INDEFINITE INTEGRATION

Definition: If f(x) and g(x) are two functions such that $f'(x) = g(x)$ then f(x) is called antiderivative or primitive of g(x) with respect to x.

Note 1: If f(x) is an antiderivative of g(x) then f(x) + c is also an antiderivative of g(x) for all $c \in R$.

Definition: If F(x) is an antiderivative of f(x) then F(x) + c, $c \in R$ is called indefinite integral of f(x) with respect to x. It is denoted by $\int f(x)\,dx$. The real number c is called constant of integration.

Note:

1. The integral of a function need not exists. If a function f(x) has integral then f(x) is called an integrable function.

2. The process of finding the integral of a function is known as Integration.

3. The integration is the reverse process of differentiation.

Corollary:

If f(x), g(x) are two integrable functions then $\int (f \pm g)(x)\,dx = \int f(x)dx \pm \int g(x)dx$

Corollary:

If $f_1(x), f_2(x), \ldots, f_n(x)$ are integrable functions then $\int (f_1 + f_2 + \ldots + f_n)(x)dx = \int f_1(x)dx + \int f_2(x)dx + \ldots + \int f_n(x)dx$.

Corollary:

If f(x), g(x) are two integrable functions and k, l are two real numbers then $\int (kf + lg)(x)dx = k\int f(x)dx + l\int g(x)dx$.

Integration by Substitution

Theorem: If $\int f(x)dx = g(x)$ and $a \neq 0$ then $\int f(ax+b)dx = \frac{1}{a}g(ax+b)+c$.

Proof:

Put ax + b = t.

Then $\frac{d}{dx}(ax+b) = \frac{dt}{dx} \Rightarrow a \cdot dx = dt \Rightarrow dx = \frac{1}{a}dt$

$\therefore \int f(ax+b)dx = \int f(t) \cdot \frac{1}{a}dt$

$= \frac{1}{a}\int f(t)dt = \frac{1}{a}g(t)+c = \frac{1}{a}g(ax+b)+c$

E.g. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, (n \neq -1)$

Theorem: If f(x) is a differentiable function then $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c$.

Proof:

Put f(x) = t $\Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$

$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{1}{t}dt = \log|t|+c = \log|f(x)|+c$

Theorem: $\int \tan x\, dx = \log|\sec x|+c$ for $x \neq (2n+1)\frac{\pi}{2}, n \in Z$.

Proof:

$\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d(\cos x)}{\cos x}dx$

$= -\log|\cos x|+c = \log\frac{1}{|\cos x|}+c = \log|\sec x|+c$

Theorem: $\int \cot x\, dx = \log|\sin x|+c$ for $x \neq n\pi$, $n \in Z$.

Proof:

$\int \cot x\, dx = \int \frac{\cos x}{\sin x}dx = \log|\sin x|+c$

Theorem: $\int \sec x\,dx = \log|\sec x + \tan x| + c = \log|\tan(\pi/4 + x/2) + c$ for $x \neq (2n+1)\frac{\pi}{2}, n \in Z$.

Proof:

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx = \log|\sec x + \tan x| + c$$

$$= \log\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c = \log\left|\frac{1+\sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{1-\cos(\pi/2+x)}{\sin(\pi/2+x)}\right| + c$$

$$= \log\left|\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right| + c$$

$$= \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$$

Theorem: $\int \csc x\,dx = \log|\csc x - \cot x| + c = \log|\tan x/2| + c$ for $x \neq n\pi, n \in Z$.

Proof:

$$\int \csc x\,dx = \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x}dx$$

$$= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}dx = \log|\csc x - \cot x| + c$$

$$= \log\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| + c = \log\left|\frac{1-\cos x}{\sin x}\right| + c$$

$$= \log\left|\frac{2\sin^2 x/2}{2\sin x/2\cos x/2}\right| + c = \log|\tan x/2| + c$$

Theorem: If f(x) is differentiable function and $n \neq -1$ then $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$.

Proof:

Put f(x) = t $\Rightarrow$ f′(x) dx = dt

$\therefore \int [f(x)]^n f'(x)dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c$ **Note :** $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

Theorem: If $\int f(x)dx = F(x)$ and $g(x)$ is a differentiable function then $\int (fog)(x)g'(x)dx = F[g(x)] + c$.

Proof :

$g(x) = t \Rightarrow g'(x)\, dx = dt$

$$\therefore \int (fog)(x)g'(x)dx = \int f[g(x)]g'(x)dx$$

$$= \int f(t)dt = F(t) + c = F[g(x)] + c$$

Integration of Some Standard Functions

Theorem: $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c$ for $x \in (-a, a)$.

Proof:

Put $x = a \sin \theta$. Then $dx = a \cos\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}dx = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} a\cos\theta d\theta$$

$$= \int \frac{1}{a\sqrt{1-\sin^2\theta}} a\cos\theta d\theta = \int \frac{1}{\cos\theta}\cos\theta d\theta$$

$$= \int d\theta = \theta + c = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c$$

Theorem: $\int \frac{1}{\sqrt{a^2 + x^2}}dx = \mathrm{Sinh}^{-1}\left(\frac{x}{a}\right) + c$ for $x \in \mathbb{R}$.

Proof:

Put $x = a \sinh\theta$. Then $dx = a \cosh \theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 + x^2}}dx = \int \frac{1}{\sqrt{a^2 + a^2\sinh^2\theta}} a\cosh\theta d\theta$$

$$= \int \frac{a\cosh\theta}{a\cosh\theta} = \int d\theta = \theta + c = \mathrm{Sinh}^{-1}\left(\frac{x}{a}\right) + c$$

Theorem:

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \mathrm{Cosh}^{-1}\left(\frac{x}{a}\right) + c \text{ for } x \in (-\infty, -a) \cup (a, \infty).$$

Proof :

Put $x = a \cosh \theta$. Then $dx = a \sinh\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{\sqrt{a^2 \cosh^2 \theta - a^2}} a \sinh \theta \, d\theta$$

$$= \int \frac{a \sinh \theta}{a \sinh \theta} d\theta = \int d\theta = \theta + c = \mathrm{Cosh}^{-1}\left(\frac{x}{a}\right) + c$$

Theorem:

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c \text{ for } x \in R.$$

Proof:

Put $x = a \tan \theta$. Then $dx = a \sec^2\theta \, d\theta$

$$\therefore \int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a^2 + a^2 \tan^2 \theta} a \sec^2 \theta \, d\theta$$

$$= \int \frac{1}{a^2 (1 + \tan^2 \theta)} a \sec^2 \theta \, d\theta = \frac{1}{a} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta$$

$$= \frac{1}{a} \int d\theta = \frac{1}{a} \theta + c = \frac{1}{a} \mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c$$

Theorem: $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + c \text{ for } x \neq \pm a$

Proof:

$$\int \frac{1}{a^2 - x^2} dx = \int \frac{1}{(a + x)(a - x)} dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{a + x} + \frac{1}{a - x}\right) dx = \frac{1}{2a} [\log|a + x| - \log|a - x|] + c$$

$$= \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + c$$

Theorem: $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c \text{ for } x \neq \pm a$

Proof:

$$\int \frac{1}{x^2 - a^2} dx = \int \frac{1}{(x - a)(x + a)} dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{x - a} - \frac{1}{x + a}\right) dx = \frac{1}{2a} [\log|x - a| - \log|x + a|] + c$$

$$= \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c$$

Theorem: $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$ for $x\in(-a, a)$.

Proof:

Put $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$

$\therefore\int\sqrt{a^2-x^2}dx=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\, d\theta$

$=\int a\sqrt{1-\sin^2\theta}\,a\cos\theta d\theta=a^2\int\cos^2\theta\, d\theta$

$=a^2\int\frac{1+\cos 2\theta}{2}d\theta=\frac{a^2}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]+c$

$=\frac{a^2}{2}\left[\theta+\frac{1}{2}2\sin\theta\cos\theta\right]+c=\frac{a^2}{2}\left[\theta+\sin\theta\sqrt{1-\sin^2\theta}\right]$

$=\frac{a^2}{2}\left[\text{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+c$

$=\frac{a^2}{2}\text{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c$

Theorem:

$\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\text{Sinh}^{-1}\left(\frac{x}{a}\right)+c$ for $x\in R$.

Proof:

Put $x = \sinh\theta$. Then $dx = a\cosh\theta\, d\theta$

$\therefore\int\sqrt{a^2+x^2}dx=\int\sqrt{a^2+a^2\sinh^2\theta}\,a\cosh\theta d\theta$

$=\int a\sqrt{1+\sinh^2\theta}\,a\cosh\theta d\theta=a^2\int\cosh^2\theta\, d\theta$

$=a^2\int\frac{1+\cosh 2\theta}{2}d\theta=\frac{a^2}{2}\left[\theta+\frac{1}{2}\sinh 2\theta\right]+c$

$=\frac{a^2}{2}\left[\theta+\frac{1}{2}2\sinh\theta\cosh\theta\right]+c$

$=\frac{a^2}{2}\left[\theta+\sinh\theta\sqrt{1+\sinh^2\theta}\right]+c$

$=\frac{a^2}{2}\left[\text{Sinh}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}\right]+c$

$=\frac{a^2}{2}\text{Sinh}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{a^2+x^2}+c$

Theorem: $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$ for $x \in [a, \infty)$.

Proof:

Put x = a coshθ. Then dx = a sinhθ dθ

$$\therefore \int \sqrt{x^2 - a^2}dx = \int \sqrt{a^2\cosh^2\theta - a^2}\, a\sinh\theta d\theta$$

$$= \int a\sqrt{\cosh^2\theta - 1}\, a\sinh\theta d\theta = a^2 \int \sinh^2\theta d\theta$$

$$= a^2 \int \frac{\cosh 2\theta - 1}{2} d\theta = \frac{a^2}{2}\left[\frac{1}{2}\sinh 2\theta - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\frac{1}{2} 2\sinh\theta\cosh\theta - \theta\right] + c$$

$$= \frac{a^2}{c}\left[\cosh\theta\sqrt{\cosh^2\theta - 1} - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1} - \text{Cosh}^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$$

Very Short Answer Questions

Evaluate the following integrals.

1. $\int (x^3 - 2x^2 + 3)dx$

Sol. $\int (x^3 - 2x^2 + 3)dx = \int x^3 dx - \int 2x^2 dx + 3\int dx = \frac{x^4}{4} - \frac{2}{3}x^3 + 3x + c$

2. $\int 2x\sqrt{x}\,dx$

Sol. $\int 2x\sqrt{x}\,dx = 2\int x^{3/2}dx = \frac{2x^{5/2}}{(5/2)} + c = \frac{4}{5}x^{5/2} + c$

3. $\int \sqrt[3]{2x^2}\,dx$

Sol. $\int \sqrt[3]{2x^2}\,dx = \int 2^{1/3} \cdot x^{2/3}\,dx$

$$= 2^{1/2} \cdot \frac{x^{5/3}}{(5/3)} + c = \sqrt[3]{2} \cdot \frac{3}{5}x^{5/3} + c$$

4. $\int \frac{x^2 + 3x - 1}{2x}dx, x \in I \subset R \setminus \{0\}$

Sol. $\int \frac{x^2 + 3x - 1}{2x}dx = \int \left(\frac{x^2}{2x} + \frac{3}{2} - \frac{1}{2x}\right)dx$

$$= \int \frac{x}{2}dx + \frac{3}{2}\int dx - \frac{1}{2}\int \frac{1}{x}dx$$

$$= \frac{x^2}{4} + \frac{3}{2}x - \frac{1}{2}\log|x| + c$$

5. $\int \frac{1-\sqrt{x}}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{1-\sqrt{x}}{x}dx = \int \frac{dx}{x} - \int \frac{\sqrt{x}}{x}dx$

$$= \log|x| - \frac{x^{-\frac{1}{2}+1}}{(1/2)} + c = \log|x| - 2\sqrt{x} + c$$

6. $\int\left(1+\frac{2}{x}-\frac{3}{x^2}\right)$ **dx on I ⊂ R \ {0}.**

Sol. $\int\left(1+\frac{2}{x}-\frac{3}{x^2}\right)dx=\int dx+2\int\frac{dx}{x}-3\int x^{-2}dx$

$=x+2\log|x|+\frac{3}{x}+c$

7. $\int\left(x+\frac{4}{1+x^2}\right)dx$ **on R.**

Sol. $\int\left(x+\frac{4}{1+x^2}\right)dx=\int xdx+4\int\frac{1}{1+x^2}dx$

$=\frac{x^2}{2}+4\tan^{-1}x+c$

8. $\int\left(e^x-\frac{1}{x}+\frac{2}{\sqrt{x^2-1}}\right)dx$ **.**

Sol. $\int\left(e^x-\frac{1}{x}+\frac{2}{\sqrt{x^2-1}}\right)dx$

$=\int e^x dx-\int\frac{1}{x}dx+2\int\frac{1}{\sqrt{x^2-1}}dx$

$=e^x-\log|x|+2\log|x+\sqrt{x^2-1}|+c$

9. $\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx$

Sol. $\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx=\int\frac{1}{1-x^2}dx+\int\frac{1}{1+x^2}dx$

$=\tanh^{-1}x+\tan^{-1}x+c$

10. $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1+x^2}}\right)$dx **on (–1, 1).**

Sol. $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1+x^2}}\right)dx$

$=\int\frac{1}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1+x^2}}dx$

$=\sin^{-1}x+2\sinh^{-1}x+c$

11. $\int e^{\log(1+\tan^2 x)} dx$

Sol. $\int e^{\log(1+\tan^2 x)} dx = = \int e^{\log(\sec^2 x)} dx$

$= \int \sec^2 x \, dx = \tan x + c$

12. $\int \frac{\sin^2 x}{1+\cos 2x} dx$

Sol. $\int \frac{\sin^2 x}{1+\cos 2x} dx$

$= \int \frac{\sin^2 x}{\cos^2 x} dx = \int \tan^2 x dx$

$= \int (1+\sec^2 x) dx = x + \tan x + c$

13. $\int e^{2x} dx,\ x \in R$ **.**

Sol. $\int e^{2x} dx = \frac{e^{2x}}{2} + C$

14. $\int \sin 7x \, dx, x \in R$

Sol. $\int \sin 7x \, dx = -\frac{\cos 7x}{7} + C$

15. $\int \frac{x}{1+x^2} dx,\ x \in R$

Sol. $\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{2x\,dx}{1+x^2} = \frac{1}{2} \log(1+x^2) + C$

16. $\int 2x \sin(x^2+1) dx, x \in R$

Sol. $\int 2x \sin(x^2+1) dx$

Put $x^2+1 = t \Rightarrow 2x\,dx = dt$

$\int 2x \cdot \sin(x^2+1) dx = \int \sin t \, dt = -\cot t + C$

$= -\cos(x^2+1) + C$

17. $\int \frac{(\log x)^2}{x} dx$.

Sol. $\int \frac{(\log x)^2}{x} dx$

put $\log x = t \Rightarrow dt = \frac{1}{x} dx$

$$\int \frac{(\log x)^2}{x} dx = \int t^2 \cdot dt = \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$$

18. $\int \frac{e^{Tan^{-1}x}}{1+x^2} dx$ **on** $I \subset (0, \infty)$.

Sol. $\int \frac{e^{Tan^{-1}x}}{1+x^2} dx$

put $\tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$$\int \frac{e^{Tan^{-1}x}}{1+x^2} dx = \int e^t \cdot dt = e^t + C = e^{\tan^{-1} x} + C$$

18. $\int \frac{\sin(Tan^{-1}x)}{1+x^2} dx, x \in R$

Sol. $\int \frac{\sin(Tan^{-1}x)}{1+x^2} dx$

put $\tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$$\int \frac{\sin(Tan^{-1}x)}{1+x^2} dx = \int \sin t \, dt$$

$$= -\cos t + t = -\cos(\tan^{-1} x) + C$$

19. $\int \frac{1}{8+2x^2} dx$ **on R.**

Sol. $\int \frac{1}{8+2x^2} dx = \frac{1}{2} \int \frac{dx}{x^2 + 2^2}$

$$= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C = \frac{1}{4} \tan^{-1}\left(\frac{x}{2}\right) + C$$

20. $\int \frac{3x^2}{1+x^6}dx$**, on R.**

Sol. $\int \frac{3x^2}{1+x^6}dx$

put $x^3 = t \Rightarrow 3x^2dx = dt$

$$\int \frac{3x^2dx}{1+x^6} = \int \frac{dt}{1+t^2}$$

$$= \tan^{-1}(t) + C = \tan^{-1}(x^3) + C$$

21. $\int \frac{2}{\sqrt{25+9x^2}}dx$ **on R.**

Sol. $\int \frac{2}{\sqrt{25+9x^2}}dx = \frac{2}{3}\int \frac{dx}{\sqrt{x^2+\left(\frac{5}{3}\right)^2}}$

$$= \frac{2}{3}\sinh^{-1}\left(\frac{x}{5/3}\right) + C = \frac{2}{3}\sinh^{-1}\left(\frac{3x}{5}\right) + C$$

22. $\int \frac{3}{\sqrt{9x^2-1}}dx$ on $\left(\frac{1}{3},\infty\right)$

Sol. $\int \frac{3}{\sqrt{9x^2-1}}dx = \int \frac{dx}{\sqrt{x^2-\left(\frac{1}{3}\right)^2}}$

$$= \cosh^{-1}\left(\frac{x}{1/3}\right) + C = \cosh^{-1}(3x) + C$$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

23. $\int \sin mx \cos nx \, dx$ **on R, $m \neq n$, m and n are positive integers.**

Sol: $\int \sin mx \cos nx \, dx = \frac{1}{2}\int 2\sin mx \cos nx \, dx$

$$= \frac{1}{2}\int [\sin(mx+nx) + \sin(mx-nx)dx$$

$$= \frac{1}{2}\int [\sin(m+n)x + \sin(m-n)x]dx$$

$$= \frac{-1}{2(m+n)}\cos(m+n)x - \frac{1}{2(m-n)}\cos(m-n)x + c$$

$$= \frac{-1}{2}\left[\frac{\cos(m+n)x}{m+n} + \cos\frac{(m-n)x}{m-n}\right] + c$$

24. $\int \sin mx \sin nx\, dx$ on R, m ≠ n, m and n are positive integers.

Sol: $\int \sin mx \sin nx\, dx = \frac{1}{2}\int 2\sin mx \sin nx\, dx$

$$= \frac{1}{2}\int [\cos(mx-nx) - \sin(mx+nx)dx$$

$$= \frac{1}{2}\int [\cos(m-n)x - \cos(m+n)x]dx$$

$$= \frac{1}{2(m-n)}\sin(m-n)x - \frac{1}{2(m+n)}\sin(m+n)x + c$$

$$= \frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} - \sin\frac{(m+n)x}{m+n}\right] + c$$

25. $\int \cos mx \cos nx\, dx$ on R, m ≠ n, m and n are positive integers.

Sol: $\int \cos mx \cos nx\, dx = \frac{1}{2}\int 2\cos mx \cos nx\, dx$

$$= \frac{1}{2}\int [\cos(mx+nx) + \cos(mx-nx)]dx$$

$$= \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]dx$$

$$= \frac{-1}{2(m+n)}\sin(m+n)x - \frac{1}{2(m-n)}\sin(m-n)x$$

$$= -\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \sin\frac{(m-n)x}{m-n}\right] + c\,.$$

26. $\int \frac{e^x}{e^{x/2}+1}dx$ **on R.**

Sol. $t = 1 + e^{x/2} \Rightarrow dt = \frac{1}{2}e^{x/2}dx$

$$\int \frac{e^x}{e^{x/2}+1}dx = 2\int \frac{e^{x/2}\left(\frac{1}{2}e^{x/2}dx\right)}{e^{x/2}+1}$$

$= 2\int \frac{(t-1)dt}{t} = 2\int \left(1-\frac{1}{t}\right)dt = 2(t - \log t) + C$

$= 2(1+e^{x/2} - \log(1+e^{x/2})) + C$

27. Evaluate $\int \left(x+\frac{1}{x}\right)^3 dx, x>0$**.**

Sol: $\int \left(x+\frac{1}{x}\right)^3 dx = \int \left[x^3 + \frac{1}{x^3} + \left(x+\frac{1}{x}\right)\right]dx$

$= \int x^3 dx + 3\int x\,dx + 3\int \frac{dx}{x} + \int \frac{dx}{x^3} + c$

$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\log|x| - \frac{1}{2x^2} + c.$

28. Find $\int \sqrt{1+\sin 2x}\,dx$ **on R.**

Sol: $\int \sqrt{1+2\sin x\cos x}\,dx$

$= \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\,dx$

$= \int \sqrt{(\sin x + \cos x)^2}\,dx$

$= \int (\sin x + \cos x)dx$

If $2n\pi - \frac{\pi}{4} \le x \le 2n\pi + \frac{3\pi}{4}$ for some $n \in Z$

$= \int -(\sin x + \cos x)dx$ other wise

$\therefore \int \sqrt{1+2\sin x\cos x}\,dx = -\cos x + \sin x + c$

If $2n\pi - \frac{\pi}{4} \le x \le 2n\pi + \frac{3\pi}{4}$

= cos x – sin x + c

If $2n\pi + \frac{3\pi}{4} \le x \le 2n\pi + \frac{7\pi}{4}$.

29. Find $\int \left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx$ **on I where I = (0, ∞).**

Sol: Let $x + \frac{1}{x} = t$ then $\left(1-\frac{1}{x^2}\right)dx = dt$

$\therefore \int \left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}dx = \int e^t dt$

$= e^t + c = e^{\left(x+\frac{1}{x}\right)} + c.$

Short Answer Questions

Evaluate the following integrals.

1. $\int (1-x^2)^3 dx$

Sol. $\int (1-x^2)^3 dx = \int (1-3x^2+3x^4-x^6)dx$

$$= x - x^3 + \frac{3}{5}x^5 - \frac{x^7}{7} + c$$

2. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right)dx$

Sol. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right)dx =$

$$= 3\int \frac{dx}{\sqrt{x}} - 2\int \frac{dx}{x} + \frac{1}{3}\int x^{-2}dx$$

$$= 3(2\sqrt{x}) - 2\log|x| - \frac{1}{3x} + c$$

$$= 6\sqrt{x} - 2\log|x| - \frac{1}{3x} + c$$

3. $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx$

Sol. $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx = \int \frac{x+1+2\sqrt{x}}{x^2}dx$

$$= \int \frac{x}{x^2}dx + \int \frac{dx}{x^2} + 2\int \frac{\sqrt{x}}{x^2}dx$$

$$= \int \frac{dx}{x} + \int \frac{dx}{x^2} + 2\int x^{-3/2}dx$$

$$= \log|x| - \frac{1}{x} + \frac{2x^{-1/2}}{(-1/2)} + c$$

$$= \log|x| - \frac{1}{x} - \frac{4}{\sqrt{x}} + c$$

4. $\int \frac{(3x+1)^2}{2x}dx$

Sol. $\int \frac{(3x+1)^2}{2x}dx = \int \frac{9x^2+6x+1}{2x}dx$

$= \frac{9}{2}\int x dx + 3\int dx + \frac{1}{2}\int \frac{1}{x}dx$

$= \frac{9}{2}\cdot\frac{x^2}{2} + 3x + \frac{1}{2}\log|x| + c$

$= \frac{9}{4}x^2 + 3x + \frac{1}{2}\log|x| + c$

5. $\int \left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx$

Sol. $\int \left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx = \int \frac{4x^2-4x+1}{9x}dx$

$= \frac{4}{9}\int x\,dx - \frac{4}{9}\int dx + \frac{1}{9}\int \frac{dx}{x}$

$= \frac{4}{9}\frac{x^2}{2} - \frac{4}{9}x + \frac{1}{9}\log|x| + c$

$= \frac{4}{18}x^2 - \frac{4}{9}x + \frac{1}{9}\log|x| + c$

6. $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right)dx$ **on (1, ∞)**

Sol. $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right)dx = \int \frac{1}{\sqrt{x}}dx + 2\int \frac{1}{\sqrt{x^2-1}}dx + \frac{3}{2}\int \frac{1}{x^2}dx$

$= 2\sqrt{x} + 2\cosh^{-1}x + \frac{3}{2x} + c$

7. $\int (\sec^2 x - \cos x + x^2)dx,$

Sol. $\int (\sec^2 x - \cos x + x^2)dx$

$= \int \sec^2 x dx - \int \cos x\,dx + \int x^2 dx = \tan x - \sin x + \frac{x^3}{3} + c$

8. $\int\left(\sec x\tan x+\frac{3}{x}-4\right)dx$

Sol. $\int\left(\sec x\tan x+\frac{3}{x}-4\right)dx$

$=\int \sec x\tan x\,dx+3\int\frac{dx}{x}-4\int dx$

$=\sec x+3\log|x|-4x+c$

9. $\int\left(\sqrt{x}-\frac{2}{1-x^2}\right)dx$ **on (0, 1).**

Sol. $\int\left(\sqrt{x}-\frac{2}{1-x^2}\right)dx=\int\sqrt{x}\,dx-2\int\frac{dx}{1-x^2}$

$=\frac{x^{3/2}}{(3/2)}-2\tanh^{-1}x+c$

$=\frac{2}{3}x\sqrt{x}-2\tanh^{-1}x+c$

10. $\int\left(x^3-\cos x+\frac{4}{\sqrt{x^2+1}}\right)dx, x\in R$

Sol. $\int\left(x^3-\cos x+\frac{4}{\sqrt{x^2+1}}\right)dx$

$=\int x^3dx-\int\cos x\,dx+4\int\frac{1}{\sqrt{x^2+1}}dx$

$=\frac{x^4}{4}-\sin x+4\sinh^{-1}x+c$

11. $\int\left(\cosh x+\frac{1}{\sqrt{x^2+1}}\right)dx, x\in R$

Sol. $\int\left(\cosh x+\frac{1}{\sqrt{x^2+1}}\right)dx$

$=\int\cosh x dx+\int\frac{dx}{\sqrt{x^2+1}}$

$=\sinh x+\sinh^{-1}x+c$

12. $\int\left(\sinh x+\frac{1}{(x^2-1)^{1/2}}\right)dx,$

Sol. $\int\left(\sinh x+\frac{1}{(x^2-1)^{1/2}}\right)dx$

$=\int \sinh x dx+\int\frac{dx}{\sqrt{x^2-1}}$

$=\cosh x+\log(x+\sqrt{x^2-1})+c$

13. $\int\frac{(a^x-b^x)^2}{a^x b^x}dx$

Sol. $\int\frac{(a^x-b^x)^2}{a^x b^x}dx$

$=\int\frac{a^{2x}+b^{2x}-2a^x b^x}{a^x\cdot b^x}dx$

$=\int\frac{a^{2x}}{a^x\cdot b^x}dx+\int\frac{b^{2x}}{a^x\cdot b^x}dx-2\int\frac{a^x b^x}{a^x\cdot b^x}dx$

$=\int\left(\frac{a}{b}\right)^x dx+\int\left(\frac{b}{x}\right)^2 dx-2\int dx$

$=\frac{(a/b)^x}{\log(a/b)}+\frac{(b/a)^x}{\log(b/a)}-2x+c$

$=\frac{1}{(\log a-\log b)}\left[\left(\frac{a}{b}\right)^x-\left(\frac{b}{a}\right)^x\right]-2x+c$

14. $\int\sec^2 x\csc^2 x\,dx$ **.**

Sol. $\int\sec^2 x\csc^2 x\,dx=\int\frac{1}{\cos^2 x\sin^2 x}dx$

$=\int\frac{\sin^2 x+\cos^2 x}{\cos^2 x\cdot\sin^2 x}dx=\int\frac{1}{\cos^2 x}dx+\int\frac{1}{\sin^2 x}dx=\int\sec^2 x\,dx+\int\csc^2 x\,dx=\tan x-\cot x+C$

15. $\int \frac{1+\cos^2 x}{1-\cos 2x} dx$.

Sol. $\int \frac{1+\cos^2 x}{1-\cos 2x} dx = \int \frac{1+\cos^2 x}{2\sin^2 x} dx$

$= \frac{1}{2}\int \frac{1}{\sin^2 x} dx + \frac{1}{2}\int \cot^2 x\, dx$

$= \frac{1}{2}\int \operatorname{cosec}^2 x\, dx + \frac{1}{2}\int (\csc^2 x - 1) dx$

$= \int \csc^2 x\, dx - \frac{1}{2}\int dx = -\cot x - \frac{x}{2} + C$

16. $\int \sqrt{1-\cos 2x}\, dx$

Sol. $\int \sqrt{1-\cos 2x}\, dx = \int \sqrt{2\sin^2 x} dx$

$= \int \sqrt{2} \sin x dx = -\sqrt{2} \cos x + C$

17. $\int \frac{1}{\cosh x + \sinh x} dx$ **on R.**

Sol. $\int \frac{1}{\cosh x + \sinh x} dx = \int \frac{(\cosh x - \sinh x)}{(\cosh x + \sinh x)(\cosh x - \sinh x)} dx = \int \frac{\cosh x - \sinh x}{\cosh^2 x - \sinh^2 x} dx$

$= \int (\cosh x - \sinh x) dx = \sinh x - \cosh x + C$

18. $\int \frac{1}{1+\cos x} dx$ **on R**

Sol. $\int \frac{1}{1+\cos x} dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} dx$

$= \int \left(\frac{1-\cos x}{1-\cos^2 x}\right) dx = \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$

$= \int \csc^2(x) dx - \int \csc x \cot x dx$

$= -\cot x + \csc x + C$

19. $\int (3x-2)^{1/2}\, dx$

Sol. given integral $= \int (3x-2)^{1/2}\, dx$

put $3x - 2 = t \Rightarrow 3\, dx = dt$

$\int (3x-2)^{1/2} dx = \frac{1}{3}\int t^{1/2} dt$

$$= \frac{1}{3}\frac{t^{3/2}}{3/2} + C = \frac{2}{9}(3x-2)^{3/2} + C$$

20. $\int \frac{1}{7x+3} dx$ **on** $I \subset R \setminus \left\{-\frac{3}{7}\right\}$

Sol. $\int \frac{1}{7x+3} dx$

Put 7x + 3=t $\Rightarrow$ 7 dx = dt

$$\int \frac{1}{7x+3} dx = \frac{1}{7}\int \frac{dt}{t}$$

$$= \frac{1}{7}\log|t| + C = \frac{1}{7}\log|7x+3| + C$$

21. $\int \frac{\log(1+x)}{1+x} dx$ **on (–1, ∞).**

Sol. $\int \frac{\log(1+x)}{1+x} dx$

Put 1 + x = t $\Rightarrow$ dx= dt

$$\int \frac{\log(1+x)}{1+x} dx = \int \frac{\log t}{t} \cdot dt$$

$$= \frac{(\log t)^2}{2} + C = \frac{1}{2}\left[\log(1+x)^2\right] + C$$

22. $\int (3x^2-4)x\,dx$ on R **.**

Sol. $\int (3x^2-4)x\,dx$

put $3x^2 - 4 = t \Rightarrow 6x\,dx = dt$

$$\int (3x^2-4)x\,dx = \frac{1}{6}\int t\,dt$$

$$= \frac{1}{6} \cdot \frac{t^2}{2} + C = \frac{(3x^2-4)^2}{12} + C$$

23. $\int \frac{dx}{\sqrt{1+5x}} dx$ on $\left(-\frac{1}{5}, \infty\right)$

Sol. $\int \frac{dx}{\sqrt{1+5x}}$

Put 1 + 5x = t^2 ; 5dx = 2t dt, dx = $\frac{2}{5} t\,dt$

$$\int \frac{dx}{\sqrt{1+5x}} = \frac{2}{5}\int \frac{t\,dt}{t} = \frac{2}{5}\int dt$$

$$= \frac{2}{5}t + C = \frac{2}{5}\sqrt{1+5x} + C$$

24. $\int (1-2x^3)x^2 dx$ **on R.**

Sol. $\int (1-2x^3)x^2 dx$

put $1-2x^3 = t \Rightarrow -6x^2 dx = dt$

$$\int (1-2x^3)x^2 dx = -\frac{1}{6}\int t\,dt$$

$$= -\frac{1}{6}\cdot\frac{t^2}{2} + C = \frac{-(1-2x^3)^2}{12} + C$$

25. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$ **on** $I \subset R \setminus \left\{n\pi - \frac{\pi}{4} : n \in Z\right\}$

Sol. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

put $1+\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\int \frac{\sec^2 x}{(1+\tan x)^3} dx = \int \frac{dt}{t^3} = \int t^{-3} dt$$

$$= \frac{t^{-2}}{(-2)} + C = -\frac{1}{2t^2} + C = -\frac{1}{2(1+\tan x)^2} + C$$

26. $\int x^3 \sin x^4 dx$ on R

Sol. $\int x^3 \sin x^4 dx$

Put $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\int x^3 \sin x^4 dx = \frac{1}{4}\int \sin t \cdot dt$$

$$= -\frac{1}{4}\cos t + C = -\frac{1}{4}\cdot\cos x^4 + C$$

27. $\int \frac{\cos x}{(1+\sin x)^2} dx$ **on** $I \subset R \setminus \left\{2n\pi + \frac{3\pi}{2} : n \in Z\right\}$

Sol. $\int \frac{\cos x}{(1+\sin x)^2} dx$

Put $1+\sin x = t \Rightarrow \cos x dx = dt$

$$\int \frac{\cos x}{(1+\sin x)^2} dx = \int \frac{dt}{t^2}$$

$$= -\frac{1}{t} + C = -\frac{1}{1+\sin x} + C$$

28. $\int \sqrt[3]{\sin x}\cos x\,dx$ **on [2nπ, (2n + 1)π, (n ∈ Z)].**

Sol. $\int \sqrt[3]{\sin x}\cos x\,dx$

Put sin x=t ⇒ cos x dx =dt

$$\int \sqrt[3]{\sin x}\cos x\,dx = \int \sqrt[3]{t}\cdot dt$$

$$= \frac{t^{4/3}}{(4/3)} + C = \frac{3}{4}t^{4/3} + C = \frac{3}{4}(\sin x)^{4/3} + C$$

29. $\int 2x\,e^{x^2}\,dx$ **on R.**

Sol. $\int 2x\,e^{x^2}\,dx$

Let x^2 =t⇒ 2x dx =dt

$$\int 2x\,e^{x^2}\,dx = \int e^t\,dt = e^t + C = e^{x^2} + C$$

30. $\int \frac{e^{\log x}}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{e^{\log x}}{x}dx$

Put log x= t $\Rightarrow \frac{1}{x}dx = dt$

$$\int \frac{e^{\log x}}{x}dx = \int e^t\cdot dt = e^t + C$$

$$= e^{\log x} + C = x + C$$

31. $\int \frac{x^2}{\sqrt{1-x^6}}dx$ **on I = (–1, 1).**

Sol. $\int \frac{x^2}{\sqrt{1-x^6}}dx$

Put x^3=t ⇒ $3x^2$ dx =dt

$$\int \frac{x^2}{\sqrt{1-x^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{3}\sin^{-1}t + C = \frac{1}{3}\sin^{-1}(x^3) + C$$

32. $\int \frac{2x^3}{1+x^8}dx$ **on R.**

Sol. let $x^{4=t} \Rightarrow \quad 4x^3\,dx = dt$

$$\int \frac{2x^3}{1+x^8}dx = \frac{1}{2}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\tan^{-1} t + C = \frac{1}{2}\tan^{-1}(x^4) + C$$

33. $\int \frac{x^8}{1+x^{18}}dx$

Sol. $\int \frac{x^8}{1+x^{18}}dx = \int \frac{x^8}{1+\left(x^9\right)^2}dx$ on R.

Put $x^9 = t \Rightarrow \quad 9x^8\,dx = dt$

$$\int \frac{x^8}{1+x^{18}}dx = \int \frac{x^8}{1+(x^9)^2}dx = \frac{1}{9}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{9}\tan^{-1} t + C = \frac{1}{9}\tan^{-1}(x^9) + C$$

34. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$ **on**

Sol. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $x\,e^x = t$

$$(x \cdot e^x + e^x)dx = e^x(1+x)dx = dt$$

$$\text{G.I.} = \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt$$

$$= \tan t + C = \tan(x \cdot e^x) + C$$

35. $\int \frac{\csc^2 x}{(a+b\cot x)^5}dx$ **on** $I \subset R \setminus \{x \in R : a + b\cot x = 0\}$**, where** $a, b \in R, b \neq 0$**.**

Sol.

$$\text{G.I.} = \int \frac{\csc^2 x}{(a+b\cot x)^5}dx$$

Put $a + b\cot x = t \Rightarrow \quad -b\csc^2 x\,dx = dt$

$$\int \frac{\csc^2 x}{(a+b\cot x)^5}dx = -\frac{1}{b}\int \frac{dt}{t^5} = -\frac{1}{b}\int t^{-5}dt$$

$$= -\frac{1}{b}\frac{t^{-4}}{-4} + C = \frac{1}{4bt^4} + C = \frac{1}{4b(a+b\cot x)^4} + C$$

36. $\int e^x \sin e^x dx$ **on R.**

Sol. $e^x = t \Rightarrow e^x\, dx = dt$

$$\int e^x \sin e^x dx = \int \sin t\, dt$$

$$= -\cot + C = -\cos(e^x) + C$$

37. $\int \frac{\sin(\log x)}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{\sin(\log x)}{x}dx$

put $\log x = t \Rightarrow dt = \frac{1}{x}dx = dt$

$$\int \frac{\sin(\log x)}{x}dx = \int \sin t\, dt$$

$$= -\cot t + C = -\cos(e^x) + C$$

38. $\int \frac{1}{x\log x}dx$ **on (0, ∞)**

Sol. $\int \frac{1}{x\log x}dx$

Put $\log x = t \Rightarrow dt = \frac{1}{x}dx = dt$

$$\int \frac{1}{x\log x}dx = \int \frac{1}{t}dt = \log t + C = \log(\log x + C)$$

39. $\int \frac{(1+\log x)^n}{x}dx$ **on (0, ∞), n ≠ –1.**

Sol. $\int \frac{(1+\log x)^n}{x}dx$

Put $1 + \log x = t, \Rightarrow \frac{1}{x}dx = dt$

$$\int \frac{(1+\log x)^n}{x}dx = \int t^n dt = \frac{t^{n+1}}{n+1} + C$$

$$= \frac{(1+\log x)^{n+1}}{n+1} + C$$

40. $\int \frac{\cos(\log x)}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{\cos(\log x)}{x}dx$

Put log x =t⇒ dt = $\frac{1}{x}dx$ =dt

$$\int \frac{\cos(\log x)}{x}dx = \int \cos t\,dt$$
$$= \sin t + C = \sin(\log x) + C$$

41. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ **on (0, ∞)**

Sol. let $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \Rightarrow 2dt = \frac{dx}{\sqrt{x}}$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t dx$$
$$= 2\sin t + C = 2\sin\sqrt{x} + C$$

42. $\int \frac{2x+1}{x^2+x+1}dx$ **on R.**

Sol. $\int \frac{2x+1}{x^2+x+1}dx$

put $x^2 + x + 1$=t ⇒(2x + 1)dx =dt

$$\int \frac{2x+1}{x^2+x+1}dx = \int \frac{dt}{t}$$
$$= \log|t| + C = \log|x^2+x+1| + C$$

43. $\int \frac{ax^{n-1}}{bx^n+C}dx$**, where n ∈ N, a, b, c are real numbers, b ≠ 0 and** $x \in I \subset \left\{x \in R : x^n \neq -\frac{c}{b}\right\}$

Sol. $\int \frac{ax^{n-1}}{bx^n+C}dx$

let $bx^n + C$=t⇒ $nbx^{n-1}dx$=dt, $x^{n-1}dx = \frac{1}{nb}dt$

$$\int \frac{ax^{n-1}}{bx^n+C}dx = \frac{a}{nb}\int \frac{dt}{t} = \frac{a}{nb}\log|t| + dt$$
$$= \frac{a}{nb}\log|bx^n + c| + k$$

44. $\int \frac{1}{x\log x[\log(\log x)]}dx$ **on (1, ∞)**

Sol. G.I. $\int \frac{1}{x\log x[\log(\log x)]}dx$

Put log(log x)=t, $\frac{1}{\log x}\cdot\frac{1}{x}dx = dt$

$$\int \frac{1}{x\log x[\log(\log x)]}dx = \int \frac{dt}{t}$$

$$= \log|t| + C = \log|\log(\log x)| + C$$

45. $\int \coth x dx$ **on R.**

Sol. sinh x=t⇒ cosh x dx=dt

$$\int \coth x dx = \int \frac{dt}{t} = \log|t| + C$$

$$= \log|\sinh x| + C$$

46. $\int \frac{1}{\sqrt{1-4x^2}}dx$ on $\left(-\frac{1}{2},\frac{1}{2}\right)$

Sol. $\int \frac{1}{\sqrt{1-4x^2}}dx = \frac{1}{2}\int \frac{dx}{\sqrt{(1/2)^2 - x^2}}$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{1/2}\right) + C = \frac{1}{2}\sin^{-1}(2x) + C$$

47. $\int \frac{dx}{\sqrt{25+x^2}}$ **on R**

Sol. $\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{dx}{\sqrt{x^2+5^2}} = \sinh^{-1}\left(\frac{x}{5}\right) + C$

48. $\int \frac{1}{(x+3)\sqrt{x+2}}dx$ **on I ⊂ (–2, ∞)**

Sol. put $x + 2 = t^2$, $dx = 2t\,dt$

$$\int \frac{1}{(x+3)\sqrt{x+2}}dx = \int \frac{2t\,dt}{t(t^2+1)} = 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1}(t) + C = 2\tan^{-1}(\sqrt{x+2}) + C$$

49. $\int \frac{1}{1+\sin 2x} dx$ **on** $I \subset R \backslash \left\{ \frac{n\pi}{2} + (-1)^n \frac{\pi}{4} : n \in Z \right\}$

Sol. $\int \frac{1}{1+\sin 2x} dx = \int \frac{dx}{1+\frac{2\tan x}{1+\tan^2 x}}$

$= \int \frac{(1+\tan^2 x)dx}{1+\tan^2 x + 2\tan x} = \int \frac{\sec^2 x dx}{(1+\tan x)^2}$

put $1+\tan x = t \Rightarrow \sec^2 x dx = dt$

$\int \frac{1}{1+\sin 2x} dx = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{1+\tan x} + C$

50. $\int \frac{x^2+1}{x^4+1} dx$ on R.

Sol: $\int \frac{x^2+1}{x^4+1} dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$

$= \int \frac{\left[1+\frac{1}{x^2}\right]}{\left[x-\frac{1}{x}\right]^2+2} . dx$

$(\because a^2+b^2 = (a+b)^2 - 2ab)$

Take $x - \frac{1}{x} = t$ then $\left(1+\frac{1}{x^2}\right) dx = dt$

$\therefore \int \frac{x^2+1}{x^4+1} dx = \int \frac{dt}{t^2+2} = \int \frac{dt}{t^2+(\sqrt{2})^2}$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c.$

51. $\int \frac{dx}{\cos^2 x + \sin 2x}$ on $I \subset R / \left[\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\} \cup \left\{2n\pi + \mathrm{Tan}^{-1}\frac{1}{2} : n \in Z\right\}\right]$

Sol: $\int \frac{dx}{\cos^2 x + \sin 2x} = \int \frac{dx}{\cos^2 x + 2\sin x \cos x}$

$= \int \frac{(\sin^2 x + \cos^2 x)}{\cos^2 x + 2\sin x \cos x} dx$

$= \int \frac{1 + \tan^2 x}{1 + 2\tan x} dx = \int \frac{\sec^2 x dx}{1 + 2\tan x}$

Let 1 + 2 tan x = t then 2 $\sec^2$ x dx = dt

$\Rightarrow \sec^2 x dx = \frac{1}{2} dt$

$\therefore \int \frac{dx}{\cos^2 x + \sin 2x} = \frac{1}{2} \int \frac{dt}{t}$

$= \frac{1}{2} \log|t| + c$

$= \frac{1}{2} \log|1 + 2\tan x| + c$.

52. $\int \sqrt{1 - \sin 2x}\, dx$ on $I \subset \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right], n \in Z$.

Sol: $\int \sqrt{1 - \sin 2x}\, dx$

$= \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x}\, dx$

$= \int \sqrt{(\sin x - \cos x)^2}\, dx$

$= \int \sqrt{(\cos x - \sin x)^2}\, dx$

$= \int (\cos x - \sin x) dx$

$= \int \cos x\, dx - \int \sin x\, dx$

$= \sin x + \cos x + c$

For $x \in \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right]$

$\int \sqrt{1 - \sin 2x}\, dx = (\sin x + \cos x) + c$.

53. $\int \sqrt{1+\cos 2x}\, dx$ on $I \subset \left\{2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right\}, n \in Z$.

Sol: $\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{1+2\cos^2 x - 1}\, dx$

$= \int \sqrt{2\cos^2 x}\, dx$

$= \sqrt{2}\int \cos x\, dx + c$

$= \sqrt{2}\sin x + c$

For $x \in \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right]$

54. $\int \frac{\cos x + \sin x}{\sqrt{1+\sin 2x}} dx$ on $I \subset \left[2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right], n \in Z$.

Sol: $\int \frac{\cos x + \sin x}{\sqrt{1+\sin 2x}} dx$

$= \int \frac{(\cos x + \sin x)}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}}$

$= \int \frac{\cos x + \sin x}{\sqrt{(\cos x + \sin x)^2}} dx$

$= \int \left(\frac{\cos x + \sin x}{\cos x + \sin x}\right) dx = \int dx = x + c$, For $x \in \left[2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right]$

55. $\int \frac{\sin 2x}{(a + b\cos x)^2} dx$ on $\begin{cases} R, \text{if } |a| > |b| \\ I \subset \{x \in R : a + b\cos x \neq 0\}, \text{if } |a| < |b| \end{cases}$.

Sol: $\int \frac{\sin 2x}{(a+b\cos x)^2} dx = \int \frac{2\sin x \cos x}{(a+b\cos x)^2} dx$

Let a + b cos x = t, then –b sin x dx = dt

$\Rightarrow \sin x\, dx = -\frac{1}{b} dt$

Also b cos x = t – a

$\Rightarrow \cos x = \frac{t-a}{b}$

$\therefore \int \frac{\sin 2x}{(a+b\cos x)^2} dx = -\frac{2}{b^2}\int \left(\frac{t-a}{t^2}\right) dt$

$= -\frac{2}{b^2}\left[\int \frac{1}{t} dt - a\int \frac{1}{t^2} dt\right]$

$= -\frac{2}{b^2}\left[\log(|t|) + \frac{a}{t}\right] + c$

$$= -\frac{2}{b^2}\left[\log|a+b\cos x| + \frac{a}{a+b\cos x}\right] + c.$$

56. $\int \frac{\sec x}{(\sec x+\tan x)^2}dx$ on $I \subset R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$.

Sol: $\int \frac{\sec x}{(\sec x+\tan x)^2} \quad = \int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^3}dx$

Let sec x + tan x = t

then (sec x tan x + $\sec^2$ x) dx = dt

$\Rightarrow \sec x(\sec x+\tan x)dx = dt$

$\therefore \int \frac{\sec x}{(\sec x+\tan x)^2}dx$

$= \int \frac{dt}{t^3} = \int t^{-3}dt = \frac{t^{-2}}{-2}$

$= -\frac{1}{2t^2} = -\frac{1}{2(\sec x+\tan x)^2} + c$

57. $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$ on R, $a \neq 0$, $b \neq 0$.

Sol: $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$

Dividing numerator and denominator by $\cos^2$ x,

$= \int \frac{\sec^2 x dx}{a^2.\tan^2 x + b^2}$

Let tan x = t, then $\sec^2$ x dt = dt

$\therefore \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \int \frac{dt}{a^2t^2+b^2}$

$= \frac{1}{a^2}\int \frac{dt}{t^2+\left(\frac{b}{a}\right)^2}$

$= \frac{1}{a^2}\frac{1}{\left(\frac{b}{a}\right)}\tan^{-1}\frac{t}{\left(\frac{b}{a}\right)} = \frac{1}{ab}\mathrm{Tan}^{-1}\frac{at}{b}$

$= \frac{1}{ab}\mathrm{Tan}^{-1}\left(\frac{a\tan x}{b}\right) + c.$

58. $\int \frac{\sec^2 x}{\sqrt{a+b\tan x}}dx$ **, a, b are positive real numbers, on** $I \subset R \setminus \left(\left\{x\in R : \tan x < -\frac{a}{b}\right\} \cup \left\{\frac{(2n+1)\pi}{2} : n\in Z\right\}\right)$

Put $a+b\tan x = t$

Ans. $\frac{2}{b}\sqrt{a+b\tan x}+C$

59. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ **on** $I \subset R \setminus (\{a+n\pi : n\in Z\}\cup\{b+n\pi : n\in Z\})$.

Sol. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

[**Hint:** $\int \cot x dx = \log|\sin x|+C$]

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}$$

$$\int\left\{\frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}\right\}dx$$

$$=\frac{1}{\sin(b-a)}\int\{\cot(x-b)-\cot(x-a)\}dx$$

$$=\frac{1}{\sin(b-a)}[\log|\sin(x-b)|-\log|\sin(x-a)|]+C$$

$$=\frac{1}{\sin(b-a)}\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$$

60. $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$ **on** $I\subset R \setminus \left(\left\{a+\frac{(2n+1)\pi}{2} : n\in Z\right\}\cup\left\{b+(2n+1)\frac{\pi}{2} : n\in Z\right\}\right)$

Sol. $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}dx$$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin(\overline{x-b}-\overline{x-a})}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}$$

$$\int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \{\tan(x-b) - \tan(x-a)\} dx$$

$$= \frac{1}{\sin(a-b)} [\log|\sec(x-b)| - \log|\sec(x-a)|] + C$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\sec(x-b)}{\sec(x-a)}\right| + C$$

61. $\int \sqrt{1+\sec x}\, dx$ **on** $\left[\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right], n \in Z$.

Sol. $\int \sqrt{1+\sec x}\, dx = \sqrt{\frac{\sec^2 x - 1}{\sec x - 1}} dx$

$$= \int \frac{\tan x}{\sqrt{\sec x - 1}} dx = \int \frac{\frac{\sin x}{\cos x}}{\sqrt{\frac{1-\cos x}{\cos x}}} dx$$

$$= \int \frac{\sin x}{\sqrt{\cos x}\sqrt{1-\cos x}} dx$$

Put $\cos x = t \Rightarrow \sin x\, dx = -dt$

$$= \int \frac{-dt}{\sqrt{t}\sqrt{1-t}} = -\int \frac{1}{\sqrt{t-t^2}} dt$$

$$= -\int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(t-\frac{1}{2}\right)^2}}$$

$$= -\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right) + C = -\left[t^2 - t + \frac{1}{4} - \frac{1}{4}\right]$$

$$= -\sin^{-1}(2t-1) + C = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2$$

$$= -\sin^{-1}[2\cos x - 1] + C$$

62. $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx$

Sol. put $a\cos^2 x + b\sin^2 x = t$

$$\left(a(2\cos x)(-\sin x) + b(2\sin x\cos x)\right)dx = dt$$

$$= \sin 2x(b-a)dx$$

$$\sin 2x \cdot dx = \frac{1}{(b-a)} dt$$

$$\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx = \frac{1}{(b-a)} \int \frac{dt}{t}$$

$$= \frac{1}{(b-a)} \log|t| + C$$

$$= \frac{1}{(b-a)} \log|a\cos^2 x + b\sin^2 x| + C$$

63. $\int \frac{\cot(\log x)}{x} dx$**,** $x \in I \subset (0,\infty) \setminus \{e^{n\pi} : n \in Z\}$**.**

Sol. Put $\log x = t \Rightarrow dt = \frac{1}{x} dx = dt$

$$\int \frac{\cot(\log x)}{x} dx = \int \cot t\, dt = \log(\sin t) + C$$

$$= \log(\sin(\log x)) + C$$

64. $\int e^x \cdot \cot e^x dx, x \in I \subset R \setminus \{\log n\pi : n \in Z\}$

Sol. Put $e^x = t \Rightarrow e^x\, dx = dt$

$$\int e^x \cdot \cot e^x dx = \int \cot t\, dt = \log|\sin t| + C$$

$$= \log(\sin e^x) + C$$

65. $\int \sec x (\tan x) \sec^2 x\, dx$**,**

Sol. $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\int \sec x(\tan x)\sec^2 x\, dx = \int \sec t \cdot dt$$

$$= \log\tan\left(\frac{\pi}{4} + \frac{t}{2}\right) + C$$

$$= \log\left(\tan\left(\frac{\pi}{4} + \frac{\tan x}{2}\right)\right) + C$$

66. $\int \sqrt{\sin x}\cos x\,dx$ **on** $[2n\pi, (2n+1)\pi]$**,** $n \in Z$**.**

Sol. $t = \sin x \Rightarrow dt = \cos x\,dx$

$$\int \sqrt{\sin x}\cdot\cos x\,dx = \int \sqrt{t}\;dt = \frac{2}{3}t^{3/2} + C$$

$$= \frac{2}{3}(\sin x)^{3/2} + C$$

67. $\int \tan^4 x\sec^2 x\,dx, x \in I \subset R \setminus \left\{\frac{(2n+1)\pi}{2} : n \in Z\right\}$

Sol. $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\int \tan^4 x\sec^2 x\,dx = \int t^4 dt$$

$$= \frac{t^5}{5} + C = \frac{(\tan x)^5}{t} + C$$

68. $\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx$**,** $x \in I \subset R \setminus [-4,1]$**.**

Sol. Let $x^2 + 3x - 4 = t \Rightarrow (2x + 3)dx = dt$

$$\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C$$

$$= 2\sqrt{x^2+3x-4} + C$$

69. $\int \csc^2 x\sqrt{\cot x}\,dx$ on $\left(0, \frac{\pi}{2}\right)$

Sol. put $\cot x = t \Rightarrow -\csc^2 x\,dx = dt$

$$\int \csc^2 x\sqrt{\cot x}\,dx = -\int \sqrt{t}\,dt$$

$$= -\frac{2}{3}t\sqrt{t} + C = -\frac{2}{3}\cot(x)^{3/2} + C$$

70. $\int \sec x\log(\sec x + \tan x)dx$ on $\left(0, \frac{\pi}{2}\right)$

Sol. $\log(\sec x + \tan x) = t$

$$\Rightarrow \frac{(\sec x\cdot\tan x + \sec^2 x)dx}{(\sec x + \tan x)} = dt = \sec x\;dx$$

$$\int \sec x\cdot\log(\sec x + \tan x)dx = \int t\,dt$$

$$= \frac{t^2}{2} + C = \frac{(\log(\sec x + \tan x))^2}{2} + C$$

71. $\int \sin^3 x\, dx$ **on R.**

Sol. since $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\int \sin^3 x\, dx = \frac{3}{4}\int \sin x - \frac{1}{4}\int \sin 3x\, dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

$$= \frac{1}{12}(\cos 3x - 9\cos x) + C$$

72. $\int \cos^3 x dx$ **on R.**

Sol. since $\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\int \cos^3 x\, dx = \frac{3}{4}\int \cos x\, dx + \frac{1}{4}\int \cos 3x\, dx$$

$$= \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

$$= \frac{1}{12}(9\sin x + \sin 3x) + C$$

73. $\int \cos x \cos 2x\, dx$ **on R.**

Sol. $\cos 2x \cos x = \frac{1}{2}(2\cos 2x \cdot \cos x)$

$$\int \cos x = \cos 2x\, dx = \frac{1}{2}\int (\cos 3x + \cos x) dx$$

$$= \frac{1}{2}\int \cos 3x dx + \frac{1}{2}\int \cos x dx$$

$$= \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C = \frac{\sin 3x + 3\sin x}{6} + C$$

74. $\int \cos x \cos 3x\, dx$ **on R.**

Sol. $\cos 3x \cos x = \frac{1}{2}(2\cos 3x \cdot \cos x)$

$$\frac{1}{2}(\cos 4x + \cos 2x)$$

$$\int \cos x \cos 3x\, dx = \frac{1}{2}\int \cos 4x\, dx + \frac{1}{2}\cos 2x\, dx$$

$$= \frac{1}{2}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right) + C$$

$$= \frac{1}{8}(\sin 4x + 2\sin 2x) + C$$

75. $\int \cos^4 x\, dx$ **on R.**

Sol. $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}(2 + 4\cos 2x + 1 + \cos 4x)$$

$$= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$

$$= \frac{1}{8}\left(3\int dx + 4\int \cos 2x\, dx + \int \cos 4x\, dx\right)$$

$$= \frac{1}{8}\left(3x + 4\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right) + C$$

$$= \frac{1}{32}(12x + 8\sin 2x + \sin 4x) + C$$

76. $\int x\sqrt{4x+3}\, dx$ **on** $\left(-\frac{3}{4}, \infty\right)$**.**

Sol. put $4x + 3 = t^2 \Rightarrow 4dx = 2t\, dt$

$$dx = \frac{1}{2} t\, dt \Rightarrow x = \frac{t^2 - 3}{4}$$

$$\int x\sqrt{4x+3}\, dx = \int \frac{t^2-3}{4} \cdot t \cdot \frac{1}{2} t\, dt$$

$$= \frac{1}{8}\int (t^4 - 3t^2) dt = \frac{1}{8}\left(\frac{t^5}{5} - t^3\right) + C$$

$$= \frac{(4x+3)^{5/2}}{40} - \frac{1}{8}(4x+3)^{3/2} + C$$

77. $\int \frac{dx}{\sqrt{a^2-(b+cx)^2}}$ **on** $\{x \in R : |b+cx| < a\}$, **where a, b, c are real numbers c ≠ 0 and a > 0.**

Sol. $\int \frac{dx}{\sqrt{a^2-(b+cx)^2}} = \int \frac{dx}{c\sqrt{\left(\frac{a}{c}\right)^2-\left(\frac{b}{c}-x\right)^2}}$

$$= \frac{1}{c}\sin^{-1}\left(\frac{\left(\frac{b}{c}+x\right)}{\left(\frac{a}{c}\right)}\right)+K = \frac{1}{c}\sin^{-1}\left(\frac{b+cx}{a}\right)+K$$

78. $\int \frac{dx}{a^2+(b+cx)^2}$ **on R, where a, b, c are real numbers, c ≠ 0 and a > 0.**

Sol. $\int \frac{dx}{a^2+(b+cx)^2} = \frac{1}{c^2}\int \frac{dx}{\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}+x\right)^2}$

$$= \frac{1}{a^2\cdot\frac{a}{c}}\tan^{-1}\left(\frac{\frac{b}{c}+x}{\frac{a}{c}}\right)+C$$

$$= \frac{1}{ac}\tan^{-1}\left(\frac{b+cx}{a}\right)+C$$

79. $\int \frac{dx}{1+e^x}, x \in R$

Sol. $\int \frac{dx}{1+e^x} = \int\left(\frac{1+e^x-e^x}{1+e^x}\right)dx$

$$= \int\left(1-\frac{e^x}{1+e^x}\right)dx = x-\log(1+e^x)+C$$

80. $\int \frac{x^2}{(1+bx)^2}dx$, $x \in I \subset R \setminus \left\{-\frac{a}{b}\right\}$, **where a, b are real numbers, b ≠ 0.**

Sol. Put a + bx=t ,⇒ b dx =dt⇒ dx = $\frac{1}{b}\cdot dt$

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b}\int \frac{\left(\frac{t-a}{b}\right)^2}{t^2}dt$$

$$= \frac{1}{b^3}\int \frac{t^2-2at+a^2}{t^2}dt$$

$$= \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) + C$$

$$= \frac{1}{b^3}\left(t - 2a\log|t| - \frac{a^2}{t}\right) + C$$

$$= \frac{1}{b^3}\left[(a+bx) - 2a\log|a+bx| - \frac{a^2}{(a+bx)}\right] + C$$

81. $\int \frac{x^2}{\sqrt{1-x}}dx, x \in (-\infty, 1)$

Sol. Put $1 - x = t^2, -dx = 2t\,dt$

$$\int \frac{x^2}{\sqrt{1-x}}dx = \int (1-t^2)^2 \cdot \frac{-2t}{t}dt$$

$$= 2\int (1 - 2t^2 + t^4)dt = -2\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) + C$$

$$= -2\left(\sqrt{1-x} - \frac{2}{3}(1-x)^{3/2} + \frac{1}{5}(1-x)^{5/2}\right) + C$$

82. $\int \frac{1-\tan x}{1+\tan x}dx$ **for** $x \in I \subset R \setminus \left\{n\pi - \frac{\pi}{4} : n \in Z\right\}$

Sol. $\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}dx$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x}dx$$

$\cos x + \sin x = t$

$\Rightarrow dt = -\sin x + \cos x\,dx$

$$\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{dt}{t} = \log|t| + C$$

$$= \log|\cos x + \sin x| + C$$

83. Evaluate $\int \frac{1}{a\sin x + b\cos x}dx$ **where a, b ∈ and $a^2 + b^2 \neq 0$ on r.**

Sol. We can find real numbers r and θ such that a = cosθ, b = r sin θ

Then $r = \sqrt{a^2+b^2}, \cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$

$a\sin x + b\cos x = r\cdot\cos\theta\sin x + r\sin\theta\cos x$

$= r[\cos\theta\sin x + \sin\theta\cos x] = r\sin(x+\theta)$

$\int \frac{1}{a\sin x + b\cos x}dx = \frac{1}{r}\int\frac{1}{\sin(x+\theta)}dx$

$= \frac{1}{r}(\csc(x+\theta))dx = \frac{1}{r}\log\left|\tan\frac{1}{2}(x+\theta)\right| + C$

$= \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\frac{1}{2}(x+\theta)\right| + C$

For all x ∈ I where I is an internal disjoint with {nπ– θ : n ∈ Z}.

84. $\int \sin x\sin 2x\sin 3x\ dx$ **on R.**

Sol: Consider sin x sin 2x sin 3x

$= \frac{1}{2}(2\sin x\sin 2x\sin 3x)$

$= \frac{1}{2}[\cos(3x-2x) - \cos(3x+2x)]\sin x$

$= \frac{1}{2}[\sin x\cos x - \sin x\cos 5x]$

$= \frac{1}{4}[2\sin x\cos x - 2\sin x\cos 5x]$

$= \frac{1}{4}[\sin 2x - [\sin(5x+x) + \sin(x-5x)]$

$= \frac{1}{4}[\sin 2x - [\sin 6x - \sin 4x)]$

$= \frac{1}{4}[\sin 2x - \sin 6x + \sin 4x]$

$\therefore \int \sin x\sin 2x\sin 3x\ dx = \frac{1}{4}\int\sin 2x\,dx - \frac{1}{4}\int\sin 6x\,dx$

$= -\frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x + c$

$= \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + c$

85. $\int \frac{\sin x}{\sin(a+x)} dx$ on $I \subset R - \{n\pi - a : n \in Z\}$.

Sol: $\int \frac{\sin x}{\sin(a+x)} dx = \int \frac{\sin(x+a-a)}{\sin(x+a)} dx$

$= \int \left[\frac{\sin(x+a)\cos a - \cos(x+a)\sin a}{\sin(x+a)} \right] dx$

$= \cos a \int dx - \sin a \int \frac{\cos(x+a)}{\sin(x+a)} dx$

$= x\cos a - \sin a - \log|\sin(x+a)| + c.$