefore \frac{1}{2}\int \frac{(7-2x)}{\sqrt{7x-x^2-10}}dx=\frac{1}{2}\int \frac{dt}{\sqrt{t}}=\frac{1}{2}\int t^{-1/2}dt$

$=\frac{1}{2}\frac{t^{1/2}}{1/2}=\sqrt{t}=\sqrt{7x-x^2-10}$ …(2)

Consider the second integral and

$7x-x^2-10=-(x^2-7x+10)$

$=-\left(x^2-2\left(\frac{7}{2}\right)x+\frac{49}{4}-\frac{49}{4}+10\right)$

$=-\left[\left(x-\frac{7}{2}\right)^2-\frac{9}{4}\right]$

$=-\left[\left(x-\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]$

$=\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2$

$\therefore \int \frac{dx}{\sqrt{7x-x^2-10}}=\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}}$

$=\sin^{-1}\left(\frac{x-\frac{7}{2}}{\frac{3}{2}}\right)=\sin^{-1}\left(\frac{2x-7}{3}\right)$ …(3)

$\therefore$ From (1), (2) and (3)

$\int \frac{5-x}{\sqrt{7x-x^2-10}}dx=$

$\sqrt{7x-x^2-10}+\frac{3}{2}\sin^{-1}\left(\frac{2x-7}{3}\right)+c$

12. $\int \sqrt{\frac{1+x}{1-x}}\,dx$ on $(-1,1)$

Sol: $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{(1+x)^2}{1-x}}\,dx$

$= \int \sqrt{\frac{1+x}{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{xdx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \frac{1}{2}\int \frac{(-2x)dx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \sqrt{1-x^2} + c$

13. $\int \frac{dx}{(1-x)\sqrt{3-2x-x^2}}$ on (–1, 3).

Sol: Put $1 - x = 1/t \Rightarrow x = 1-\frac{1}{t}$

$\therefore dx = +\frac{1}{t^2}dt$

Also $3-2x-x^2 = 3-\left(1-\frac{1}{t}\right)-\left(1-\frac{1}{t}\right)^2$

$= 3+\frac{2}{t}-2-\left(\frac{1}{t^2}-\frac{2}{t}+1\right)$

$= \frac{4}{t}-\frac{1}{t^2} = \frac{4t-1}{t^2}$

$\therefore \int \frac{dx}{(1-x)\sqrt{3-2x-x^2}} = \int \frac{\left(\frac{1}{t^2}\right)dt}{\left(\frac{1}{t}\right)\sqrt{\frac{4t-1}{t^2}}}$

$= \int \frac{dt}{\sqrt{4t-1}} = \frac{2\sqrt{4t-1}}{4}+c$

$= \frac{1}{2}\sqrt{4t-1}+c$

$= \frac{1}{2}\sqrt{4\left(\frac{1}{1-x}\right)-1}+c$

$$= \frac{1}{2}\sqrt{\frac{4-1+x}{1-x}} + c = \frac{1}{2}\sqrt{\frac{3+x}{1-x}}.$$

14. $\int \frac{dx}{(x+2)\sqrt{x+1}}$

Sol. $\int \frac{dx}{(x+2)\sqrt{x+1}}$

put x+1 $=t^2$

$dx = 2tdt$ and $x+2=1+t^2$

$$G.I. = \int \frac{2t}{(1+t^2)t} dt$$

$$= 2.\tan^{-1} t + c = 2\tan^{-1}\sqrt{x+1} + c$$

15. $\int \frac{dx}{(2x+3)\sqrt{x+2}}$

Sol. put x+2 $=t^2$

$dx = 2tdt$ and $2x+3=2t^2-1$

$$G.I. = \int \frac{2t}{(2t^2-1)t} dt = 2\int \frac{1}{(\sqrt{2}\,t)^2 - 1} dt$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}\,t-1}{\sqrt{2}\,t+1} + c$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2x+4}-1}{\sqrt{2x+4}+1} + c$$

16. $\int \frac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}}$

Sol. $\int \frac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}} = \int \frac{dx}{\left(1+\sqrt{x}\right)\sqrt{x}\sqrt{1-x}}$

put $x=\sin^2 t \Rightarrow dx = 2\sin t.\cos t.dt$

$$= \int \frac{2\sin t.\cos t.dt}{(1+\sin t)\sin t.\cos t} = 2\int \frac{1}{1+\sin t} dt = 2.(\tan t - \sec t) + c$$

$$= 2\int \frac{1}{1+\sin t} dt = 2.(\tan t - \sec t) + c$$

17. $\int \frac{dx}{(x+1)\sqrt{2x^2+3x+1}}$

Sol. $\int \frac{dx}{(x+1)\sqrt{2x^2+3x+1}}$

put $x+1 = \frac{1}{t} \Rightarrow x = \frac{1-t}{t}$ *and* $dx = \frac{-1}{t^2} dt$

$$g.i. = \int \frac{1}{\frac{1}{t}.\sqrt{2\left(\frac{1-t}{t}\right)^2 + 3\left(\frac{1-t}{t}\right) + 1}} . \frac{-1}{t^2} dt$$

$$= \int \frac{-1}{\sqrt{2+2t^2-4t+3t-3t^2+t^2}} dt$$

$$= -\int \frac{1}{\sqrt{2-t}} dt = 2\sqrt{2-t} + c$$

$$= 2\sqrt{2-\frac{1}{x+1}} + c = 2\sqrt{\frac{2x+1}{x+1}} + c$$

18. $\int \sqrt{e^x - 4}\,dx$

Sol. $\int \sqrt{e^x - 4}\,dx = \int \frac{e^x - 4}{\sqrt{e^x - 4}}\,dx$

$$= \int \frac{e^x}{\sqrt{e^x - 4}}\,dx - \int \frac{4}{\sqrt{e^x - 4}}\,dx$$

$$= 2\sqrt{e^x - 4} - 4\int \frac{e^{-x/2}}{\sqrt{1 - 4e^{-x}}}dx$$

$$= 2\sqrt{e^x - 4} + 4\int \frac{e^{-x/2}}{\sqrt{1 - 4\left(e^{-x/2}\right)^2}}dx$$

$$= 2\sqrt{e^x - 4} + 4.\int \frac{e^{-x/2}}{\sqrt{1 - \left(2e^{-x/2}\right)^2}}dx$$

$$= 2\sqrt{e^x - 4} + 4\sin^{-1} 2e^{-x/2} + c$$

$$= 2\sqrt{e^x - 4} + 4\tan^{-1} \frac{2e^{-x/2}}{\sqrt{1 - 4e^{-x}}} + c$$

$$= 2\sqrt{e^x - 4} + 4\tan^{-1} \frac{2}{\sqrt{e^x - 4}} + c$$

19. $\int \sqrt{1 + \sec x}\,dx$

$$\int \sqrt{1 + \sec x}\,dx = \int \sqrt{\frac{1 + \cos x}{\cos x}}\,dx = \int \sqrt{\frac{2\cos^2 \frac{x}{2}}{1 - 2\sin^2 \frac{x}{2}}}\,dx$$

$$= \int \frac{\sqrt{2}.\cos \frac{x}{2}}{\sqrt{1 - \left(\sqrt{2}\sin \frac{x}{2}\right)^2}}dx = 2\int \frac{\frac{1}{2}\sqrt{2}.\cos \frac{x}{2}}{\sqrt{1 - \left(\sqrt{2}\sin \frac{x}{2}\right)^2}}dx$$

$$= 2\sin^{-1}\left(\sqrt{2}\sin \frac{x}{2}\right) + c$$

20. $\int \frac{1}{1+x^4}dx$

Sol. $\int \frac{1}{1+x^4}dx$

$$=\frac{1}{2}\int \frac{2}{1+x^4}dx=\frac{1}{2}\int \frac{1+x^2+1-x^2}{1+x^4}dx$$

$$=\frac{1}{2}\int \left(\frac{1+x^2}{1+x^4}+\frac{1-x^2}{1+x^4}\right)dx$$

$$=\frac{1}{2}\int\left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}+\frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}}\right)dx$$

$$=\frac{1}{2}\left(\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{2}\right)^2}dx-\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-\left(\sqrt{2}\right)^2}dx\right)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\log\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+c$$

$$=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\frac{x^2-1}{x\sqrt{2}}-\frac{1}{2}\log\frac{x^2+1-\sqrt{2}}{x^2+1+\sqrt{2}}\right)+c$$

21. Evaluate $\int\frac{dx}{5+4\cos x}$.

Sol: Put $\tan\frac{x}{2}=t$ then $dx=\frac{2\,dt}{1+t^2}$

And $\cos x=\frac{1-t^2}{1+t^2}$

$$\int\frac{dx}{5+4\cos x}=\int\frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{2dt}{9+t^2} = 2\int \frac{dt}{3^2+t^2}$$

$$= 2\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{t}{3}\right)$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + c$$

22. $\int \frac{dx}{3\cos x + 4\sin x + 6}$

Sol: Let $\tan\frac{x}{2} = t$ then $dx = \frac{2\,dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore \int \frac{dx}{3\cos x + 4\sin x + 6}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) + 6}$$

$$= \int \frac{2\,dt}{3-3t^2+8t+6+6t^2}$$

$$= \int \frac{2dt}{3t^2+8t+9}$$

$$= \frac{2}{3}\int \frac{dt}{t^2+\frac{8}{3}t+3}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2 + 3 - \frac{16}{9}}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2 + \frac{11}{9}}$$

$$= \frac{2}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{2}{3}\frac{3}{\sqrt{11}}\tan^{-1}\left(\frac{t+\frac{4}{3}}{\frac{\sqrt{11}}{3}}\right)$$

$$= \frac{2}{\sqrt{11}}\text{Tan}^{-1}\left(\frac{3t+4}{\sqrt{11}}\right) + c.$$

23. Evaluate $\int \frac{1}{\sqrt{\sin^{-1}x}\sqrt{1-x^2}}dx$ **on I = (0, 1).**

Sol: Let $\sin^{-1} x = t$ then $\frac{1}{\sqrt{1-x^2}}dx = dt$

$$\therefore \int \frac{1}{\sqrt{\sin^{-1}x}\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{t}}dt$$

$$= \int t^{-1/2}dt$$

$$= 2\sqrt{t} + c = 2\sqrt{\sin^{-1}x} + c.$$

24. Find $\int \frac{x}{\sqrt{1-x}}dx$**, x ∈ I = (0, 1).**

Sol: Let $1 - x = t^2$ over (0, 1)

Then $-dx = 2t\,dt$ and $x = 1 - t^2$

$$\therefore \int \frac{x}{\sqrt{1-x}}dx = -\int \frac{(1-t^2)2tdt}{t}$$

$$= -2\int (1-t)^2 dt = -2\left[t - \frac{t^3}{3}\right]$$

$$= -2\left[\sqrt{1-x} - \frac{(1-x)^{3/2}}{3}\right]$$

$$= \frac{2}{3}(1-x)^{3/2} - 2\sqrt{1-x} + c$$

25. Evaluate $\int \frac{dx}{(x+5)\sqrt{x+4}}$ **on** $(-4,\infty)$**.**

Sol: Let x + 4 = t^2 then dx = 2t dt

Defined over (–4, ∞)

$$\therefore \int \frac{dx}{(x+5)\sqrt{x+4}} = \int \frac{2t\,dt}{(t^2+1)t} = 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1} t + c$$

$$= 2\tan^{-1}(\sqrt{x+4}) + c$$

26. Evaluate $\int \frac{dx}{\sqrt{x^2+2x+10}}$**.**

Sol: $\sqrt{x^2+2x+10} = \sqrt{x^2+2x+1+9}$

$$= \sqrt{(x+1)^2+3^2}$$

$$\therefore \int \frac{dx}{\sqrt{x^2+2x+10}} = \int \frac{dx}{\sqrt{(x+1)^2+3^2}}$$

Take x + 1 = t ⇒ dx = dt

$$= \int \frac{dt}{\sqrt{t^2+3^2}} = \sinh^{-1}\left(\frac{t}{3}\right) + c$$

$$= \sinh^{-1}\left(\frac{x+1}{3}\right) + c$$

$$\left(\text{use} \int \frac{dx}{\sqrt{a^2-x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)\right).$$