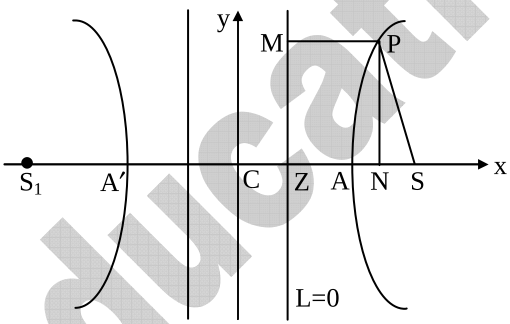
HYPERBOLA

Equation of a Hyperbola in Standard From.

The equation of a hyperbola in the standard form is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Proof: Let S be the focus, e be the eccentricity and L = 0 be the directrix of the hyperbola.

Let P be a point on the hyperbola. Let M, Z be the projections of P, S on the directrix L = 0 respectively. Let N be the projection of P on SZ. Since e > 1, we can divide SZ both internally and externally in the ratio e : 1. Let A, A′ be the points of division of SZ in the ratio e : 1 internally and externally respectively. Let AA′ = 2a. Let C be the midpoint of AA′. The points A, A′ lie on the hyperbola and $\frac{SA}{AZ}=e$, $\frac{SA'}{A'Z}=e$.



$\therefore$ SA = eAZ, SA′ = eA′Z.

Now SA + SA′ = eAZ + eA′Z

$\Rightarrow$ CS – CA + CS + CA′ = e(AZ + A′Z)

$\Rightarrow$ 2CS = eAA′ ($\because$ CA = CA′)

$\Rightarrow$ 2CS = e2a $\Rightarrow$ CS = ae

Aslo SA′ – SA = eA′Z – eAZ

$\Rightarrow$ AA′ = e(A′Z – AZ)

$\Rightarrow$ 2a = e[CA′ + CZ – (CA – CZ)]

$\Rightarrow 2^a$ = e 2CZ ($\because$ CA = CA′) $\Rightarrow$ CZ = $\frac{a}{e}$.

Take CS, the principal axis of the hyperbola as x-axis and Cy perpendicular to CS as y-axis. Then S = (ae, 0).

Let $P(x_1, y_1)$.

Now PM = NZ = CN – CZ = $x_1 - \frac{a}{e}$.

P lies on the hyperbola $\Rightarrow \frac{PS}{PM} = e$

$\Rightarrow PS = ePM \Rightarrow PS^2 = e^2PM^2$

$\Rightarrow (x_1 - ae)^2 + (y_1 - 0)^2 = e^2\left(x_1 - \frac{a}{e}\right)^2$

$\Rightarrow (x_1 - ae)^2 + y_1^2 = (x_1e - a)^2$

$\Rightarrow x_1^2 + a^2e^2 - 2x_1ae + y_1^2 = x_1^2e^2 + a^2 - 2x_1ae$

$\Rightarrow x_1^2(e^2 - 1) - y_1^2 = a^2(e^2 - 1)$

$\Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{a^2(e^2-1)} = 1 \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

Where $b^2 = a^2(e^2 - 1)$

The locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore$ The equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Nature of the Curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let C be the curve represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.Then

i) $(x, y) \in C \Leftrightarrow (x, -y) \in C$ and $(x, y) \in C \Leftrightarrow (-x, y) \in C$.

Thus the curve is symmetric with respect to both the x-axis and the y-axis. Hence the coordinate axes are two axes of the hyperbola.

ii) $(x, y) \in C \Leftrightarrow (-x, -y) \in C$.

Thus the curve is symmetric about the origin O and hence O is the midpoint of every chord of the hyperbola through O. Therefore the origin is the center of the hyperbola.

iii) $(x, y) \in C$ and $y = 0 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$.

Thus the curve meets x-axis (Principal axis) at two points A(a, 0), A′(–a, 0). Hence hyperbola has two vertices. The axis AA′ is called transverse axis. The length of transverse axis is AA′ = 2a.

iv) $(x, y) \in C$ and $x = 0 \Rightarrow y^2 = -b^2 \Rightarrow y$ is imaginary.

Thus the curve does not meet the y-axis. The points B(0, b), B′(0, –b) are two points on y-axis. The axis BB′ is called conjugate axis. BB′ = 2b is called the length of conjugate axis.

v) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = \frac{b}{a}\sqrt{x^2 - a^2} \Rightarrow y$ has no real value for $-a < x < a$.

Thus the curve does not lie between x = –a and x = a.

Further $x \to \infty \Rightarrow y \to \pm\infty$ and

$x \to -\infty \Rightarrow y \to \pm\infty$.

Thus the curve is not bounded (closed) on both the sides of the axes.

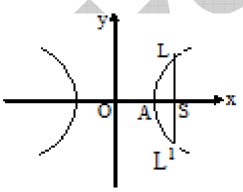
vi) The focus of the hyperbola is S(ae, 0). The image of S with respect to the conjugate axis is S′(–ae, 0). The point S′ is called second focus of the hyperbola.

vii) The directrix of the hyperbola is x = a/e. The image of x = a/e with respect to the conjugate axis is x = –a/e. The line x = –a/e is called second directrix of the hyperbola corresponding to the second focus S′.

Theorem: The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

Proof:

Let LL′ be the length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



If SL = l, then L = (ae, l)

L lies on the hyperbola $\Rightarrow \frac{(ae)^2}{a^2} - \frac{l^2}{b^2} = 1$

$$\Rightarrow \frac{l^2}{b^2} = e^2 - 1 \Rightarrow l^2 = b^2(e^2 - 1)$$

$$\Rightarrow l^2 = b^2 \times \frac{b^2}{a^2} \Rightarrow l = \frac{b^2}{a} \Rightarrow SL = \frac{b^2}{a}$$

$$\therefore LL' = 2SL = \frac{2b^2}{a}.$$

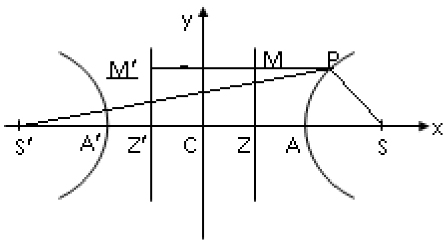
Theorem: The difference of the focal distances of any point on the hyperbola is constant i.e., if P is appoint on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci S and S′ then $|PS' - PS| = 2a$

Proof:

Let e be the eccentricity and L = 0, L′ = 0 be the directrices of the hyperbola.

Let C be the centre and A, A′ be the vertices of the hyperbola.

$\therefore$ AA′ = 2a.



Foci of the hyperbola are S(ae, 0), S′(–ae, 0).

Let $P(x_1, y_1)$ be a point on the hyperbola.

Let M, M′ be the projections of P on the directrices L = 0, L′ = 0 respectively.

$$\therefore \frac{SP}{PM} = e, \frac{S'P}{PM'} = e.$$

Let Z, Z′ be the points of intersection of transverse axis with directrices.

$$\therefore MM' = ZZ' = CZ + CZ' = 2a/e$$

$$PS' - PS = ePM' - ePM = e(PM' - PM)$$
$$= e(MM') = e(2a/e) = 2a$$

Notation: We use the following notation in this chapter.

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1, \; S_1 \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1,$$

$$S_{11} \equiv S(x_1, y_1) = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1, \; S_{12} \equiv \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} - 1.$$

Note: Let $P(x_1, y_1)$ be a point and $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be a hyperbola. Then

i) P lies on the hyperbola $S = 0 \Leftrightarrow S_{11} = 0$

ii) P lies inside the hyperbola $S = 0 \Leftrightarrow S_{11} > 0$

iii) P lies outside the hyperbola $S = 0 \Leftrightarrow S_{11} < 0$

Theorem: The equation of the chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ on the hyperbola $S = 0$ is $S_1 + S_2 = S_{12}$.

Theorem: The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$.

Theorem: The condition that the line $y = mx + c$ may be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$.

Note: The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as $y = mx \pm \sqrt{a^2m^2 - b^2}$.

The point of contact is $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$ where $c^2 = a^2m^2 - b^2$.

Theorem: Two tangents can be drawn to a hyperbola from an external point.

Note: If m_1, m_2 are the slopes of the tangents through P, then m_1, m_2 become the roots of $(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$.

Hence $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$, $m_1m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$.

Theorem: The point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ lies on the circle $x^2 + y^2 = a^2 - b^2$.

Proof:

Equation of any tangent to the hyperbola is:

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Suppose $P(x_1, y_1)$ is the point of intersection of tangents.

P lies on the tangent $\Rightarrow y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2} \Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 - b^2}$

$$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 - b^2$$

$$\Rightarrow y_1^2 + m^2x_1^2 - 2mx_1y_1 - a^2m^2 + b^2 = 0$$

$$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$$

This is a quadratic in m giving the values for m say m_1 and m_2.

The tangents are perpendicular:

$$\Rightarrow m_1m_2 = -1 \Rightarrow \frac{y_1^2 + b^2}{x_1^2 - a^2} = -1$$

$$\Rightarrow y_1^2 + b^2 = -x_1^2 + a^2 \Rightarrow x_1^2 + y_1^2 = a^2 - b^2$$

$P(x_1, y_1)$ lies on the circle $x^2 + y^2 = a^2 - b^2$.

Definition: The point of intersection of perpendicular tangents to a hyperbola lies on a circle, concentric with the hyperbola. This circle is called director circle of the hyperbola.

Definition: The feet of the perpendiculars drawn from the foci to any tangent to the hyperbola lies on a circle, concentric with the hyperbola. This circle is called auxiliary circle of the hyperbola.

Corollary: The equation to the auxiliary circle of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2$.

Theorem: The equation to the chord of contact of $P(x_1, y_1)$ with respect to the hyperbola $S = 0$ is $S_1 = 0$.

Midpoint of a Chord:

Theorem: The equation of the chord of the hyperbola S = 0 having $P(x_1, y_1)$ as it's midpoint is $S_1 = S_{11}$.

Pair of Tangents:

Theorem: The equation to the pair of tangents to the hyperbola S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

Asymptotes:

Definition: The tangents of a hyperbola which touch the hyperbola at infinity are called asymptotes of the hyperbola.

Note:

1. The equation to the pair of asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

2. The equation to the pair of asymptotes and the hyperbola differ by a constant.

3. Asymptotes of a hyperbola passes through the centre of the hyperbola.

4. Asymptotes are equally inclined to the axes of the hyperbola.

5. Any straight line parallel to an asymptote of a hyperbola intersects the hyperbola at only one point.

Theorem: The angle between the asymptotes of the hyperbola S = 0 is $2\tan^{-1}(b/a)$.

Proof:

The equations to the asymptotes are $y=\pm\frac{b}{a}x$.

If θ is an angle between the asymptotes, then

$$\tan\theta=\frac{\frac{b}{a}-\left(-\frac{b}{a}\right)}{1+\left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)}=\frac{2\left(\frac{b}{a}\right)}{1-\frac{b^2}{a^2}}=\frac{2\tan\alpha}{1-\tan^2\alpha}=\tan 2\alpha \text{ Where } \tan\alpha=\frac{b}{a}.$$

$$\therefore \theta=2\alpha=2\text{Tan}^{-1}\frac{b}{a}.$$

Parametric Equations:

A point (x, y) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ represented as x = a secθ, y = b tan θ in a single parameter θ. These equations x = a secθ, y= b tanθ are called parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The point (a secθ, b tanθ) is simply denoted by θ.

Note: A point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ can also be represented by (a coshθ, b sinhθ). The equations x = a coshθ, y = sinhθ are also called parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Theorem: The equation of the chord joining two points α and β on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is:

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}.$$

Theorem: The equation of the tangent at P(θ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$.

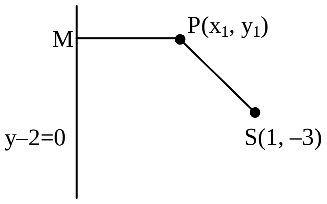
Theorem: The equation of the normal at P(θ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$.

Very Short Answer Questions

1. One focus of a hyperbola is located at the point (1, –3) and the corresponding directrix is the line y = 2. Find the equation of the hyperbola if its eccentricity is 3/2.

Sol. Focus S(1, –3) and directrix L= y – 2 = 0.

Eccentricity e = 3/2.



Let $P(x_1, y_1)$ be any point on the hyperbola. Let PM be the perpendicular from P to the directrix .

Then $SP = e \cdot PM \Rightarrow SP^2 = e^2 \cdot PM^2$

$$(x_1-1)^2+(y_1+3)^2=\frac{9}{4}\left|\frac{y_1-2}{\sqrt{1+0}}\right|^2$$

$$x_1^2+1-2x_1+y_1^2+9+6y_1=\frac{9}{4}(y_1-2)^2$$

$$4x_1^2+4y_1^2-8x_1+24y_1+40=9(y_1^2+4-4y_1)=9y_1^2-36y_1+36$$
$$4x_1^2-5y_1^2-8x_1+60y_1+4=0$$

Locus of $P(x_1, y_1)$ is

$4x^2 – 5y^2 – 8x + 60y + 4 = 0$.

2. If the lines 3x – 4y = 12 and 3x + 4y = 12 meets on a hyperbola S = 0 then find the eccentricity of the hyperbola S = 0.

Sol. Given lines 3x – 4y = 12, 3x + 4y = 12

The combined equation of the lines is

(3x – 4y)(3x + 4y) = 144

$9x^2 – 16y^2 = 144$

$$\frac{x^2}{\frac{144}{9}}-\frac{y^2}{\frac{144}{16}}=1 \Rightarrow \frac{x^2}{16}-\frac{y^2}{9}=1$$

$a^2=16, b^2=9$

Eccentricity e = $\sqrt{\frac{a^2+b^2}{a^2}}$

$$=\sqrt{\frac{16+9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$

3. Find the equations of the hyperbola whose foci are (±5, 0), the transverse axis is of length 8.

Sol. Foci are (±5, 0),

$\therefore$ SS'=2ae = 10. $\Rightarrow$ ae = 5.

Length of transverse axis is 2a = 8 $\Rightarrow$ a = 4

$\therefore$ e = 5/4

$b^2 = a^2(e^2 - 1) = 16\left(\frac{25}{9}-1\right)=9$

Equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$

$9x^2 - 16y^2 = 144.$

4. Find the equation of the hyperbola, whose asymptotes are the straight lines x + 2y +3 = 0, 3x + 4y + 5 = 0 and which passes through the point (1, –1).

Sol. Combined equation of the asymptotes is

$(x + 2y + 3)(3x + 4y + 5) = 0$

$\therefore$ Equation of the hyperbola can be taken as

$(x + 2y + 3)(3x + 4y + 5) + k = 0$

Given the hyperbola is passing through p(1, –1)

$\Rightarrow (1 - 2 + 3)(3 - 4 + 5) + k = 0$

$\Rightarrow 8 + k = 0 \Rightarrow k = -8$

Equation of the hyperbola is

$(x + 2y + 3)(3x + 4y + 5) – 8 = 0$

$3x^2 + 6xy + 9x + 4xy + 8y^2 + 12y + 5x + 10y + 15 – 8 = 0$

$3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0.$

5. If $3x – 4y + k = 0$ is a tangent to $x^2 – 4y^2 = 5$, find value of k.

Sol. Equation of the hyperbola $x^2 – 4y^2 = 5$

$$\frac{x^2}{5}-\frac{y^2}{(5/4)}=1\Rightarrow a^2=5, b^2=\frac{5}{4}$$

Equation of the line is $3x – 4y + k = 0$

$$4y = 3x + k \Rightarrow y=\frac{3}{4}x+\frac{k}{4} \text{ ---- (1)}$$

$$m=\frac{3}{4}, c=\frac{k}{4}$$

If (1) is a tangent to the hyperbola then

$c^2 = a^2m^2 – b^2$

$$\Rightarrow \frac{k^2}{16}=5\cdot\frac{9}{16}-\frac{5}{4}$$

$$\Rightarrow k^2=45-20=25\Rightarrow k=\pm5$$

6. Find the product of lengths of the perpendiculars from any point on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ to its asymptotes.

Sol. Equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$

$a^2 = 16, b^2 = 9$

Product of the perpendiculars from any point on the hyperbola to its asymptotes

$$=\frac{a^2b^2}{a^2+b^2}=\frac{16\times9}{16+9}=\frac{144}{25}$$

7. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.

Sol. Eccentricity e = 5/4

If e and e_1 are the eccentricity of a hyperbola and its conjugate hyperbola, then

$$\frac{1}{e^2}+\frac{1}{e_1^2}=1$$

$$\frac{16}{25}+\frac{1}{e_1^2}=1$$

$$\frac{1}{e_1^2}=1-\frac{16}{25}=\frac{9}{25}\Rightarrow e_1^2=\frac{25}{9}\Rightarrow e_1=\frac{5}{3}.$$

8. Find the equation of the hyperbola whose asymptotes are 3x = ±5y and the vertices and (±5, 0).

Sol: The equation of asymptotes are given by

$3x - 5y = 0$ and $3x + 5y = 0$.

∴ The equation of hyperbola is of the form $(3x - 5y)(3x + 5y) = k$

$\Rightarrow 9x^2 - 25y^2 = k$

If the hyperbola passes through the vertex (±5, 0) then

$9(25) = k \Rightarrow k = 225$

Hence the equation of asymptotes of hyperbola is $9x^2 - 25y^2 = 225$.

9. Find the equation of normal at $\theta = \frac{\pi}{3}$ to the hyperbola $3x^2 - 4y^2 = 12$.

Sol: The given equation of hyperbola is

$3x^2 - 4y^2 = 12$

$$\Rightarrow \frac{x^2}{4}-\frac{y^2}{3}=1$$

The equation of normal at P(a sec θ, b tanθ) to the hyperbola S = 0 is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

∴ Equation of normal at

$\theta = \frac{\pi}{3}$ When $a^2 = 4, b^2 = 3$

$$\frac{2x}{\sec\frac{\pi}{3}} + \frac{\sqrt{3}y}{\tan\frac{\pi}{3}} = 4 + 3$$

$$\Rightarrow \frac{2x}{2} + \frac{\sqrt{3}y}{\sqrt{3}} = 7$$

$$\Rightarrow x + y = 7.$$

10. If the angle between asymptotes is 30° then find its eccentricity.

Sol: Angle between asymptotes of hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\sec^{-1} e$.

$$\therefore 2\sec^{-1} e = 30° \Rightarrow \sec^{-1} e = 15°$$

$$\Rightarrow e = \sec 15° = \frac{1}{\cos 15°} = \frac{1}{\cos(45° - 30°)}$$

$$= \frac{1}{\cos 45° \cos 30° + \sin 45° \sin 30°}$$

$$= \frac{1}{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}}$$

$$= \frac{2\sqrt{2}}{\sqrt{3}+1} = \frac{2\sqrt{2}(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{2\sqrt{2}(\sqrt{3}-1)}{2}$$

$$= \sqrt{2}(\sqrt{3}-1)$$

$$= \sqrt{6} - \sqrt{2}.$$

Short Answer Questions

1). Find the centre, foci, eccentricity, equation of the directrices, length of the lacus rectum of the following hyperbola.

i) $16y^2 - 9x^2 = 144$

Sol.

Equation of the hyperbola is $16y^2 - 9x^2 = 144$

$$\frac{y^2}{9} - \frac{x^2}{16} = 1 \Rightarrow a^2 = 16, b^2 = 9$$

Centre c(0, 0)

Eccentricity $\sqrt{\frac{a^2 + b^2}{b^2}} = \sqrt{\frac{16+9}{9}} = \frac{5}{3}$

Foci are (0, ±be) = (0, ±5)

Equation of the directrices are

$$y = \pm\frac{b}{e} \Rightarrow y = \pm 3 \cdot \frac{3}{5} \Rightarrow 5y = \pm 9$$

Length of the latus rectum =

$$2 \cdot \frac{a^2}{b} = 2 \cdot \frac{16}{3} = \frac{32}{3}$$

ii) $x^2 - 4y^2 = 4$

Sol. Equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{1} = 1$

$a^2 = 4, b^2 = 1$

Centre c(0, 0)

Eccentricity $= \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{4+1}{4}} = \frac{\sqrt{5}}{2}$

Foci are (±ae, 0) = $(\pm\sqrt{5}, 0)$

Equations of directrices are

$$x = \pm\frac{a}{e} = \pm 2 \cdot \frac{2}{\sqrt{5}} \Rightarrow \sqrt{5}x = \pm 4 \Rightarrow \sqrt{5}x \pm 4 = 0$$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \cdot 1}{2} = 1$

iii) $5x^2 - 4y^2 + 20x + 8y = 4$

Sol. Given equation is

$5x^2 - 4y^2 + 20x + 8y = 4$

$5(x^2 + 4x) - 4(y^2 - 2y) = 4$

$5((x + 2)^2 - 4) - 4((y - 1)^2 - 1) = 4$

$5(x + 2)^2 - 4(y - 1)^2 = 20$

$$\frac{(x+2)^2}{4} - \frac{(y-1)^2}{5} = 1$$

$a^2 = 4, b^2 = 5 \Rightarrow a < b$

Centre C(–2, +1)=(h, k).

Eccentricity = $\sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{4+5}{4}} = \frac{3}{2}$

$ae = 3$

Foci are $(h \pm ae, k)$

$= (-2\pm3, 1) = (-5, 1)$ and $(1, 1)$

Equations of directrices are

$$x - h = \pm\frac{a}{e} \Rightarrow x + 2 = \pm2 \cdot \frac{2}{3}$$

$$3x + 6 = \pm4 \Rightarrow 3x + 10 = 0 \text{ or } 3x + 2 = 0$$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \cdot 5}{2} = 5$

iv) $9x^2 - 16y^2 + 72x - 32y - 16 = 0$

Sol. Equation of the hyperbola is:

$9x^2 - 16y^2 + 72x - 32y - 16 = 0$

$\Rightarrow 9(x^2 + 8x) - 16(y^2 + 2y) = 16$

$\Rightarrow 9(x^2 + 8x + 16) - 16(y^2 + 2y + 1)$

$= 16 + 144 - 16$

$\Rightarrow 9(x + 4)^2 - 16(y + 1)^2 = 144$

$$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

Comparing with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$a^2 = 16, b^2 = 9, h = -4, k = -1$

Centre $(h, k) = (-4, -1)$

$$e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

Foci = $(h \pm ae, k) = \left(-4 \pm 4 \cdot \frac{5}{4}, 1\right)$

$$= (-4 \pm 5, -1) = (1, -1) \text{ and } (-9, -1)$$

Equation of the directrices are:

$$x + 4 = \pm 4 \cdot \frac{4}{5} = \pm \frac{16}{5}$$

$5x + 20 = \pm 16$

Equation of the directrices are:

$5x + 4 = 0$ and $5x + 36 = 0$

Length of the latus rectum = $2\frac{b^2}{a} = 2 \cdot \frac{9}{4} = \frac{9}{2}$

2. Find the equation to the hyperbola whose foci are (4, 2) and (8, 2) and eccentricity is 2.

Sol. Foci are $S(4, 2)$ and $S^1(8, 2)$ and eccentricity $e = 2$.

Centre C = the midpoint of the foci.

$$= \left(\frac{4+8}{2}, \frac{2+2}{2}\right) = (6, 2)$$

$SS^1 = 2ae = 8 - 4 = 4$

$\Rightarrow ae = 2$

$e = 2 \Rightarrow a.2 = 2 \Rightarrow a = 1.$

$b^2 = a^2(e^2 - 1) = 1(4 - 1) = 3$

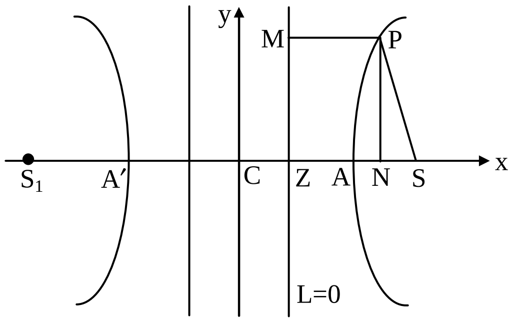
Equation of the hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1$$

3. Find the equation of the hyperbola of given length of transverse axis is 6 whose vertex bisects the distance between the centre and the focus.

Sol.



Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Given CA = AS

$a = ae - a \Rightarrow 2a = ae \Rightarrow e = 2$

Length of transverse axis is $2a = 6 \Rightarrow a = 3$

$b^2 = a^2(e^2 - 1) = 9(4 - 1) = 27$

Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{x^2}{9} - \frac{y^2}{27} = 1 \Rightarrow 3x^2 - y^2 = 27$$

4. Find the equations of the tangents to the hyperbola x^2 – $4y^2$ =4 which are (i) Parallel and (ii) Perpendicular to the line x + 2y = 0.

Sol. Equation of the hyperbola is $x^2 - 4y^2 = 4$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1 \Rightarrow a^2 = 4, b^2 = 1$$

i) Given line is x + 2y = 0

Since tangent is parallel to x + 2y = 0, slope of the tangent is $m = -\frac{1}{2}$

$$c^2 = a^2m^2 - b^2 = 4 \cdot \frac{1}{4} - 1 = 1 - 1 = 0$$

c = 0

Equation of the parallel tangent is:

$$y = mx + c = -\frac{1}{2}x$$

$\Rightarrow 2y = -x \Rightarrow x + 2y = 0$

ii) The tangent is perpendicular to x + 2y = 0

Slope of the tangent $m = \frac{-1}{(-1/2)} = 2$

$c^2 = a^2m^2 - b^2 = 4 \cdot 4 - 1 = 15$

$c = \pm\sqrt{15}$

Equation of the perpendicular tangent is

$y = 2x \pm \sqrt{15}$.

5. Find the equations of tangents drawn to the hyperbola $2x^2 - 3y^2 = 6$ through (–2, 1).

Sol. Equation of the hyperbola is $2x^2 - 3y^2 = 6$

$$\Rightarrow \frac{x^2}{3} - \frac{y^2}{2} = 1$$

Let m be the slope of the tangent.

The tangent id passing through p(–2, 1).

Equation of the tangent is

$y - 1 = m(x + 2) = mx + 2m$

$y = mx + (2m + 1) \quad \dots (1)$

Since (1) is a tangent to the hyperbola,

$c^2 = a^2m^2 - b^2$

$\Rightarrow (2m + 1)^2 = 3m^2 - 2$

$\Rightarrow 4m^2 + 4m + 1 = 3m^2 - 2$

$\Rightarrow m^2 + 4m + 3 = 0 \Rightarrow (m + 1)(m + 3) = 0$

$\Rightarrow m = -1 \text{ or } -3$

Case 1: $m = -1$

Equation of the tangent is

$y = -x - 1 \Rightarrow x + y + 1 = 0$

Case 2: $m = -3$

Equation of the tangent is

$y = -3x - 5 \Rightarrow 3x + y + 5 = 0$

6. Prove that the product of the perpendicular distances from any point on a hyperbola to its asymptotes is constant.

Sol: Let $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be the given hyperbola.

Let P = (a secθ, b tanθ) be any point on S = 0.

The equations of asymptotes of hyperbola

S = 0 are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$

$\Rightarrow bx + ay = 0 \quad \dots(1)$ and

$bx - ay = 0 \quad \dots(2)$

Let PM be the length of the perpendicular drawn from P(a sec θ, b tan θ) on the line (2).

$\therefore PM = \frac{|ba\sec\theta + ab\tan\theta|}{\sqrt{a^2 + b^2}}$

Let PN be the length of the perpendicular drawn from P(a sec θ, b tan θ) on the line (2).

$$\therefore PN = \frac{|ba\sec\theta - ab\tan\theta|}{\sqrt{a^2+b^2}}$$

$$\therefore (PM)\cdot(PN)$$

$$= \frac{|ba\sec\theta + ab\tan\theta|}{\sqrt{a^2+b^2}}\frac{|ba\sec\theta - ab\tan\theta|}{\sqrt{a^2+b^2}}$$

$$= \frac{b^2a^2\sec^2\theta - a^2b^2\tan^2\theta}{a^2+b^2}$$

$$= \frac{a^2b^2(\sec^2\theta - \tan^2\theta)}{a^2+b^2}$$

$$= \frac{a^2b^2}{a^2+b^2}(\because \sec^2\theta - \tan^2\theta = 1)$$

= constant.

∴ The product of the perpendicular distances from any point on a hyperbola to its asymptotes is a constant.

7. If e, e_1 be the eccentricity of a hyperbola and its conjugate hyperbola then $\frac{1}{e^2}+\frac{1}{e_1^2}=1$.

Sol. Equation of the hyperbola is

$$S = \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$e = \sqrt{\frac{a^2+b^2}{a^2}} \Rightarrow e^2 = \frac{a^2+b^2}{a^2}$$

$$\therefore \frac{1}{e^2} = \frac{a^2}{a^2+b^2} \quad \ldots\ldots (1)$$

Equation of the conjugate hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \Rightarrow \frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

$$e_1 = \sqrt{\frac{a^2+b^2}{b^2}} \Rightarrow e_1^2 = \frac{a^2+b^2}{b^2} \Rightarrow \frac{1}{e_1^2} = \frac{b^2}{a^2+b^2} \quad \ldots\ldots (2)$$

Adding (1) and (2)

$$\frac{1}{e^2}+\frac{1}{e_1^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}=1$$

Long Answer Questions

1. Tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ makes angles θ_1, θ_2 with transverse axis of a hyperbola. Show that the point of intersection of these tangents lies on the curve $2xy = k(x^2 - a^2)$ when $\tan\theta_1 + \tan\theta_2 = k$.

Sol. Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Inclinations of the tangents are θ_1, θ_2.

$\Rightarrow$ Slopes of the tangents are

$m_1=\tan\theta_1$ and $m_2=\tan\theta_2$

Equation of the tangent to the hyperbola is

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

Let $p(x_1, y_1)$ be the point of intersection of the tangents, then

$$y_1=mx_1\pm\sqrt{a^2m^2-b^2}$$
$$y_1-mx_1=\pm\sqrt{a^2m^2-b^2}$$

Squiring on both side

$$(y_1-mx_1)^2=a^2m^2-b^2$$
$$y_1^2+m^2x_1^2-2mx_1y_1-a^2m^2+b^2=0$$
$$m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2+b^2)=0$$

Which is a quadratic equation in m. Therefore it has two roots from m , say m_1, m_2

$$m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$$

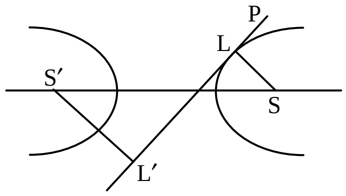
$$\tan\theta_1+\tan\theta_2=\frac{2x_1y_1}{x_1^2-a^2}$$

$$\Rightarrow k = \frac{2x_1y_1}{x_1^2 - a^2} \text{ or } 2x_1y_1 = k(x_1^2 - a^2)$$

Therefore locus of $p(x_1, y_1)$ is $2xy = k(x^2 - a^2)$

2. Show that the feet of the perpendiculars drawn from foci to any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on the auxiliary circle of the hyperbola.

Sol. Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Equation of the tangent to the hyperbola is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y - mx = \pm\sqrt{a^2m^2 - b^2} \quad \ldots (1)$$

Slope of the perpendicular line = -1/m

Equation of the line perpendicular to (1) and passing through foci ($\pm$ae, 0) is

$$y = -\frac{1}{m}(x \pm ae) \Rightarrow my = -(x \pm ae)$$

$$x + my = \pm ae \quad \ldots\ldots (2)$$

Squaring and adding (1) and (2)

$(y^2 - mx)^2 + (x + my)^2 = a^2m^2 - b^2 + a^2e^2$

$\Rightarrow y^2 + m^2x^2 - 2mxy + x^2 + m^2y^2 + 2mxy$

$= a^2m^2 - a^2(e^2 - 1) + a^2e^2$

$\Rightarrow (x^2 + y^2)(1 + m^2) = a^2m^2 - a^2e^2 + a^2 + a^2e^2$

$= a^2(1 + m^2)$

$\Rightarrow x^2 + y^2 = a^2$ which is the auxiliary circle.

3. Prove that the poles of normal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ lie on the curve $\frac{a^6}{x^2}-\frac{b^6}{y^2}=(a^2+b^2)^2$.

Sol. Equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Let $P(x_1, y_1)$ be the pole.

Equation of the polar is $S_1=0$

$$\Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \ldots (1)$$

Equation of the normal to the hyperbola is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2 \ldots.. (2)$$

(1) And (2) are representing the

$$\therefore \frac{\left(\frac{x_1}{a^2}\right)}{\left(\frac{a}{\sec\theta}\right)}=\frac{\left(-\frac{y_1}{b^2}\right)}{\left(\frac{b}{\tan\theta}\right)}=\frac{1}{a^2+b^2}$$

$$\frac{x_1\sec\theta}{a^3}=\frac{y_1\tan\theta}{-b^3}=\frac{1}{a^2+b^2}$$

$$(a^2+b^2)\sec\theta=\frac{a^3}{x_1} \quad \ldots (i)$$

$$(a^2+b^2)\tan\theta=-\frac{b^3}{y_1} \quad \ldots (ii)$$

$(i)^2-(ii)^2 \Rightarrow$

$$(a^2+b^2)^2(\sec^2\theta-\tan^2\theta)=\frac{a^6}{x_1^2}-\frac{b^6}{y_1^2}$$

Locus of $P(x_1, y_1)$ is $\frac{a^6}{x^2}-\frac{b^6}{y^2}=(a^2+b^2)^2$

4. Show that the equation $\frac{x^2}{9-c}+\frac{y^2}{5-c}=1$ represents

i) An ellipse if 'c' is a real constant less than 5.

ii) A hyperbola if 'c' is any real constant between 5 and 9.

iii) Show that each ellipse in (i) and each hyperbola (ii) has foci at the two points (±2, 0), independent of the value of 'c'.

Sol: Given equation $\frac{x^2}{9-c}+\frac{y^2}{5-c}=1$ … (1)

Represents an ellipse if $9 - c > 0$ and $5 - c > 0$

$\Rightarrow 9 > c$ and $5 > c$

$\Rightarrow c < 9$ and $c < 5 \Rightarrow c < 5$

∴ c is a real constant and less than 5 if (1) represents an ellipse.

i) The equation of hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the given equation (1) represents hyperbola if $9 - c > 0$ and $5 - c < 0$.

$\Rightarrow 9 > c$ and $5 < c$

$\Rightarrow 5 < c < 9$

∴ (1) Represents hyperbola if C is a real constant such that $5 < c < 9$.

ii) If $\frac{x^2}{9-c}+\frac{y^2}{5-c}=1$ represents ellipse then $a^2 = 9 - c$ and $b^2 = 5 - c$.

Eccentricity $b^2 = a^2(1 - e^2)$

$\Rightarrow 5-c=(9-c)(1-e^2)$

$\Rightarrow 1-e^2=\frac{5-c}{9-c}$

$\Rightarrow e^2=1-\frac{5-c}{9-c}=\frac{9-c-5+c}{9-c}=\frac{4}{9-c}$

$\Rightarrow e=\frac{2}{\sqrt{9-c}}$

∴ Foci of ellipse = (±ae, 0)

$$=\left(\pm\sqrt{9-c}\left(\frac{2}{\sqrt{9-c}}\right),0\right)$$

$$=(\pm 2,0)$$

If $\frac{x^2}{9-c}+\frac{y^2}{5-c}=1$ represents an hyperbola then $a^2 = 9 - c$ and $b^2 = -(5 - c) = c - 5$ and eccentricity in this case is

$$e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{9-c-5+c}{9-c}}$$

$$=\sqrt{\frac{4}{9-c}}=\frac{2}{\sqrt{9-c}}$$

$\therefore$ Foci of hyperbola = $(\pm ae, 0)$

$$=\left(\pm\sqrt{9-c}\left(\frac{2}{\sqrt{9-c}}\right),0\right)$$

$$=(\pm 2,0)$$

Hence the each ellipse in (i) and each hyperbola in (ii) has foci at the two points $(\pm 2, 0)$ independent of value of C.

5. Show that the angle between the two asymptotes of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\text{Tan}^{-1}\left(\frac{b}{a}\right)$ or 2 $\text{Sec}^{-1}(e)$.

Sol: Let the equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The asymptotes of hyperbola are $y=\pm\frac{b}{a}x$ where $m_1 = \frac{b}{a}$ and $m_2 = -\frac{b}{a}$. If θ is the angle between asymptotes of the hyperbola then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{b}{a}+\frac{b}{a}}{1-\frac{b^2}{a^2}}\right|$$

$$=\left(\frac{2ab}{a^2}\right)\left(\frac{a^2}{a^2-b^2}\right)=\frac{2ab}{a^2-b^2}$$

Now $\sec^2\theta = 1 + \tan^2\theta = 1+\frac{4a^2b^2}{(a^2-b^2)^2}$.

$$= \frac{(a^2-b^2)^2+4a^2b^2}{(a^2-b^2)^2} = \frac{(a^2+b^2)^2}{(a^2-b^2)^2}$$

$$\Rightarrow \sec\theta = \frac{a^2+b^2}{a^2-b^2} \Rightarrow \cos\theta = \frac{a^2-b^2}{a^2+b^2}$$

Also we have $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$$= \sqrt{\frac{1+\frac{a^2-b^2}{a^2+b^2}}{2}} = \sqrt{\frac{a^2}{a^2+b^2}}$$

$$= \sqrt{\frac{a^2}{a^2+a^2(e^2-1)}} = \frac{a}{ae} = \frac{1}{e}$$

$$\Rightarrow \sec\frac{\theta}{2} = e$$

$$\frac{\theta}{2} = \text{Sec}^{-1}(e)$$

$$\Rightarrow e = 2\text{Sec}^{-1}(e)$$

$\therefore$ Angle between asymptotes of hyperbola is $\theta = 2\ \text{Sec}^{-1}(e)$.

Also we have $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$$= \sqrt{\frac{1-\frac{a^2-b^2}{a^2+b^2}}{1+\frac{a^2-b^2}{a^2+b^2}}} = \sqrt{\frac{(a^2+b^2)-(a^2-b^2)}{(a^2+b^2)+(a^2-b^2)}} = \frac{b}{a}$$

$$\frac{\theta}{2} = \text{Tan}^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow \theta = 2\text{Tan}^{-1}\left(\frac{b}{a}\right)$$

Hence angle between asymptotes is $2\text{Tan}^{-1}\left(\frac{b}{a}\right)$ or 2 $\text{Sec}^{-1}(e)$.

6. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are

(i) Parallel and (ii) Perpendicular to the line y = x – 7.

Sol: Equation of given hyperbola is $\frac{x^2}{4} - \frac{y^2}{3} = 1$.

So that $a^2 = 4$, $b^2 = 3$ and equation to the given line y = x – 7 and slope is 1.

i) Slope of the tangents which are parallel to the given line is '1'.

$\therefore$ Equation of tangents are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = x \pm \sqrt{4-3} \text{ and}$$

$$\Rightarrow y = x \pm 1$$

ii) Slope of the tangent which is perpendicular to the given line is –1.

$\therefore$ Equations of tangents which are perpendicular to the given line are

$$y = (-1)x \pm \sqrt{4(-1)^2 - 3}$$

$$\Rightarrow x + y = \pm 1.$$

7. A circle on the rectangular hyperbola xy =1 in the points (x_r, y_r), (r = 1, 2, 3, 4). Prove that $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = 1$.

Sol: Let the circle $x^2 + y^2 = a^2$.

Since $\left(t, \frac{1}{t}\right)(t \neq 0)$ lies on xy = 1, the points of intersection of the circle and the hyperbola are

Given by $t^2 + \frac{1}{t^2} = a^2$.

$$\Rightarrow t^4 - a^2t^2 + 1 = 0$$

$$\Rightarrow t^4 + 0\cdot t^3 - a^2t^2 + 0\cdot t + 1 = 0 \quad \ldots (1)$$

If t_1, t_2, t_3 and t_4 are the roots of above biquadratic then $t_1 t_2 t_3 t_4 = 1$ … (2)

If $(x_r, y_r) = \left(t_r, \frac{1}{t_r}\right)$, (r = 1, 2, 3, 4)

Then $x_1 x_2 x_3 x_4 = t_1 t_2 t_3 t_4 = 1$ from (2).

Similarly $y_1y_2y_3y_4 = \frac{1}{t_1}\frac{1}{t_2}\frac{1}{t_3}\frac{1}{t_4}$

$$= \frac{1}{t_1t_2t_3t_4} = \frac{1}{1} = 1.$$

8. (i) If the line $lx + my + n = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then show that $a^2l^2 - b^2m^2 = n^2$.

(ii) If the $lx + my = 1$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then show that $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \left(a^2 + b^2\right)^2$.

Sol: (i) Let the line $lx + my + n = 0$ ...... (1)

is a tangent to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(\theta)$.

Then the equation of tangent at $P(\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$ ...(2)

Since (1) and (2) represent the same line,

$$\frac{l}{\left(\frac{\sec\theta}{a}\right)} = \frac{m}{-\left(\frac{\tan\theta}{b}\right)} = \frac{n}{-1}$$

$$\therefore \sec\theta = \frac{-al}{n} \text{ and } \tan\theta = \frac{bm}{n}$$

$$\therefore \sec^2\theta - \tan^2\theta = \frac{a^2l^2}{n^2} - \frac{b^2m^2}{n^2}$$

$$\Rightarrow 1 = \frac{a^2l^2}{n^2} - \frac{b^2m^2}{n^2}$$

$$\Rightarrow a^2l^2 - b^2m^2 = n^2$$

ii) Let $lx + my = 1$ .... (1)

Be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(\theta)$.

The equation of normal at $P(\theta)$ to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = (a^2 + b^2) \quad \text{...... (2)}$$

From (1) and (2) eliminating θ, we get

$$\frac{l}{\left(\frac{a}{\sec\theta}\right)} = \frac{m}{\left(\frac{b}{\tan\theta}\right)} = \frac{1}{(a^2+b^2)}$$

$$\Rightarrow \frac{l\sec\theta}{a} = \frac{m\tan\theta}{b} = \frac{1}{a^2+b^2}$$

$$\Rightarrow \sec\theta = \frac{a}{l(a^2+b^2)}, \tan\theta = \frac{b}{m(a^2+b^2)}$$

$$\therefore \sec^2\theta - \tan^2\theta = \frac{a^2}{l^2(a^2+b^2)^2} - \frac{b^2}{m^2(a^2+b^2)^2}$$

$$\Rightarrow 1 = \frac{a^2}{l^2(a^2+b^2)^2} - \frac{b^2}{m^2(a^2+b^2)^2}$$

$$\Rightarrow \frac{a^2}{l^2} - \frac{b^2}{m^2} = \left(a^2+b^2\right)^2$$

9. Prove that the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lies on the circle $x^2+y^2 = a^2-b^2$.

Sol: Let $P(x_1, y_1)$ be a point of intersection of two perpendicular tangents to the hyperbola

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

The equation of any tangent to S = 0 is of the form $y = mx \pm \sqrt{a^2m^2 - b^2}$. If this passes through (x_1, y_1) then

$$y_1 - mx_1 = \pm\sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2m^2 - b^2$$

$$\Rightarrow (x_1^2 - a^2)m^2 - 2mx_1y_1 + (y_1^2 + b^2) = 0$$

This is a quadratic equation in m which has two roots m_1, m_2 (say) which corresponds to slopes of tangents.

Then $m_1m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$ and

The two tangents are perpendicular

$\Rightarrow m_1 m_2 = -1$

$\therefore -1 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$

$\Rightarrow x_1^2 - a^2 = -(y_1^2 + b^2)$

$\Rightarrow x_1^2 - y_1^2 = a^2 - b^2$

$\therefore$ Locus of (x_1, y_1) is $x^2 + y^2 = a^2 - b^2$.

10. If four points be taken on a rectangular hyperbola such that the chords joining any two points is perpendicular to the chord joining the other two, and if α, β, γ and δ be the inclinations to either asymptote of the straight lines joining these points to the center, prove that $\tan\alpha \tan\beta \tan\gamma \tan\delta = 1$.

Sol: Let the equation of rectangular hyperbola be $x^2 - y^2 = a^2$.

By rotating the X-axis and Y-axis about the origin through an angle π/4 in the clockwise direction the equation $x^2 - y^2 = a^2$ will be transformed to $xy = c^2$.

Let $\left(ct_r, \frac{c}{t_r}\right)$, r = 1, 2, 3, 4 ($t_1 \neq 0$) be four points on the curve. Let the chord joining

$A = \left(ct_1, \frac{c}{t_1}\right), B = \left(ct_2, \frac{c}{t_2}\right)$ be perpendicular to the chord joining $C = \left(ct_3, \frac{c}{t_3}\right)$ and $D = \left(ct_4, \frac{c}{t_4}\right)$.

Then slope of $\overleftrightarrow{AB} = \frac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2} = -\frac{1}{t_1 t_2}$

Similarly slope of $\overleftrightarrow{CD} = -\frac{1}{t_3 t_4}$

Since $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$ we have

$\left(-\frac{1}{t_1 t_2}\right)\left(-\frac{1}{t_3 t_4}\right) = -1$

$\Rightarrow t_1 t_2 t_3 t_4 = -1 \quad \ldots (1)$

We have the coordinate axis as the asymptotes of the curves.

If $\overleftrightarrow{OA}, \overleftrightarrow{OB}, \overleftrightarrow{OC}$ and $\overleftrightarrow{OD}$ makes angles α,β,γ, δ with positive direction of X-axis then $\tan\alpha \tan\beta \tan\gamma$ and $\tan\delta$ are the slopes.

Then $\tan\alpha = \dfrac{\dfrac{c}{t_1} - 0}{ct_1 - 0} = \dfrac{1}{t_1^2}$

Similarly

$$\tan\beta = \frac{1}{t_2^2}, \tan\gamma = \frac{1}{t_3^2} \text{ and } \tan\delta = \frac{1}{t_4^2}$$

$$\therefore \tan\alpha \tan\beta \tan\gamma \tan\delta = \frac{1}{t_1^2 t_2^2 t_3^2 t_4^2} = 1$$

(From (1))

If $\overleftrightarrow{OA}, \overleftrightarrow{OB}, \overleftrightarrow{OC}$ and $\overleftrightarrow{OD}$ make angles α, β, γ and δ with the other asymptote the Y-axis then $\cot\alpha, \cot\beta, \cot\gamma$ and $\cot\delta$ are the respectively slopes.

So that $\cot\alpha, \cot\beta, \cot\gamma \cot\delta = \tan\alpha \tan\beta \tan\gamma \tan\delta = 1.$

11. Prove that the product of the perpendicular distance from any point on a hyperbola to its asymptotes is constant.

Sol. Equation of the hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

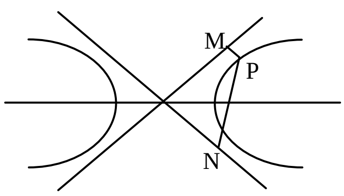
Any point on the hyperbola is P(a secθ, b tanθ)

Equation of the asymptotes are $\dfrac{x}{a} = \pm\dfrac{y}{b}$

i.e. $\dfrac{x}{a} - \dfrac{y}{b} = 0$ and $\dfrac{x}{a} + \dfrac{y}{b} = 0$

PM = Perpendicular distance from P on

$$\frac{x}{a} - \frac{y}{b} = 0 = \frac{|\sec\theta - \tan\theta|}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2}}}$$



PN = Perpendicular distance from P on

$$\frac{x}{a}+\frac{y}{b}=0=\frac{|\sec\theta+\tan\theta|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$PM\cdot PN=\frac{|\sec\theta-\tan\theta|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\frac{|\sec\theta+\tan\theta|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$=\frac{|\sec^2\theta-\tan^2\theta|}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}=\frac{1}{\frac{a^2+b^2}{a^2b^2}}$$

$$\frac{a^2b^2}{a^2+b^2} = \text{constant}$$

12. Find the centre eccentricity, foci directrices and length of the latus rectum of the following hyperbola (i) $4x^2 - 9y^2 - 8x - 32 = 0$,

(ii) $4(y + 3)^2 - 9(x - 2)^2 = 1$.

Sol.i) $4x^2 - 9y^2 - 8x - 32 = 0$

$4(x^2 - 2x) - 9y^2 = 32$

$4(x^2 - 2x + 1) - 9y^2 = 36$

$$\frac{(x-1)^2}{9}-\frac{y^2}{4}=1$$

Centre of the hyperbola is (1, 0)

$a^2 = 9, b^2 = 4 \Rightarrow a = 3, b = 2$

$$e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{9+4}{9}}=\frac{\sqrt{13}}{3}$$

Foci are $\left(1\pm 3\cdot\frac{\sqrt{13}}{3},0\right)=(1\pm\sqrt{13},0)$

Equations of directrices are:

$$x=1\pm\frac{3\cdot 3}{\sqrt{13}}\Rightarrow x=1\pm\frac{9}{\sqrt{13}}$$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \cdot 4}{3} = \frac{8}{3}$

ii) The equation of the hyperbola is:

$4(y + 3)^2 – 9(x – 2)^2 = 1$

$$\frac{y-(-3)^2}{1/4} - \frac{(x-2)^2}{1/9} = 1$$

Centre =(2, –3) =(h, k)

Semi transverse axis = b = 1/2

Semi conjugate axis = a = 1/3

$$e = \sqrt{\frac{a^2+b^2}{b^2}} = \sqrt{\frac{(1/9)+(1/4)}{(1/4)}} = \sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{3}$$

Foci are (h, k±be) =

$$\left(2, -3 \pm \frac{1}{2} \cdot \frac{\sqrt{13}}{3}\right) = \left(2, -3 \pm \frac{\sqrt{13}}{6}\right)$$

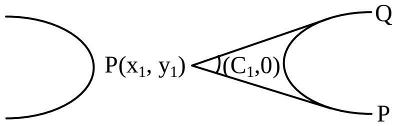
Equations of the directrices are:

$$y = k \pm \frac{b}{e} = -3 \pm \frac{1}{2} \cdot \frac{3}{\sqrt{3}}$$

$$y = -3 \pm \frac{3}{2\sqrt{13}}$$

Length of latus rectum= $\frac{2a^2}{b} = \frac{2 \cdot \frac{1}{9}}{1/2} = \frac{4}{9}$.

13. Prove that the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ lies on the circle $x^2+a^2 = a^2 - b^2$.



Sol. Equation of the hyperbola is

$$S=\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Let $P(x_1, y_1)$ be the point of intersection of two perpendicular tangents to the hyperbola.

Equation of the tangent is

$$y = mx \pm \sqrt{a^2m^2-b^2}$$

This tangent passes through $P(x_1, y_1)$

$$\Rightarrow y_1 = mx_1 \pm \sqrt{a^2m^2-b^2}$$

$$\Rightarrow (y_1-mx_1)^2 = a^2m^2-b^2$$

$$\Rightarrow y_1^2+m^2x_1^2-2mx_1y_1 = a^2m^2-b^2$$

$$\Rightarrow m^2x_1^2-a^2m^2-2mx_1y_1+y_1^2+b^2=0$$

$$\Rightarrow m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2+b^2)=0$$

Which is a quadratic in m. Therefore it has two roots from m, say m_1, m_2 which are the slopes of the tangents passing through P.

The tangents are perpendicular $\Rightarrow m_1m_2 = -1$

$$\frac{y_1^2+b^2}{x_1^2-a^2}=-1 \Rightarrow y_1^2+b^2=-x_1^2+a^2$$

$$x_1^2+y_1^2=a^2-b^2$$

Locus of $P(x_1, y_1)$ is $x^2 + y^2 = a^2 - b^2$

This circle is called director circle of the hyperbola.