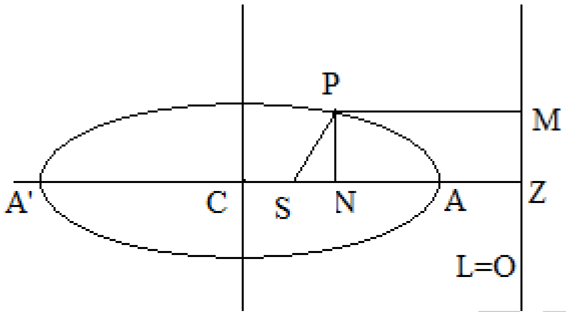
ELLIPSE

A conic is said to be an ellipse if it's eccentricity e is less than 1.

Equation of an Ellipse in Standard Form:

The equation of an ellipse in the standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.(a<b)$

Proof:



Let S be the focus, e be the eccentricity and L = 0 be the directrix of the ellipse. Let P be a point on the ellipse. Let M, Z be the projections (foot of the perpendiculars) of P, S on the directrix L = 0 respectively. Let N be the projection of P on SZ. Since e < 1, we can divide SZ both internally and externally in the ratio e: 1. Let A, A′ be the points of division of SZ in the ratio e: 1 internally and externally respectively. Let AA′ = 2a. Let C be the midpoint of AA′. The points A, A′ lie on the ellipse and $\frac{SA}{AZ}=e, \frac{SA'}{A'Z}=e$.

$\therefore$ SA = eAZ, SA′ = eA′Z

Now SA + SA′ = eAZ + eA′Z

$\Rightarrow$ AA′ = e(AZ + A′Z)

$\Rightarrow 2a = e(CZ - CA + A'C + CZ)$

$\Rightarrow 2a = e \cdot 2CZ \ (\because CA = A'C)$

$\Rightarrow CZ = a/e$

Also $SA' - SA = eA'Z - eAZ$

$\Rightarrow A'C + CS - (CA - CS) = e(A'Z - AZ)$

$\Rightarrow 2CS = eAA'$ $(\because CA = A'C)$

$\Rightarrow 2CS = e2a \Rightarrow CS = ae$

Take CS, the principal axis of the ellipse as x-axis and Cy perpendicular to CS as y-axis. Then S(ae,0) and the ellipse is in the standard form. Let $P(x_1,y_1)$.

Now PM = NZ = CZ – CN = $\frac{a}{e} - x_1$

P lies on the ellipse:

$$\Rightarrow \frac{PS}{PM} = e \Rightarrow PS = ePM \Rightarrow PS^2 = e^2PM^2$$

$$\Rightarrow (x_1 - ae)^2 + (y_1 - 0)^2 = e^2\left(\frac{a}{e} - x_1\right)^2$$

$$\Rightarrow (x_1 - ae)^2 + y_1^2 = (a - x_1e)^2$$

$$\Rightarrow x_1^2 + a^2e^2 - 2x_1ae + y_1^2 = a^2 + x_1^2e^2 - 2x_1ae$$

$$\Rightarrow (1 - e^2)x_1^2 + y_1^2 = (1 - e^2)a^2$$

$$\Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2(1 - e^2)} = 1 \Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

Where $b^2 = a^2(1 - e^2) > 0$

The locus of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$\therefore$ The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Nature of the Curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let C be the curve represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then

i) The curve is symmetric about the coordinate axes.

ii) The curve is symmetric about the origin O and hence O is the midpoint of every chord of the ellipse through O. Therefore the origin is the centre of the ellipse.

iii) Put y = 0 in the equation of the ellipse $\Rightarrow x^2 = a^2 \Rightarrow x = \pm a$.

Thus the curve meets x-axis (Principal axis) at two points A(a, 0), A′(–a, 0). Hence the ellipse has two vertices. The axis AA′ is called major axis. The length of the major axis is AA′ = 2a

iv) Put $x = 0 \Rightarrow y^2 = b^2 \Rightarrow y = \pm b$. Thus, the curve meets y-axis (another axis) at two points B(0, b), B′(0, –b). The axis BB′ is called minor axis and the length of the minor axis is BB′ = 2b.

V) The focus of the ellipse is S(ae, 0). The image of S with respect to the minor axis is S′(–ae,0). The point S′ is called second focus of the ellipse.

Vi) The directrix of the ellipse is x = a/e. The image of x = a/e with respect to the minor axis is x = –a/e. The line x = –a/e is called second directrix of the ellipse.

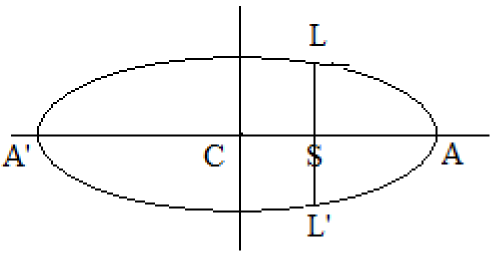
Vii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Thus y has real values only when $-a \le x \le a$. Similarly x has real values only when $-b \le y \le b$. Thus the curve lies completely with in the rectangle $x = \pm a$, $y = \pm b$. Therefore the ellipse is a closed curve.

Theorem: The length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b > 0)$ is $\frac{2b^2}{a}$.

The length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0 < a < b)$ is $\frac{2a^2}{b}$.



Proof:

Let LL' be the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Focus S =(ae, 0)

If SL = l, then L = (ae, l)

L lies on the ellipse $\Rightarrow \frac{(ae)^2}{a^2}+\frac{l^2}{b^2}=1$

$$\Rightarrow e^2+\frac{l^2}{b^2}=1 \Rightarrow \frac{l^2}{b^2}=1-e^2=\frac{b^2}{a^2} \Rightarrow l^2=\frac{b^4}{a^2}$$

$$\Rightarrow l=\frac{b^2}{a} \Rightarrow SL=\frac{b^2}{a} \quad \therefore LL'=2SL=\frac{2b^2}{a}$$

Note: The coordinates of the four ends of the latus recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b > 0)$ are $L=\left(ae,\frac{b^2}{a}\right), L'=\left(ae,-\frac{b^2}{a}\right), L_1=\left(-ae,\frac{b^2}{a}\right), L_1'=\left(-ae,-\frac{b^2}{a}\right)$.

Note: The coordinates of the four ends of the latus recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0 < a < b)$ are $L=\left(\frac{a^2}{b},be\right), L'=\left(-\frac{a^2}{b},be\right), L_1=\left(\frac{a^2}{b},-be\right), L_1'=\left(-\frac{a^2}{b},-be\right)$.

Theorem: If P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S and S′ then

PS + PS′ = 2a.

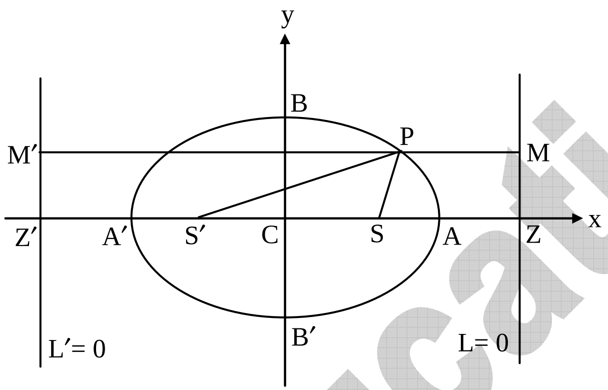
Proof: Let e be the eccentricity and L = 0, L′ = 0 be the directrices of the ellipse.

Let C be the centre and A, A′ be the vertices of the ellipse.

$\therefore$ AA′ = 2a.

Foci of the ellipse are S(ae, 0), S′(–ae, 0).

Let $P(x_1, y_1)$ be a point on the ellipse.



Let M, M′ be the projections of P on the directrices L = 0, L′ = 0 respectively.

$\therefore \frac{SP}{PM}=e, \frac{S'P}{PM'}=e$.

Let Z, Z′ be the points of intersection of major axis with directrices.

$\therefore$ MM′ = ZZ′ = CZ + CZ′ = 2a/e.

PS + PS′ = ePM + ePM′

= e(PM + PM′) = e(MM′) = e(2a/e) = 2a.

Theorem: Let $P(x_1, y_1)$ be a point and $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ be an ellipse. Then

(i) P lies on the ellipse $\Leftrightarrow S_{11} = 0$,

(ii) P lies inside the ellipse $\Leftrightarrow S_{11} < 0$,

(iii) P lies outside the ellipse $\Leftrightarrow S_{11} > 0$.

Theorem: The equation of the tangent to the ellipse S = 0 at $P(x_1, y_1)$ is $S_1 = 0$.

Theorem: The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1, y_1)$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

Proof:

The equation of the tangent to S = 0 at P is $S_1 = 0$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

The equation of the normal to S = 0 at P is

$$\frac{y_1}{b^2}(x-x_1)-\frac{x_1}{a^2}(y-y_1)=0$$

$$\Rightarrow \frac{xy_1}{b^2}-\frac{yx_1}{a^2}=\frac{x_1y_1}{b^2}-\frac{x_1y_1}{a^2}$$

$$\Rightarrow \frac{a^2b^2}{x_1y_1}\left(\frac{xy_1}{b^2}-\frac{yx_1}{a^2}\right)=\frac{a^2b^2}{x_1y_1}\left(\frac{x_1y_1}{b^2}-\frac{x_1y_1}{a^2}\right)$$

$$\Rightarrow \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

Theorem: The condition that the line y = mx + c may be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$.

Proof:

Suppose y = mx + c … (1) is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $P(x_1, y_1)$ be the point of contact.

The equation of the tangent at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0 \quad \ldots (2)$$

Now (1) and (2) represent the same line.

$$\therefore \frac{x_1}{a^2m}=\frac{y_1}{b^2(-1)}=\frac{-1}{c}\Rightarrow x_1=\frac{-a^2m}{c}, y_1=\frac{b^2}{c}.$$

P lies on the line y = mx + c $\Rightarrow y_1 = mx_1 + c$

$$\Rightarrow \frac{b^2}{c}=m\left(\frac{-a^2m}{c}\right)+c \Rightarrow b^2=-a^2m^2+c^2$$

$$\Rightarrow c^2=a^2m^2+b^2.$$

Note: The equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be taken as $y=mx\pm\sqrt{a^2m^2+b^2}$. The point of contact is $\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$ where $c^2 = a^2m^2 + b^2$.

Theorem: Two tangents can be drawn to an ellipse from an external point.

Director Circle:

The points of intersection of perpendicular tangents to an ellipse S = 0 lies on a circle, concentric with the ellipse.

Proof:

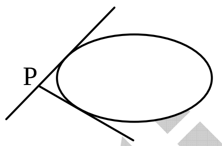
Equation of the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Let P(x_1, y_1) be the point of intersection of perpendicular tangents drawn to the ellipse.

Let $y = mx \pm \sqrt{a^2m^2 + b^2}$ be a tangent to the ellipse S = 0 passing through P.

Then $y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2}$



$$\Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 + b^2}$$

$$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$\Rightarrow y_1^2 + m^2x_1^2 - 2x_1y_1m = a^2m^2 + b^2$$

$$\Rightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - b^2) = 0 \quad \ldots (1)$$

If m_1, m_2 are the slopes of the tangents through P then m_1, m_2 are the roots of (1).

The tangents through P are perpendicular.

$$\Rightarrow m_1m_2 = -1 \Rightarrow \frac{y_1^2 - b^2}{x_1^2 - a^2} = -1$$

$$\Rightarrow y_1^2 - b^2 = -x_1^2 + a^2 \Rightarrow x_1^2 + y_1^2 = a^2 + b^2$$

$\therefore$ P lies on $x^2 + y^2 = a^2 + b^2$ which is a circle with centre as origin, the centre of the ellipse.

Auxiliary Circle:

Theorem: The feet of the perpendiculars drawn from either of the foci to any tangent to the ellipse S = 0 lies on a circle, concentric with the ellipse.(called auxiliary circle)

Proof:

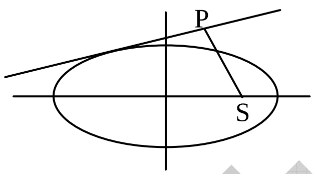
Equation of the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Let $P(x_1, y_1)$ be the foot of the perpendicular drawn from either of the foci to a tangent.

The equation of the tangent to the ellipse S = 0 is $y = mx \pm \sqrt{a^2m^2 + b^2}$ … (1)

The equation to the perpendicular from either foci $(\pm ae, 0)$ on this tangent is $y = -\frac{1}{m}(x \pm ae)$ … (2)

Now P is the point of intersection of (1) and (2).



$$\therefore\ y = mx \pm \sqrt{a^2m^2 + b^2},\ y_1 = -\frac{1}{m}(x_1 \pm ae)$$

$$\Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 + b^2},\ my_1 + x_1 = \pm ae$$

$$\Rightarrow (y_1 - mx_1)^2 + (my_1 + x_1)^2 = a^2m^2 + b^2 + a^2e^2$$

$$\Rightarrow y_1^2 + m^2x_1^2 - 2x_1y_1m + m^2y_1^2 + x_1^2 + 2x_1y_1m = a^2m^2 + a^2(1 - e^2) + a^2e^2$$

$$\Rightarrow x_1^2(m^2 + 1) + y_1^2(1 + m^2) = a^2m^2 + a^2$$

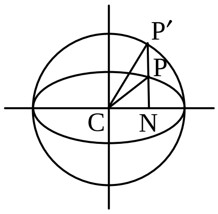
$$\Rightarrow (x_1^2 + y_1^2)(m^2 + 1) = a^2(m^2 + 1) \Rightarrow x_1^2 + y_1^2 = a^2$$

$\therefore$ P lies on $x^2 + y^2 = a^2$ which is a circle with centre as origin, the centre of the ellipse.

Theorem: The equation to the chord of contact of $P(x_1, y_1)$ with respect to the ellipse $S = 0$ is $S_1 = 0$.

Eccentric Angle Definition:

Let P(x, y) be a point on the ellipse with centre C. Let N be the foot of the perpendicular of P on the major axis. Let NP meets the auxiliary circle at P′. Then ∠NCP′ is called eccentric angle of P. The point P′ is called the corresponding point of P.



Parametric Equations: If P(x, y) is a point on the ellipse then $x = a\cos\theta$, $y = b\sin\theta$ where θ is the eccentric angle of P. These equations $x = a\cos\theta$, $y = b\sin\theta$ are called **parametric equations** of the ellipse. The point $P(a\cos\theta, b\sin\theta)$ is simply denoted by θ.

Theorem: The equation of the chord joining the points with eccentric angles α and β on the ellipse S = 0 is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

Proof:

Given points on the ellipse are $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$.

Slope of $\overleftrightarrow{PQ}$ is $\frac{b\sin\alpha-b\sin\beta}{a\cos\alpha-a\cos\beta}=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$

Equation of $\overleftrightarrow{PQ}$ is:

$$y-\sin\alpha=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}(x-a\cos\alpha)$$

$$\Rightarrow\frac{(x-a\cos\alpha)}{a}(\sin\alpha-\sin\beta)=\frac{y-b\sin\alpha}{b}(\cos\alpha-\cos\beta)$$

$$\Rightarrow\left(\frac{x}{a}-\cos\alpha\right)2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=\left(\frac{y}{b}-\sin\alpha\right)(-2)\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \left(\frac{x}{a}-\cos\alpha\right)\cos\frac{\alpha+\beta}{2} = -\left(\frac{y}{b}-\sin\alpha\right)\sin\frac{\alpha+\beta}{2}$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}$$

$$= \cos\alpha\cos\frac{\alpha+\beta}{2}+\sin\alpha\sin\frac{\alpha+\beta}{2}$$

$$= \cos\left(\alpha-\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Theorem: The equation of the tangent at P(θ) on the ellipse

S = 0 is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

Theorem: The Equation Of The Normal At P(θ) On The Ellipse

S = 0 Is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$.

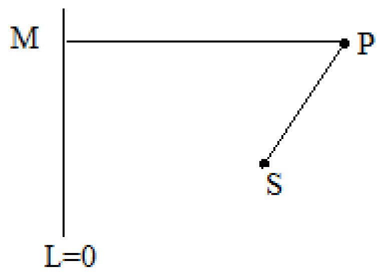
Theorem: Four normals can be drawn from any point to the ellipse and the sum of the eccentric angles of their feet is an odd multiple of π.

Very Short Answer Questions

1. Find the equation of the ellipse with focus at (1, –1) e = 2/3 and directrix is x + y + 2 = 0.

Sol. Let $P(x_1, y_1)$ be any point on the ellipse. Equation of the directrix is

$L = x + y + 2 = 0$



By definition of ellipse SP = e PM

$$SP^2 = e^2 \cdot PM^2$$

$$(x_1-1)^2 + (y_1+1)^2 = \left(\frac{2}{3}\right)^2 \left[\frac{x_1+y_1+2}{\sqrt{1+1}}\right]^2$$

$$(x_1-1)^2 + (y_1+1)^2 = \frac{4}{9}\frac{(x_1+y_1+2)^2}{2}$$

$$9\left[(x_1-1)^2 + (y_1+1)^2\right] = 2(x_1+y_1+2)^2$$

$$9\left[x_1^2 - 2x_1 + 1 + y_1^2 + 2y_1 + 1\right] = 2\left[x_1^2 + y_1^2 + 4 + 2x_1y_1 + 4x_1 + 4y_1\right]$$

$$9x_1^2 + 9y_1^2 - 18x_1 + 18y_1 + 18 = 2x_1^2 + 2y_1^2 + 4x_1y_1 + 8x_1 + 8y_1 + 8$$

$$7x_1^2 - 4x_1y_1 + 7y_1^2 - 26x_1 + 10y_1 + 10 = 0$$

Locus of $P(x_1, y_1)$ is $7x^2 - 4xy + 7y^2 - 26x + 10y + 10 = 0$

2. **Find the equation of the ellipse in the standard form whose distance between foci is 2 and length of latus rectum is 15/2.**

Sol. Latus rectum = 15/2

$$\frac{2b^2}{a} = \frac{15}{2}$$

Distance between foci is 2ae = 2

$\Rightarrow$ ae = 1

$b^2 = a^2 - a^2e^2$

$\Rightarrow b^2 = a^2 - 1$

$$\Rightarrow \frac{15}{4}a = a^2 - 1 \Rightarrow 4a^2 - 15a - 4 = 0$$

$$a = 4 \text{ or } a = -\frac{1}{4}$$

Equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{15} = 1$.

3. **Find the equation of the ellipse in the standard form such that the distance between the foci is 8 and the distance between directrices is 32.**

Sol. Distance between foci is $2ae = 8 \Rightarrow ae = 4$

Distance between directrices = 32

$$\frac{2a}{e} = 32 \Rightarrow \frac{a}{e} = 16$$

$$(ae)\left(\frac{a}{e}\right) = 64$$

$a^2 = 64$

$b^2 = a^2 - a^2e^2 = 64 - 16 = 48$

Equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{48} = 1$.

4. Find the eccentricity of the ellipse, in standard form, if its length of the latus rectum is equal to half of its major axis.

Sol.

Given, latus rectum is equal to half of its major axis

$$\Rightarrow \frac{2b^2}{a} = a$$

$$2b^2 = a^2$$

But $b^2 = a^2 (1 - e^2)$

$$2a^2 (1 - e^2) = a^2$$

$$1 - e^2 = \frac{1}{2} \Rightarrow e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$$

5. The distance of a point on the ellipse $x^2 + 3y^2 = 6$ from its centre is equal to 2. Find the eccentric angles.

Sol. Equation of the ellipse is $x^2 + 3y^2 = 6$

$$\Rightarrow \frac{x^2}{6} + \frac{y^2}{2} = 1$$

$$a = \sqrt{6}, b = \sqrt{2}$$

Any point on the ellipse is

$$P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$$

Given CP = 2 $\Rightarrow CP^2 = 4$

$6\cos^2\theta + 2\sin^2\theta = 4$

$6(1 - \sin^2\theta) + 2\sin^2\theta = 4$

$6 - 6\sin^2\theta + 2\sin^2\theta = 4$

$4\sin^2\theta = 2 \Rightarrow \sin^2\theta = \frac{2}{4} = \frac{1}{2}$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{5\pi}{4}, \frac{7\pi}{4}$$

Eccentric angles are: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

6. Find the equation of the ellipse in the standard form, if it passes through the points (–2, 2) and (3, –1).

Sol. Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It is passing through (–2, 2), (3, –1)

$(-2, 2) \Rightarrow \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \quad \ldots$ (i)

$(3, -1) \Rightarrow \quad \frac{9}{a^2} + \frac{1}{b^2} = 1 \quad \ldots$ (ii)

Solving (i) and (ii), we get

$$\frac{1}{a^2} = \frac{3}{32}, \; \frac{1}{b^2} = \frac{5}{32}$$

$$\frac{3x^2}{32} + \frac{5y^2}{32} = 1$$

$$3x^2 + 5y^2 = 32$$

7. If the ends of major axis of an ellipse are (5, 0) and (–5, 0). Find the equation of the ellipse in the standard form if its focus lies on the line 3x – 5y – 9 = 0.

Sol. Vertices $(\pm a, 0) = (\pm 5, 0) \Rightarrow a = 5$,

Focus S = (ae,o)

Focus lies on the line 3x – 5y – 9 = 0

3(ae) – 5(0) – 9 = 0

$$5e = \frac{9}{3} \Rightarrow e = \frac{3}{5}$$

$b^2 = a^2(1 - e^2)$

$$\Rightarrow b^2 = 25\left(1 - \frac{9}{25}\right) = 25\left(\frac{16}{25}\right) = 16$$

Equation of the ellipse is

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \Rightarrow 16x^2+25y^2=400$$

8. Find the equation of the tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at (–1, 2).

Sol. Given ellipse $S = x^2 + 8y^2 = 33$

Equation of the tangent is $S_1 = 0$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

$$x(-1)+8y(2)=33$$
$$\Rightarrow -x+16y=33$$
$$\Rightarrow x-16y+33=0$$

Equation of the normal is

$16x + y + k = 0$

It passes through P(–1, 2)

$-16 + 2 + k = 0 \Rightarrow k = 14$

Equation of the normal is

$16x + y + 14 = 0$.

9. Find the equation of the tangent and normal to the ellipse $x^2+2y^2-4x+12y+14=0$ at (2, –1).

Sol. Given ellipse

$S = x^2+2y^2-4x+12y+14 = 0$

Equation of the tangent is $S_1 = 0$

$xx_1 + 2yy_1 - 2(x + x_1) + 6(y + y_1) + 14 = 0$

$\Rightarrow 2x - 2y - 2(x + 2) + 6(y - 1) + 14 = 0$

$\Rightarrow 4y + 4 = 0$

$y = -1$ required equation of tangent.

Slope of tangent is 0

Equation of normal be $y+1=\frac{-1}{0}(x-2)$

$x = 2$ equation of normal.

10. Find the equation of the tangents to $9x^2 + 16y^2 = 144$ which makes equal intercepts on coordinate axes.

Sol. Equation of the ellipse is

$9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$

Equation of the tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Slope of the tangent = $-\frac{b\cos\theta}{a\sin\theta}=-1$

$$\cot\theta=\frac{a}{b}=\frac{4}{3}$$

$$\cos\theta=\pm\frac{4}{5},\sin\theta=\pm\frac{3}{5}$$

Equation of the tangent is:

$$\frac{x}{4}\left(\pm\frac{4}{5}\right)+\frac{y}{3}\left(\pm\frac{3}{5}\right)=1$$

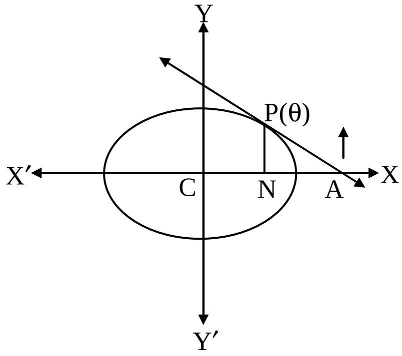
$$x\pm y\pm 5=0$$

11. If PN is the ordinate of a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the tangent at P meets the X-axis at T then show that (CN)(CT) = a^2 where C is the centre of the ellipse.

Sol: Let P(θ) = (a cos θ, b sin θ) be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of the tangent at P(θ) is

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ **(or)**

$\frac{x}{\left(\frac{a}{\cos\theta}\right)}+\frac{y}{\left(\frac{b}{\sin\theta}\right)}=1$ meets X-axis at T.



$\therefore$ X - Intercept (CT) = $\dfrac{a}{\cos\theta}$ and the ordinate of P is PN = b sin θ then its abscissa

CN = a cos θ

$\therefore (CN).(CT) = (a\cos\theta)\dfrac{a}{\cos\theta} = a^2$.

12. Find the value of k if 4x + y + k = 0 is a tangent to the ellipse $x^2 + 3y^2 = 3$.

Sol. Equation of the ellipse is $x^2 + 3y^2 = 3$

$$\frac{x^2}{3} + \frac{y^2}{1} = 1$$

$a^2 = 3,\ b^2 = 1$

Equation of the line is 4x + y + k = 0

$\Rightarrow y = -4x - k$

$\Rightarrow m = -4,\ c = -k$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2 m^2 + b^2$

$(-k)^2 = 3(-4)^2 + 1$

$k^2 = 48 + 1 = 49$

$k = \pm 7$.

13. Find the condition for the line xcosα + ysinα = p to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**.**

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ … (i)

Equation of the line is $x\cos\alpha + y\sin\alpha = p$

$y\sin\alpha = -x\cos\alpha + p$

$$y = -x\frac{\cos\alpha}{\sin\alpha}+\frac{p}{\sin\alpha}$$

$$\therefore \quad m = -\frac{\cos\alpha}{\sin\alpha}, c = \frac{p}{\sin\alpha}$$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2m^2 + b^2$

$$\frac{p^2}{\sin^2\alpha} = a^2\frac{\cos^2\alpha}{\sin^2\alpha}+b^2$$

Or $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.

14. If the length of the major axis of an ellipse is three times the length of its minor axis then find the eccentricity of the ellipse.

Sol: Let the ellipse in the standard form be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots (1)$$

Length of major axis is 'a' and length of minor axis is 'b'. Given that a = 3b

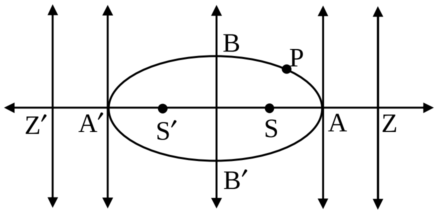
$$\Rightarrow a^2 = 9b^2 \Rightarrow a^2 = 9a^2(1-e^2)$$

$$\Rightarrow 1-e^2 = \frac{1}{9} \Rightarrow e^2 = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

$\therefore$ Eccentricity of the ellipse $= \frac{2\sqrt{2}}{3}$.

15. A man running on a race course notices that the sum of the distances of the two flag posts from him is always 10 m. and the distance between the flag posts is 8 m. Find the equation of the race course traced by the man.

Sol:



Given $AA' = 2a = 10 \Rightarrow a = 5$

(Taking flag posts located at A and A′)

Also given the distance between two fixed points S and $S' = 8$ m.

$\therefore 2ae = 8 \Rightarrow ae = 4$

$\therefore b^2 = a^2(1 - e^2)$

$\Rightarrow a^2 - a^2e^2 = 25 - 16 = 9$

$\therefore b^2 = 9$

Hence the equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

16. The orbit of the Earth is an ellipse with eccentricity 1/60 with the Sun at one of its foci, the major axis being approximately 186×10^6 miles in length. Find the shortest and longest distance of the Earth from the Sun.

Sol: Let the earth's orbit be an ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; (a > b)$

Since the major axis is 186×10^6 miles

We have $2a = 186 \times 10^6$

$\Rightarrow a = 93 \times 10^6$ miles

If e is the eccentricity of ellipse then $e = \frac{1}{60}$.

The longest and shortest distances of the Earth from the Sun are respectively

$a + ae$ and $a - ae$.

Here, the longest distance of earth from the sun = $a + ae = 93\times10^6 \times\left(1+\dfrac{1}{60}\right)$

$= 9445\times10^4$ miles

And the shortest distance of earth from the sun = a – ae $= 93\times10^6\left(1-\dfrac{1}{60}\right)$

= 9145×10^4 miles.

17. Find the equation of the ellipse referred to its major and minor axes as the coordinate axes X, Y – respectively with latus rectum of length 4 and distance between foci is $4\sqrt{2}$.

Sol: Let the equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a > b)$$

Length of the latus rectum

$$\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$$

Distance between foci, S = (ae, 0) and S′ = (–ae, 0) is $2ae = 4\sqrt{2} \Rightarrow ae = 2\sqrt{2}$

Also $b^2 = a^2(1-e^2) \Rightarrow 2a = a^2 - (ae)^2 = a^2 - 8$

$\Rightarrow a^2 - 2a - 8 = 0$

$\Rightarrow (a-4)(a+2) = 0$

$\Rightarrow a = 4 (\because a > 0)$

$\therefore b^2 = 2a = 8$

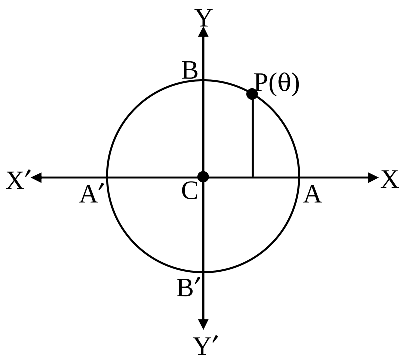
$\therefore$ Equation of ellipse is

$$\frac{x^2}{16}+\frac{y^2}{8}=1 \text{ (or) } x^2 + 2y^2 = 16.$$

18. C is the centre, AA′ and BB′ are major and minor axes of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

If PN is the ordinate of a point P on the ellipse then show that $\frac{(PN)^2}{(A'N)(AN)}+\frac{(BC)^2}{(CA)^2}$.

Sol: Let P(θ) = (a cos θ, b sin θ) be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



The PN = b sin θ and CN = a cos θ

CA = CA′ = a, CB = CB′ = b

$$\therefore \text{LHS}=\frac{(PN)^2}{(A'N)(AN)}$$

$$=\frac{(PN)^2}{(A'C+CN)(CA-CN)}$$

$$=\frac{(b\sin\theta)^2}{(a+a\cos\theta)(a-a\cos\theta)}$$

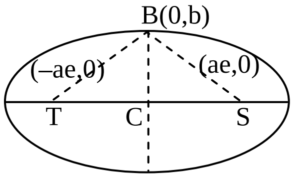
$$=\frac{b^2\sin^2\theta}{a^2\sin^2\theta}=\frac{b^2}{a^2}=\frac{BC^2}{(CA)^2}=\text{RHS}$$

19. S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.

Sol: Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;(a>b)$ be an ellipse whose foci are S and T. B is an end of the minor axis such that STB is equilateral triangle.

Then SB = ST = TB.

Also S = (ae, 0), T = (–ae, 0) and B(0, b)



Consider SB = ST $\Rightarrow (SB)^2 = (ST)^2$

$\Rightarrow (ae)^2 + b^2 = 4a^2 e^2$

$\Rightarrow a^2e^2 + a^2(1-e^2) = 4a^2e^2$

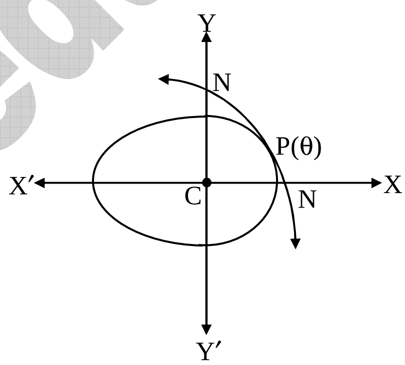
$\Rightarrow a^2e^2 + a^2 - a^2e^2 = 4a^2e^2$

$\Rightarrow 1 = 4e^2$

$\Rightarrow e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$.

20. If a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b) meets its major axis and minor axis at M and N respectively then prove that $\frac{a^2}{(CM)^2} + \frac{b^2}{(CN)^2} = 1$ where C is the centre of the ellipse.

Sol:



Let P(θ) = P(a cos θ, b sin θ) be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the equation of tangent at P(θ) is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$$\Rightarrow \frac{x}{\frac{a}{\cos\theta}} + \frac{y}{\frac{b}{\sin\theta}} = 1$$

This meets major axis at M and minor axis at N so that

$$CM = \frac{a}{\cos\theta} \text{ and } CN = \frac{b}{\sin\theta}$$

$$\Rightarrow \frac{a}{CM} = \cos\theta \text{ and } \frac{b}{CN} = \sin\theta$$

$$\therefore \frac{a^2}{(CM)^2} + \frac{b^2}{(CN)^2} = \cos^2\theta + \sin^2\theta = 1.$$

Short Answer Questions

1).Find the length of major axis, minor axis, latus rectum, eccentricity, coordinates of the centre, foci and equations of directrices of the following ellipse.

i) $9x^2 + 16y^2 = 144$

ii) $4x^2 + y^2 - 8x + 2y + 1 = 0$

iii) $x^2 + 2y^2 - 4x + 12y + 14 = 0$

Sol.I) Given equation is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$\therefore$ a = 4, b = 3 where a>b

Length of major axis = 2a = $2 \times 4 = 8$

Length of minor axis = 2b = $2 \times 3 = 6$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2\cdot 9}{4} = \frac{9}{2}$

Eccentricity = $\sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

Centre is C(0, 0)

Foci are (±ae, 0) = $(\pm\sqrt{7}, 0)$

Equations of the directrices are

$$x = \pm\frac{a}{e} \Rightarrow x = \pm 4\cdot\frac{4}{\sqrt{7}} = \pm\frac{16}{\sqrt{7}}$$
$$\Rightarrow \sqrt{7}x = \pm 16$$

ii) Given equation is $4x^2 + y^2 - 8x + 2y + 1=0$

$$4(x^2-2x)+(y^2+2y)=-1$$
$$4\left((x-1)^2-1\right)+\left((y+1)^2-1\right)=-1$$
$$4(x-1)^2+(y+1)^2=4+1-1=4$$
$$\frac{(x-1)^2}{1}+\frac{(y+1)^2}{4}=1$$

a = 1, b = 2 where a < b $\Rightarrow$ y-axis is major axis

Length of major axis = 2b = 4

Length of minor axis = 2a = 2

Length of latus rectum = $\frac{2a^2}{b}=\frac{2}{2}=1$

Eccentricity = $\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{\frac{4-1}{4}}=\frac{\sqrt{3}}{2}$

Centre is c(–1, 1)

Be = $2\cdot\frac{\sqrt{3}}{2}=\sqrt{3}$

Foci are $(-1,1\pm\sqrt{3})$

Equations of the directrices are

$$y+1=\pm\frac{b}{e}=\pm\frac{4}{\sqrt{3}}$$
$$\sqrt{3}y+\sqrt{3}=\pm4$$
$$\sqrt{3}y+\sqrt{3}\pm4=0$$

iii) Given equation is:

$x^2 + 2y^2 - 4x + 12y + 14 = 0$

$x^2 - 4x + 2(y^2 + 6y) = -14$

$\Rightarrow (x^2 - 4x + 4) + 2(y^2 + 6y + 9) = 4+18-14$

$\Rightarrow (x-2)^2 + 2(y+3)^2 = 8$

$$\Rightarrow \frac{(x-2)^2}{8}+\frac{(y+3)^2}{4}=1$$
$$\Rightarrow \frac{(x-2)^2}{(2\sqrt{2})^2}+\frac{(y+3)^2}{2^2}=1$$

$a = 2\sqrt{2}, b = 2, h = 2, k = -3$

Length of major axis = 2a = $2(2\sqrt{2}) = 4\sqrt{2}$

Length of minor axis = 2b = 2 × 2 = 4

Length of latus rectum =

$$\frac{2b^2}{a} = \frac{2 \cdot 4}{2\sqrt{2}} = 2\sqrt{2}$$

Eccentricity = $\sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{4}{8}} = \frac{1}{\sqrt{2}}$

Centre = (h, k) = (2, –3)

Foci = (h ± ae, k) = (2 ± 2, –3)

= (4, –3), (0, –3)

Equations of the directrices are:

$$x - h = \pm\frac{a}{e} \Rightarrow x - 2 = \pm\frac{2\sqrt{2}}{(1/\sqrt{2})}$$

$\Rightarrow x - 2 = \pm 4$

i.e. x = 6, x = –2

2. Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ given the following data.

i) Centre (2, –1), one end of major axis (2,–5), e = 1/3.

Sol. Centre C =(h, k) = (2, –1) ⇒ h = 2, k = –1

End of major axis A =(2, –5).

The x coordinates of centre and end of the major axis are same, therefore major axis is parallel to y axis.

$$b = CA = \sqrt{(2-2)^2 + (-5+1)^2} = \sqrt{(-4)^2} = 4$$

$$a^2 = b^2(1-e^2) = 16\left(1-\frac{1}{9}\right) = \frac{128}{9}$$

Equation of the ellipse is

$$\frac{(x-2)^2}{\frac{128}{9}}+\frac{(y+1)^2}{16}=1$$

$$\frac{9(x-2)^2}{128}+\frac{(y+1)^2}{16}=1$$

$$9(x-2)^2+8(y+1)^2=128$$

i.e. $8(x-2)^2 + 9(y+1)^2 = 128$.

ii) Centre (4, –1), one end of major axis is (–1,–1) and passing through (8, 0).

Sol. Centre C (4, –1)

ONE end of major axis is A =(–1,–1).

Y coordinates of above points are same, major axis is parallel to x axis

$$a = CA = \sqrt{(4+1)^2+(-1+1)^2}=5$$

Ellipse is passing through (8, 0)

$$\Rightarrow \frac{(8-4)^2}{25}+\frac{(0+1)^2}{b^2}=1 \Rightarrow \frac{1}{b^2}=1-\frac{16}{25}=\frac{9}{25}$$

Equation of ellipse is

$$\frac{(x-4)^2}{25}+\frac{9}{25}(y+1)^2=1$$

$$\Rightarrow (x-4)^2+9(y+1)^2=25$$

iii) Centre (0, –3), e = 2/3, semi-minor axis = 5.

Sol.

Centre C (0, –3) , e = 2/3

Semi minor axis b = 5

$\Rightarrow b^2 = a^2 - a^2 e^2$

$$\Rightarrow 25 = a^2 - a^2\frac{4}{9} = a^2\left(\frac{5}{9}\right)$$

$\Rightarrow 45 = a^2$

Equation of ellipse is

$$\frac{(x-0)^2}{45}+\frac{(y+3)^2}{25}=1$$

$$\Rightarrow \frac{x^2}{45}+\frac{(y+3)^2}{25}=1$$

iv) Centre (2, –1), e = 1/2, latus rectum = 4.

Sol.

Centre (2, –1), e = 1/2

Latus rectum = 4 $\Rightarrow \frac{2b^2}{a}=4 \Rightarrow b^2=2a$

$b^2 = a^2 - a^2 e^2$

$$\Rightarrow b^2=a^2-a^2\frac{1}{4} \Rightarrow b^2=\frac{3}{4}a^2$$

$$\Rightarrow 2a=\frac{3}{4}a^2 \Rightarrow \frac{8}{3}=a \text{ or } a^2=\frac{64}{9}$$

$$\Rightarrow b^2=\frac{16}{3}$$

Equation of the ellipse is

$$\frac{9(x-2)^2}{64}+\frac{3(y+1)^2}{16}=1$$

$9(x-2)^2 + 12(y+1)^2 = 64$

3. Find the equations of tangent and normal to the ellipse $2x^2 + 3y^2 = 11$ at the point whose ordinate is 1.

Sol. Equation of the ellipse is $S = 2x^2 + 3y^2 = 11$

Given y = 1

$2x^2 + 3 = 11 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2$

Points on the ellipse are P(2, 1) and Q(–2, 1)

Case I: P(2, 1)

Equation of the tangent is $S_1 = 0$

$$\Rightarrow 2x\cdot 2+3y\cdot 1=11 \Rightarrow 4x+3y=11$$

The normal is perpendicular to the tangent.

Equation of the normal at P can be taken as

$3x - 4y = k.$

The normal passes through P(2, 1)

$6 - 4 = k \Rightarrow k = 2$

Equation of the normal at P is $3x - 4y = 2$.

Case II: Q(–2, 1)

Equation of the tangent at Q is $S_2 = 0$

$\Rightarrow 2x(-2) + 3y.1 = 11$

$\Rightarrow -4x + 3y = 11$

$4x - 3y + 11 = 0$

Equation of the normal can be taken as

$3x + 4y = k$

The normal passes through Q(–2, 1)

$-6 + 4 = k \Rightarrow k = -2$

Equation of the normal at Q is $3x + 4y = -2$

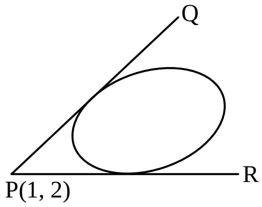
Or $3x + 4y + 2 = 0$.

4. Find the equations to the tangents to the ellipse, $x^2 + 2y^2 = 3$ drawn from the point (1, 2) and also find the angle between these tangents.

Sol. Equations of the ellipse is $x^2 + 2y^2 = 3$

$$\Rightarrow \frac{x^2}{3} + \frac{y^2}{3/2} = 1$$

$\Rightarrow a^2 = 3, b^2 = 3/2$



Let m be the slope of the tangent which is passing through P(1, 2)

Equation of the tangent is

$y - 2 = m(x - 1) = mx - m$

$y = mx + (2 - m)$

Above line is a tangent to the ellipse

$\Rightarrow c^2 = a^2m^2 + b^2$

$$(2-m)^2 = 3(m^2) + \frac{3}{2}$$

$$4 + m^2 - 4m = 3m^2 + \frac{3}{2}$$

$$2m^2 + 4m - \frac{5}{2} = 0$$

$$4m^2 + 8m - 5 = 0$$

$$(2m-1)(2m+5) = 0$$

$$m = \frac{1}{2} \text{ or } -\frac{5}{2}$$

Case I: $m = 1/2$

Equation of the tangent is

$$y = \frac{1}{2}x + 2 - \frac{1}{2} = \frac{x}{2} + \frac{3}{2}$$

$2y = x + 3$

$x - 2y + 3 = 0$

Case II: $m = -5/2$

Equation of the tangent is

$$y = -\frac{5}{2}x + (2 + \frac{5}{2}) = -\frac{5x}{2} + \frac{9}{2}$$

$2y = -5x + 9$

$5x + 2y - 9 = 0$

Angle between the tangents is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{1}{2} + \frac{5}{2}}{1 + \left(\frac{1}{2}\right)\left(-\frac{5}{2}\right)}\right| = \left|\frac{3}{1 - \frac{5}{4}}\right| = |-12| = 12$$

$\theta = \tan^{-1} 12.$

5. Find the equation of tangents to the ellipse $2x^2 + y^2 = 8$ which are parallel to $x - 2y + 4 = 0$.

Sol. Equation of the ellipse is $2x^2 + y^2 = 8$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{8} = 1$$

Equation of the tangent parallel to $x - 2y + 4 = 0$.

Is $x - 2y + k = 0$.

$$\Rightarrow y = \frac{x}{2} + \frac{k}{2}$$

Above line is a tangent to the ellipse

$$\Rightarrow c^2 = a^2m^2 + b^2$$

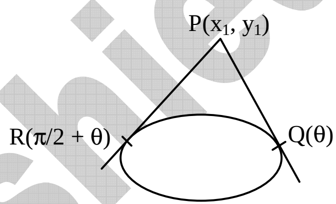
$$\Rightarrow \frac{k^2}{4} = 4.\frac{1}{4} + 8 \Rightarrow k^2 = 36 \Rightarrow k = \pm 6$$

Equation of tangents are

$$x - 2y \pm 6 = 0$$

6. Show that the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at points whose eccentric angles differ by $\pi/2$ intersect on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

Sol.



Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of the tangent at $Q(\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Equation of the tangent at $R\left(\frac{\pi}{2} + \theta\right)$ is

$$\frac{x}{a}\cos\left(\frac{\pi}{2}+\theta\right)+\frac{y}{b}\sin\left(\frac{\pi}{2}+\theta\right)=1$$

$$-\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta=1$$

Suppose P(x_1, y_1) is the point of intersection of the tangents at Q and R

$$\therefore \quad \frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta=1 \qquad \text{..... (1)}$$

$$\frac{-x_1}{a}\sin\theta+\frac{y_1}{b}\cos\theta=1 \qquad \text{..... (2)}$$

Squaring and adding (1) and (2)

$$\left(\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta\right)^2+\left(\frac{-x_1}{a}\sin\theta+\frac{y_1}{b}\cos\theta\right)^2=1+1$$

$$\frac{x_1^2}{a^2}\cos^2\theta+\frac{y_1^2}{b^2}\sin^2\theta+\frac{2x_1y_1}{ab}\cdot\cos\theta\sin\theta$$

$$+\frac{x_1^2}{a^2}\sin^2\theta+\frac{y_1^2}{b^2}\cos^2\theta-\frac{2x_1y_1}{ab}\cos\theta\sin\theta=2$$

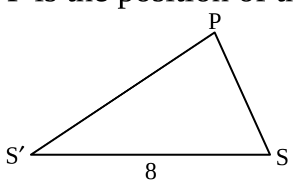
$$\frac{x_1^2}{a^2}(\cos^2\theta+\sin^2\theta)+\frac{y_1^2}{b^2}(\sin^2\theta+\cos^2\theta)=2$$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=2$$

Locus of P(x_1, y_1) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

7. A man running on a race course notices that the sum of the distances of the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. find the equation of the race course traced by the man.

Sol. S and S′ are the flags and P is the position of the man.



Given SP + S′P = 10 and SS′ = 8

The path traced by the man is an ellipse whose foci are S and S′.

$$2a = 10 \Rightarrow a = 5$$

$$SS' = 8 \Rightarrow 2ae = 8 \Rightarrow ae = 4$$

$$\Rightarrow e = \frac{4}{5}$$

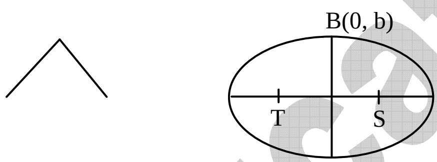
$$b^2 = a^2(1-e^2) = 25\left(1-\frac{16}{25}\right) = 9$$

Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

8. If S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.

Sol.



Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Foci are S(ae, 0), T(–ae, 0)

B(0, b) is the end of the minor axis

STB is an equilateral triangle

$SB = ST \Rightarrow SB^2 = ST^2$

$a^2e^2 + b^2 = 4a^2e^2$

$b^2 = 3a^2e^2$

$a^2(1 - e^2) = 3a^2e^2$

$1 - e^2 = 3e^2$

$4e^2 = 1 \Rightarrow e^2 = \frac{1}{4}$

Eccentricity of the ellipse: $e = \frac{1}{2}$.

9. Find the equation of the tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latus rectum in the first quadrant.

Sol. Given ellipse is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$

$$\Rightarrow e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\frac{\sqrt{7}}{4}$$

End of the latus rectum in first Quadrant

$$P\left(ae,\frac{b^2}{a}\right)=\left(\sqrt{7},\frac{9}{4}\right)$$

Equation of the tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$$x\cdot\frac{\sqrt{7}}{16}+\frac{y}{9}\left(\frac{9}{4}\right)=1$$

$$\frac{\sqrt{7}x}{16}+\frac{y}{4}=1$$

$$\sqrt{7}x+4y=16$$

Equation of the normal at P is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

$$\frac{16x}{\sqrt{7}}-\frac{9y}{(9/4)}=16-9$$

$$\frac{16x}{\sqrt{7}}-4y=7$$

$$16x-4\sqrt{7}y=7\sqrt{7}$$

10. Find the condition for the line

i) lx + my + n = 0 to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**.**

ii) lx + my + n = 0 to be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**.**

Sol. i) Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of the tangent at P(θ) is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad \ldots(1)$$

Equation of the given line is

$$lx + my = -n \quad \ldots(2)$$

(1) and (2) are representing the same line. Therefore,

$$\frac{\cos\theta}{al}=\frac{\sin\theta}{bm}=\frac{1}{-n}$$

$$\frac{\cos\theta}{al}-\frac{\sin\theta}{bm}=\frac{1}{-n}$$

$$\cos\theta=-\frac{al}{n}\sin\theta=-\frac{bm}{n}$$

$$\cos^2\theta+\sin^2\theta=1$$

$$\frac{a^2l^2}{n^2}+\frac{b^2m^2}{n^2}=1$$

$\Rightarrow$ $a^2l^2 + b^2m^2 = n^2$ is the required condition.

ii) Let lx + my + n = 0 be normal at P(a)

Equation of the normal at P(a) is :

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad \ldots (1)$$

$$Lx + my = -n \quad \ldots (2)$$

Comparing (1) and (2)

$$\frac{l}{\left(\frac{a}{\cos\theta}\right)} = \frac{m}{\left(\frac{-b}{\sin\theta}\right)} = \frac{n}{a^2 - b^2}$$

$$\frac{l\cos\theta}{a} = \frac{-m\sin\theta}{b} = \frac{-n}{a^2 - b^2}$$

$$\cos\theta = \frac{-an}{l(a^2 - b^2)},\ \sin\theta = \frac{bn}{m(a^2 - b^2)}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{a^2n^2}{l^2(a^2 - b^2)^2} + \frac{b^2n^2}{m^2(a^2 - b^2)^2} = 1$$

$\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2 - b^2)^2}{n^2}$ is the required condition.

10. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one end of the minor axis, then show that $e^4 + e^2 = 1$.

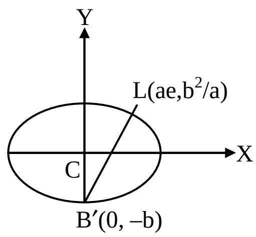
Sol. Equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

One end of the latusrectum is $L(ae, b^2/a)$

Equation of the normal at $L(ae, b^2/a)$ is

$$\frac{a^2x}{ae} - \frac{b^2y}{(b^2/a)} = a^2 - b^2 \quad \left(\because \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2\right)$$

$$\frac{ax}{e} - ay = a^2e^2$$



This normal passes through $B'(0, -b)$

$ab = a^2e^2$

$\Rightarrow$ $b = ae^2$

$b^2 = a^2e^4$

$a^2(1 - e^2) = a^2e^4$

$e^4 + e^2 = 1.$

11. Show that the points of intersection of the perpendicular tangents to an ellipse lies on a circle.

Sol: Let the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b).

Any tangent to the above ellipse is of the form $y=mx \pm \sqrt{a^2m^2+b^2}$

Let the perpendicular tangents intersect at

$P(x_1, y_1)$.

$\therefore y_1 = mx_1 \pm \sqrt{a^2m^2+b^2}$

$\Rightarrow (y_1 - mx_1)^2 = a^2m^2+b^2$

$\Rightarrow y_1^2 - 2my_1x_1 + m^2x_1^2 = a^2m^2 + b^2$

$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0$

This being a quadratic is m has two roots m_1 and m_2 which corresponds to the slopes of tangents drawn from P to ellipse then

$$m_1 m_2 = \left(\frac{y_1{}^2 - b^2}{x_1{}^2 - a^2}\right) \Rightarrow -1 = \left(\frac{y_1{}^2 - b^2}{x_1{}^2 - a^2}\right)$$

($\because$ Product of slopes = –1 for perpendicular tangents)

$\Rightarrow x_1^2 + y_1^2 = a^2 + b^2$

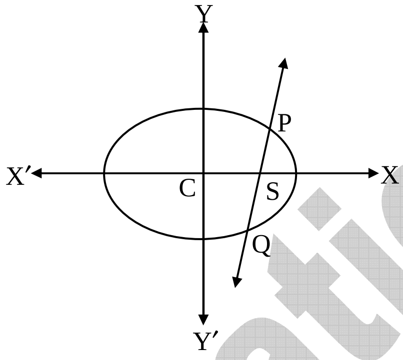
$\therefore$ Locus of (x_1, y_1) is $x^2 + y^2 = a^2 + b^2$ which is a circle.

12. If θ_1, θ_2 are the eccentric angles of the extremities of a focal chord (other than the vertices) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) and e its eccentricity. Then show that

i) $e\cos\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$

ii) $\frac{e+1}{e-1}=\cot\left(\frac{\theta_1}{2}\right)\cot\left(\frac{\theta_2}{2}\right)$

Sol:



Let $P(\theta_1)$, $Q(\theta_2)$ be the two extremities of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$

$\therefore P=(a\cos\theta_1, b\sin\theta_1), (\theta_1\neq 0)$

$Q=(a\cos\theta_2, b\sin\theta_2), (\theta_2\neq\pi)$

And focus S = (ae, 0). Now PQ is a focal chord and hence P, S, Q are collinear.

$\therefore$ Slope of $\overline{PS}$ = slope of $\overline{SQ}$

$$\Rightarrow \frac{b\sin\theta_1}{a(\cos\theta_1-e)}=\frac{b\sin\theta_2}{a(\cos\theta_2-e)}$$

$$\Rightarrow ab\sin\theta_1(\cos\theta_2-e)=ab\sin\theta_2(\cos\theta_1-e)$$

$$\Rightarrow (\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)=e(\sin\theta_1-\sin\theta_2)$$

$$\Rightarrow (\sin\theta_1-\theta_2)=e(\sin\theta_1-\sin\theta_2)$$

$$\Rightarrow 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)=e\,2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$\Rightarrow e\cos\left(\frac{\theta_1-\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$\Rightarrow e=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$$

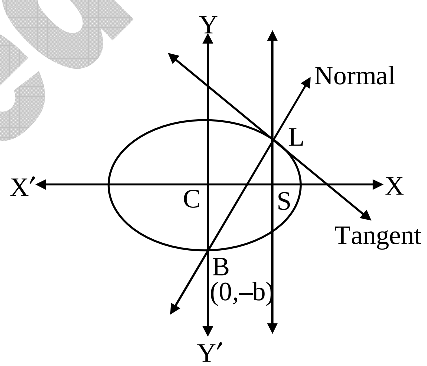
$$\therefore \frac{e+1}{e-1}=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$$

$$=\frac{2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}=\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}$$

$$\therefore \frac{e+1}{e-1}=\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}$$

13. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one end of the minor axis, then show that $e^4 + e^2 = 1$. [e is the eccentricity of the ellipse]

Sol:



Let ‘L’ be the one end of the latus rectum of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then the co-ordinates of L = $\left(ae,\frac{b^2}{a}\right)$.

$\therefore$ Equation of the normal at L is

$$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2$$

$$\Rightarrow \frac{ax}{e} - ay = a^2 - b^2$$

If this passes through the one end $B' = (0, -b)$ of the minor axis then $ab = a^2 - b^2$.

$$\Rightarrow ab = a^2 - a^2(1 - e^2)$$

$$\Rightarrow ab = a^2 e^2 \Rightarrow e^2 = \frac{ab}{a^2} = \frac{b}{a}$$

$$\Rightarrow e^4 = \frac{b^2}{a^2} = \frac{a^2(1-e^2)}{a^2} = 1 - e^2$$

$$\Rightarrow e^4 + e^2 - 1 = 0 \Rightarrow e^4 + e^2 = 1$$

14. If a circle is concentric with the ellipse, find the inclination of their common tangent to the major axis of the ellipse.

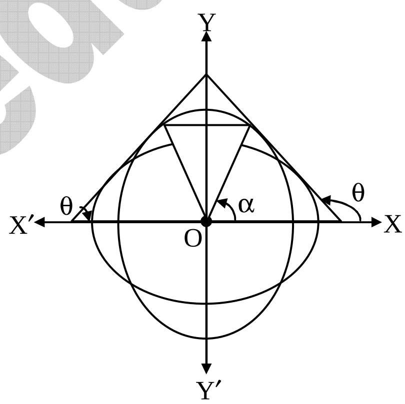
Sol: Let the circle $x^2 + y^2 = r^2$ and the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$.

$\therefore$ The major axis of ellipse is X-axis.

If $r < b < a$, then the circle lies completely in the ellipse making no common tangents.

If $b < a < r$ (ellipse lies completely in circle) no common tangent is passive.

Case (i): If $b < r < a$



Let one of the common tangent make angle θ with positive X-axis and suppose the equation of tangent to the circle be $x\cos\alpha + y\sin\alpha = r$ where α is the angle made by the radius of circle with positive X-axis.

$$\therefore \theta = \frac{\pi}{2} + \alpha \text{ (or) } \theta = \alpha - \frac{\pi}{2}$$

Since x cos α + y sin α = 4 touches the ellipse also, we have

$$a^2\cos^2\alpha + b^2\sin^2\alpha = r^2.$$

$$\therefore a^2\cos^2\left(\theta - \frac{\pi}{2}\right) + b^2\sin^2\left(\theta - \frac{\pi}{2}\right) = r^2$$

$$\text{(or) } a^2\cos^2\left(\frac{\pi}{2} + \theta\right) + b^2\sin^2\left(\frac{\pi}{2} + \theta\right) = r^2$$

$$\therefore a^2\sin^2 + b^2\cos^2\theta = r^2$$

$$\Rightarrow a^2\left(\frac{1-\cos 2\theta}{2}\right) + b^2\left(\frac{1+\cos 2\theta}{2}\right) = r^2$$

$$\Rightarrow \left(\frac{a^2+b^2}{2}\right) + \cos 2\theta\left(\frac{b^2-a^2}{2}\right) = r^2$$

$$\Rightarrow (a^2+b^2) + (b^2-a^2)\cos 2\theta = 2r^2$$

$$\cos 2\theta = \frac{a^2+b^2-2r^2}{a^2-b^2}$$

$$\Rightarrow 2\theta = \cos^{-1}\left(\frac{a^2+b^2-2r^2}{a^2-b^2}\right)$$

$$\Rightarrow \theta = \frac{1}{2}\cos^{-1}\left(\frac{a^2+b^2-2r^2}{a^2-b^2}\right)$$

Case (ii): When r = a the circle touches the ellipse at the ends of major axis of the Ellipse so that the common tangents are $x = \pm a$ and $\theta = \frac{\pi}{2}$.

Case (iii): When r = b, the circle touches the ellipse at the ends of minor axis of Ellipse so that common tangents y = ±b making θ = 0..

15. Find the coordinates of the points on the ellipse $x^2 + 3y^2 = 37$ at which the normal is parallel to the line $6x - 5y = 2$.

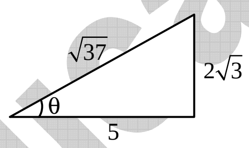
Sol. Equation of the ellipse is $x^2 + 3y^2 = 37$

$$\Rightarrow \frac{x^2}{37} + \frac{y^2}{\left(\frac{37}{3}\right)} = 1 \qquad a^2 = 37,\ b^2 = \frac{37}{3}$$

Slope of the normal $= \dfrac{a\sin\theta}{b\cos\theta} = \dfrac{\sqrt{37}\sin\theta}{\sqrt{\frac{37}{3}}\cos\theta} = \sqrt{3}\tan\theta$

The normal is parallel to $6x - 5y = 2$

$$\therefore \begin{aligned} \sqrt{3}\tan\theta &= \frac{6}{5} \\ \tan\theta &= \frac{6}{5\sqrt{3}} = \frac{2\sqrt{3}}{5} \end{aligned}$$



Case I:

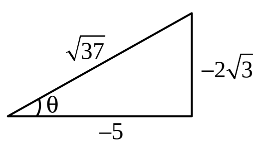
The coordinates of P are $(a\cos\theta, b\sin\theta)$

$$\left(\sqrt{37}\frac{5}{\sqrt{37}}, \frac{\sqrt{37}}{\sqrt{3}} \cdot \frac{2\sqrt{3}}{\sqrt{37}}\right) = (5, 2)$$

Case II:

The coordinates of P are $(a\cos\theta, b\sin\theta)$

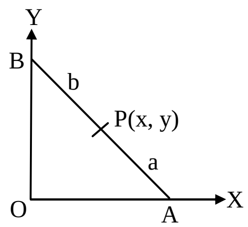
$$\left(\sqrt{37}\frac{(-5)}{\sqrt{37}}, \frac{\sqrt{37}}{\sqrt{3}} \cdot \frac{-2\sqrt{3}}{\sqrt{37}}\right) = (-5, -2)$$



Long Answer Questions

1. A line of fixed length (a + b) moves so that its ends are always on two perpendicular straight lines prove that a marked point on the line, which divides this line into portions of lengths 'a' and 'b' describes an ellipse and also find the eccentricity of the ellipse when a = 8, b = 12.

Sol.



Let the perpendicular lines as coordinate axes.

Let OA = α and OB = β so that equation of AB is $\frac{x}{\alpha}+\frac{y}{\beta}=1$.

Given length of the line AB =(a + b)

$\Rightarrow \alpha^2+\beta^2=(a+b)^2 \qquad \ldots$ (i)

Let P(x, y) be the point which divides AB in the ratio a : b

$$\Rightarrow P=\left(\frac{b\alpha}{a+b},\frac{a\beta}{a+b}\right)=(x,y)$$

$$\frac{b\alpha}{a+b}=x\Rightarrow\alpha=\frac{a+b}{b}\cdot x$$

$$\frac{a\beta}{a+b}=y\Rightarrow\beta=\frac{a+b}{a}\cdot y$$

Substituting the values of α, β in (i), we get,

$$\frac{(a+b)^2}{b^2}\cdot x^2+\frac{(a+b)^2}{a^2}\cdot y^2=(a+b)^2$$

$$\text{or } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

P describes an ellipse.

Given a = 8, b = 12, so that b > a.

Eccentricity = $\sqrt{\dfrac{b^2-a^2}{b^2}} = \sqrt{\dfrac{144-64}{144}} = \sqrt{\dfrac{80}{144}} = \dfrac{\sqrt{5}}{3}$

2. Prove that the equation of the chord joining the points α and β on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2}+\dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2}=\cos\left(\dfrac{\alpha-\beta}{2}\right)$.

Sol. The given points on the ellipse are

P(a cos α, b sin α) and Q(a cos β, b sin β)

Slope of PQ = $\dfrac{y_1-y_2}{x_1-x_2} = \dfrac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$

$$= \frac{b\left(2\cos\dfrac{\alpha+\beta}{2}\cdot\sin\dfrac{\alpha-\beta}{2}\right)}{a\left(-2\sin\dfrac{\alpha+\beta}{2}\cdot\sin\dfrac{\alpha-\beta}{2}\right)} = -\frac{b\cdot\cos\dfrac{\alpha+\beta}{2}}{a\cdot\sin\dfrac{\alpha+\beta}{2}}$$

Equation of the chord PQ is

$$y-b\sin\alpha = -\frac{b\cos\dfrac{\alpha+\beta}{2}}{a\sin\dfrac{\alpha+\beta}{2}}(x-a\cos\alpha)$$

$$\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\sin\alpha\cdot\sin\frac{\alpha+\beta}{2} = -\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\cos\alpha\cdot\cos\frac{\alpha+\beta}{2}$$

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\alpha\cdot\cos\frac{\alpha+\beta}{2}+\sin\alpha\cdot\sin\frac{\alpha+\beta}{2}$$

$$= \cos\left(\alpha-\frac{\alpha+\beta}{2}\right) = \cos\frac{\alpha-\beta}{2}$$

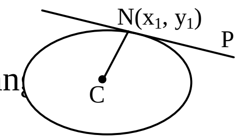
3. Show that the feet of the perpendicular drawn from the centre on any tangent to the ellipse lies on the curve $(x^2 + y^2) = a^2x^2 + b^2y^2$.

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of the tangent at P(θ) is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Slope of the tan



$$PN = \frac{-\left(\frac{\cos\theta}{a}\right)}{\left(\frac{\sin\theta}{b}\right)}=-\frac{b\cos\theta}{a\sin\theta}$$

Let N (x_1, y_1) be the foot of the perpendicular from C(0, 0) to any tangent.

slope of CN = $\frac{y_1}{x_1}$.

∴ Slope of PT × slope of CN = –1

$$-\frac{b\cos\theta}{a\sin\theta}\cdot\frac{y_1}{x_1}=-1$$

$$\frac{\cos\theta}{ax_1}=\frac{\sin\theta}{by_1}=\frac{1}{\sqrt{a^2x_1^2+b^2y_1^2}}=k$$

$$\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta=1$$

$$\cos\theta=\frac{ax_1}{k},\ \sin\theta=\frac{by_1}{k}$$

$$\frac{x_1}{a}\cdot ax_1+\frac{y_1}{b}\cdot by_1=k$$

$$x_1^2+y_1^2=k$$

N(x_1, y_1) is a point on $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

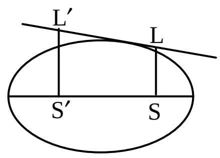
$$\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta=1$$

$$x_1^2 + y_1^2 = \sqrt{a^2x_1^2 + b^2y_1^2} \text{ (or)}$$

$$\left(x_1^2 + y_1^2\right)^2 = a^2x_1^2 + b^2y_1^2$$

Locus of $N(x_1, y_1)$ is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

4. Show that the locus of the feet of the perpendiculars drawn from foci on any tangent of the ellipse is the auxiliary circle.



Sol. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of the tangent to the ellipse is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad y - mx = \pm\sqrt{a^2m^2 + b^2} \quad \ldots (1)$$

Equation to the perpendicular from either focus ($\pm$ae, 0) on this tangent is

$$y = -\frac{1}{m}(x \pm ae)$$

$$my = -(x \pm ae)$$

$$my + x = \pm ae \quad \ldots (2)$$

Squaring and adding (1) and (2)

$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2e^2$

$y^2 + m^2x^2 - 2mxy + m^2y^2 + x^2 + 2mxy$

$= a^2m^2 + a^2 - a^2e^2 + a^2e^2$

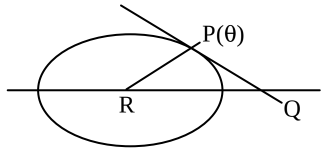
$(x^2 + y^2)(1 + m^2) = a^2(1 + m^2)$

$\Rightarrow x^2 + y^2 = a^2$

The locus is the auxiliary circle concentric with the ellipse.

5. The tangent and normal to the ellipse $x^2 + 4y^2 = 4$ at a point $P(\theta)$ meets the major axis at Q and R respectively. If $0 < \theta < \pi/2$ and QR = 2, then show that $\theta = \cos^{-1}(2/3)$.

Sol.



Equation of the ellipse is $x^2 + 4y^2 = 4$

$$\frac{x^2}{4}+\frac{y^2}{1}=1$$

Equation of the tangent at $P(\theta)$ is

$$\frac{x}{2}\cdot\cos\theta+\frac{y}{1}\sin\theta=1$$

Equation of x-axis (i.e., major axis) is $y = 0$

$$\frac{x}{2}\cdot\cos\theta=1\Rightarrow x=\frac{2}{\cos\theta}$$

Coordinates of Q are $\left(\frac{2}{\cos\theta},0\right)$

Equation of the normal at $P(\theta)$ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

$$\frac{2x}{\cos\theta}-\frac{y}{\sin\theta}=3$$

Substituting $y = 0$ we get $\frac{2x}{\cos\theta}=3$

$$x=\frac{3}{2}\cos\theta$$

Coordinates of R are $\left(\frac{3}{2}\cos\theta,0\right)$

$$QR=\left(-\frac{3}{2}\cos\theta+\frac{2}{\cos\theta}\right)=\frac{-3\cos^2\theta+4}{2\cos\theta}$$

Given QR = 2

$$\frac{-3\cos^2\theta+4}{2\cos\theta}=2$$

$$-3\cos^2\theta+4=4\cos\theta$$

$3\cos^2\theta + 4\cos\theta - 4 = 0$

$(3\cos\theta - 2)(\cos\theta + 2) = 0$

$3\cos\theta - 2 = 0 \Rightarrow \cos\theta = 2/3$

$\cos\theta + 2 = 0 \Rightarrow \cos\theta = -2$

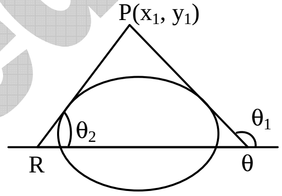
$\cos\theta = 2/3$ or -2

$$\Rightarrow \cos\theta=\frac{2}{3}$$

i.e. $\theta=\cos^{-1}\left(\frac{2}{3}\right)$

6. Show that the points of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) which one inclined at an angle θ_1 and θ_2 with its major axis such that $\cot\theta_1 + \cot\theta_2 = k^2$ lies on the curve $k^2(y^2 - b^2) = 2xy$.

Sol.



Equation of any tangent to the ellipse

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

This line passes through $P(x_1, y_1)$

$$y_1=mx_1\pm\sqrt{a^2m^2+b^2}$$

$$(y_1-mx_1)^2=a^2m^2+b^2$$

$$m^2x_1^2 + y_1^2 - 2mx_1y_1 - a^2m^2 - b^2 = 0$$

$$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0$$

This is a quadratic equation in m.

m_1 and m_2 are roots of this equation.

$$m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}, m_1m_2 = \frac{y_1^2 - b^2}{x_1^2 - a^2}$$

$$\cot\theta_1 + \cot\theta_2 = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1m_2} = \frac{2x_1y_1}{y_1^2 - b^2}$$

Given $\cot\theta_1 + \cot\theta_2 = k^2$

$$\frac{2x_1y_1}{y_1^2 - b^2} = k^2 \Rightarrow k^2(y_1^2 - b^2) = 2x_1y_1$$

Locus of $P(x_1, y_1)$ is $k^2(y^2 - b^2) = 2xy$.

7. Show that the point of intersection of perpendicular tangents to an ellipse lie on a circle.

Sol. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be the point of intersection of the tangents.

Equation of the tangent is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

This tangent is passing through $P(x_1, y_1)$

$$y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2}$$

$$y_1 - mx_1 = \pm\sqrt{a^2m^2 + b^2}$$

$$(y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$m^2x_1^2 + y_1^2 - 2mx_1y_1 - a^2m^2 - b^2x$$

$$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0$$

This is a quadratic equation in m giving two values for m say m_1 and m_2. These are the slopes of the tangents passing through (x_1,y_1).

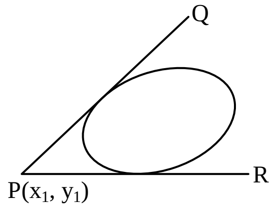
The tangents are perpendicular $\Rightarrow m_1m_2 = -1$

$$\frac{y_1^2 - b^2}{x_1^2 - a^2} = -1$$

$$y_1^2 - b^2 = -x_1^2 + a^2$$

$$x_1^2 + y_1^2 = a^2 + b^2$$

Locus of $P(x_1, y_1)$ is $x^2 + y^2 = a^2 + b^2$ which is a circle.



This circle is called Director circle of the Ellipse.

Problems for Practice

1. Find the eccentricity, coordinates of foci. Length of latus rectum and equations of directrices of the following ellipses.

i) $9x^2 + 16y^2 - 36x + 32y - 92 = 0$

ii) $3x^2 + y^2 - 6x - 2y - 5 = 0$

2. Find the equation of the ellipse referred to its major and minor axes as the coordinate axes x, y respectively with latus rectum of length 4 and the distance between foci $4\sqrt{2}$.

Ans. $x^2 + 2y^2 = 16$

3. If the length of the latus rectum is equal to half of its minor axis of an ellipse in the standard form, then find the eccentricity of the ellipse.

Ans. $e = \sqrt{3}/2$