DIFFERENTIAL EQUATIONS

An equation involving one dependent variable, one or more independent variables and the differential coefficients (derivatives) of dependent variable with respect to independent variables is called a differential equation.

Order of a Differential Equation: The order of the highest derivative involved in an ordinary differential equation is called the order of the differential equation.

Degree of a Differential Equation: The degree of the highest derivative involved in an ordinary differential equation, when the equation has been expressed in the form of a polynomial in the highest derivative by eliminating radicals and fraction powers of the derivatives is called the degree of the differential equation.

sakshieducation.com

Very Short Answer Questions

1. Find the order of the family of the differential equation obtained by eliminating the arbitrary constants b and c from $xy = ce^{x} + be^{-x} + x^{2}$.

Sol.

Equation of the curve is $xy = ce^{x} + be^{-x} + x^{2}$

Number of arbitrary constants in the given curve is 2.

Therefore, the order of the corresponding differential equation is 2.

2. **Find the order of the differential equation of the family of all circles with their centres at the origin**.

Given family of curves is $x^2 + y^2 = a^2 ---(1)$, a parameter.

Diff (1) w.r.t x,

$2x+2y.y_1 = 0$.

Hence required differential equation is $x+y.y_1 = 0$.

Order of the differential equation is 1.

3. Find the order and degree of $\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right)^{6/5} = 6y$.

Sol. Given equation is: $\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right)^{6/5} = 6y$

i.e. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3 = (6y)^{5/6}$

Order = 2, degree = 1

Short Answer Questions

1. Form the differential equation of the following family of curves where parameters are given in brackets.

i). $y = c(x-c)^2; (c)$

$y = c(x-c)^2 ------(1)$

Diff. w.r.t x,

$y_1 = c.2(x-c) ----(2)$

$\frac{(1)}{(2)} \Rightarrow \frac{y}{y_1} = \frac{x-c}{2}$

$\Rightarrow x - c = \frac{2y}{y_1}$ and $c = x - \frac{2y}{y_1}$

$from(1), y = \left(x - \frac{2y}{y_1}\right)\left(\frac{2y}{y_1}\right)^2 \Rightarrow y_1^3 = 4y(xy_1 - 2y)$

ii) $xy = ae^x + be^{-x}; (a,b)$

$xy = ae^x + be^{-x} ---(1)$

Diff. w.r.t. x,

$y + x.y_1 = ae^x - be^{-x} ---(2)$

$diff.w.r.t.$ x,

$y_1 + y_1 + xy_2 = ae^x + be^{-x} = xy$

$\therefore 2y_1 + xy_2 = xy$

Which is required differential equation.

iii) $y = (a+bx)e^{kx}; (a,b)$

$y = (a+bx)e^{kx} ---(1)$

Diff. w.r.t x,

$\Rightarrow y_1 = k(a+bx)e^{kx} + be^{kx}$

$\Rightarrow y_1 = ky + be^{kx} ---(2)$

Diff. w.r.t. x,

$$\Rightarrow y_2 = ky_1 + kbe^{kx}$$
$$\Rightarrow y_2 = ky_1 + k(y_1 - ky)$$

$\Rightarrow y_2 = 2ky_1 - k^2 y$ Which is required differential equation.

iv) $y = a\cos(nx+b); (a,b)$

Ans. $y_2 + n^2 y = 0$

2. Obtain the differential equation which corresponds to each of the following family of curves.

i) The rectangular hyperbolas which have the coordinates axes as asymptotes.

Sol. Equation of the rectangular hyperbola is $xy=c^2$ where c is arbitrary constant.

Differentiating w.r.t. x

$$x\frac{dy}{dx} + y = 0$$

ii) The ellipses with centres at the origin and having coordinate axes as axes.

Sol. Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Diff. w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow y.y_1 = -\frac{b^2}{a^2}x$$

Diff. w.r.t. x,

$$y.y_2 + y_1.y_1 = -\frac{b^2}{a^2} \Rightarrow y.y_2 + 2y_1 = \frac{y.y_1}{x}$$

$$\Rightarrow x(y.y_2 + 2y_1) = y.y_1$$

3. Form the differential equation corresponding to the family of circles of radius r given by $(x - a^2) + (y - b)^2 = r^2$, where a and b are parameters.

Sol. We have: $(x - a^2) + (y - b)^2 = r^2$ …(1)

Differentiating (1) w.r.to x

$$2(x-a)+2(y-b)\frac{dy}{dx}=0 \qquad \text{… (2)}$$

Differentiating (2) w.r.to x

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \qquad \text{… (3)}$$

From (2) $(x-a)=-(y-b)\frac{dy}{dx}$

Substituting in (1), we get

$$(y-b)^2\left(\frac{dy}{dx}\right)^2+(y-b)^2=r^2$$

$$(y-b)^2\left(\left(\frac{dy}{dx}\right)^2+1\right)=r^2 \qquad \text{… (4)}$$

From (3) $(y-h)\frac{d^2y}{dx^2}=-\left(1+\left(\frac{dy}{dx}\right)^2\right)$

$$(y-h)=-\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\left(\frac{d^2y}{dx^2}\right)}$$

Substituting in (4): $\dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2}=r^2$

i.e. $r^2\left(\frac{d^2y}{dx^2}\right)^2=\left(1+\left(\frac{dy}{dx}\right)^2\right)^3$

Long Answer Questions

1. Form the differential equations of the following family of curves where parameters are given in brackets.

i) $y = ae^{3x} + be^{4x}; (a, b)$

Sol. $y = ae^{3x} + be^{4x} ----(1)$

Differentiating w.r.to x

$y_1 = 3ae^{3x} + 4be^{4x} -----(2)$

Differentiating w.r.to x,

$y_2 = 9ae^{3x} + 16be^{4x} -----(3)$

Eliminating a,b from above equations,

$$\begin{vmatrix} y & e^{3x} & e^{4x} \\ y_1 & 3e^{3x} & 4e^{4x} \\ y_2 & 9e^{3x} & 16e^{4x} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y & 1 & 1 \\ y_1 & 3 & 4 \\ y_2 & 9 & 16 \end{vmatrix} = 0$$

$\Rightarrow y_2 - 7y_1 + 12y = 0$ Which is the required differential equation.

ii) $y = ax^2 + bx$; (a, b)

Sol.

$y = ax^2 + bx$ ---- (1)

diff. w.r.t. x,

$y_1 = y_2x + b \Rightarrow y_1x = y_2x^2 + bx ---(2)$

diff. w.r.t. x,

$y_2 = 2a$ ----------- (3)

From (2) and (3),

$$y_1 = y_2 x + b \Rightarrow y_1 x = y_2 x^2 + bx ---(4)$$

$$x^2 \frac{d^2y}{dx^2} = 2ax^2 \qquad \ldots \text{(i)}$$

$$-2x\frac{dy}{dx} = -4ax^2 - 2bx \qquad \ldots \text{(ii)}$$

$2y = 2ax^2 + 2bx$ … (iii)

Adding all three equations we get

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

iii) $ax^2 + by^2 = 1$; (a, b)

Sol.

Given equation is

$ax^2 + by^2 = 1$ ------ (1)

Differentiating w.r.t. x

$$\Rightarrow 2ax + 2byy_1 = 0$$
$$\Rightarrow ax + byy_1 = 0 ------(2)$$

Differentiating w.r.t. x

$$\Rightarrow a + b(yy_2 + y_1 y_1) = 0 \Rightarrow a + b(yy_2 + y_1^2) = 0$$

$$\Rightarrow ax + bx(yy_2 + y_1^2) = 0 \text{ ---- (3)}$$

$$(3)-(2) \Rightarrow bx(yy_2 + y_1^2) - byy_1 = 0$$
$$\Rightarrow x(yy_2 + y_1^2) - yy_1 = 0$$

iv) $xy = ax^2 + \frac{b}{x}$; (a, b)

Sol. $xy = ax^2 + \frac{b}{x}$

$x^2y = ax^2 + b$

Differentiating w.r.t. x

$x^2y_1 + 2xy = 3ax^2$

Dividing with x

$xy_1 + 2y = 3ax$ … (i)

Differentiating w.r.t. x

$xy_2 + y_1 + 2y_1 = 3a$

$xy_2 + 3y_1 = 3a$ … (ii)

Dividing (i) by (ii)

$$\frac{xy_1 + 2y}{xy_2 + 3y_1} = \frac{3ax}{3a} = x$$

Cross multiplying

$$xy_1 + 2y = x^2y_2 + 3xy$$
$$x^2y_2 + 2xy_1 - 2y = 0$$
$$x^2\left(\frac{d^2y}{dx^2}\right) + 2x\left(\frac{dy}{dx}\right) - 2y = 0$$

2. Obtain the differential equation which corresponds to each of the following family of curves.

i) The circles which touch the Y-axis at the origin.

Sol. Equation of the given family of circles is

$x^2 + y^2 + 2gx = 0$, g arbitrary const …(i)

$x^2 + y^2 = -2gx$

Differentiating w.r.t. x

$2x + 2yy_1 = -2g$ … (ii)

Substituting in (i)

$x^2 + y^2 = x(2x + 2yy_1)$ by (ii)

$= 2x^2 + 2xyy_1$

$yy^2 - 2xyy_1 - 2x^2 = 0$

$y^2 - x^2 = 2xy\frac{dy}{dx}$.

ii) The parabolas each of which has a latus rectum 4a and whose axes are parallel to x-axis.

Sol.

Equation of the given family of parabolas is

$(y - k)^2 = 4a(x - h)$ -----(i)

Where h,k are arbitrary constants

Differentiating w.r.t. x

$2(y - k)y_1 = 4a$

$(y - k)y_1 = 2a$ …(2)

Differentiating w.r.t. x

$(y - k)y_2 + y_1^2 = 0$ …(3)

From (2), $y - k = \frac{2a}{y_1}$

Substituting in (3)

$$\frac{2a}{y_1}\cdot y_2 + y_1^2 = 0 \Rightarrow 2ay_2 + y_1^3 = 0$$

iii) The parabolas having their foci at the origin and axis along the x-axis.

Sol.

Given family of parabolas is $y^2 = 4a(x + a)$ ----- (i)

Diff. w.r.t.x,

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{1}{2}yy' = a \quad \text{----- (2)}$$

From (i) and (2),

$$y^2 = 4\frac{1}{2}yy'\left(x + \frac{1}{2}yy'\right)$$

$$y^2 = 2y'x + 4\cdot\frac{1}{4}y^2y'^2 \Rightarrow y^2 = 2yy'x + y^2y'^2$$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) = y$$

Solutions of Differential Equations

Variables Separable:

Let the given equation be $\frac{dy}{dx} = f(x, y)$. If f(x, y) is a variables separable function,

i.e., f(x, y) = g(x)h(y) then the equation can be written as $\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$. By integrating both sides, we get the solution of $\frac{dy}{dx} = f(x, y)$. This method of finding the solution is known as **variables separable**.

Very Short Answer Questions

1. **Find the general solution of** $\sqrt{1-x^2}dy + \sqrt{1-y^2}dx = 0$.

Sol. Given d.e. is

$$\sqrt{1-x^2}dy + \sqrt{1-y^2}dx = 0$$

$$\sqrt{1-x^2}dy = -\sqrt{1-y^2}dx$$

Integrating both sides

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$\sin^{-1}y = -\sin^{-1}x + c$

Solution is $\sin^{-1}x + \sin^{-1}y = c$, where c is a constant.

2. **Find the general solution of** $\frac{dy}{dx} = \frac{2y}{x}$.

Sol. $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \int \frac{dy}{y} = 2\frac{dx}{x}$

Integrating both sides

$$\log c + \log y = 2\log x$$

$$\log cy = \log x^2$$

Solution is cy = x^2 where c is a constant.

Short Answer Questions

Solve the following differential equations.

1. $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$

Sol. $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$

Integrating both sides

$$\Rightarrow \int \frac{dy}{1+y^2}=\int \frac{dx}{1+x^2}$$

$\tan^{-1} y=\tan^{-1} x+\tan^{-1} c$ Where c is a constant.

2. $\frac{dy}{dx}=e^{y-x}$

Sol. $\frac{dy}{dx}=\frac{e^y}{e^x}\Rightarrow \frac{dy}{e^y}=\frac{dx}{e^x}$

Integrating both sides $\int e^{-x}dx=\int e^{-y}dy \Rightarrow -e^{-x}=-e^{-y}+c$

$e^{-y}=e^{-x}+c$ Where c is a constant.

3. $(e^x + 1)y\,dy + (y + 1)dx = 0$

Sol. $(e^x + 1)y\,dy = -(y + 1)dx$

$\frac{ydy}{y+1}=-\frac{dx}{e^x+1}$

Integrating both sides

$$\int\left(1-\frac{1}{y+1}\right)dy=\int -\frac{e^{-x}dx}{e^{-x}+1}$$

$$y-\log(y+1)=\log(e^{-x}+1)+\log c$$
$$\Rightarrow y-\log(y+1)=\log c(e^{-x}+1)$$
$$\Rightarrow y=\log(y+1)+\log c(e^{-x}+1)$$
$$y=\log c(y+1)(e^{-x}+1)$$

Solution is: $e^{y}=c(y+1)(e^{-x}+1)$.

4. $\frac{dy}{dx}=e^{x-y}+x^{2}e^{-y}$

Sol. $\frac{dy}{dx}=e^{x-y}+x^{2}e^{-y}=\frac{e^{x}}{e^{y}}+\frac{x^{2}}{e^{y}}$

Integrating both sides

$$\int e^{y}\cdot dy=\int(e^{x}+x^{2})dx$$

Solution is: $e^{y}=e^{x}+\frac{x^{3}}{3}+c$

5. tan y dx + tan x dy = 0

Sol. tan y dx = –tan x dy

$$\frac{dx}{\tan x}=\frac{-dy}{\tan y}\Rightarrow\frac{\cos x}{\sin x}dx=-\frac{\cos y}{\sin y}dy$$

Taking integration

$$\int\frac{\cos x}{\sin x}dx=-\int\frac{\cos y}{\sin y}dy$$

$$\log\sin x=-\log\sin y+\log c$$
$$\log\sin x+\log\sin y=\log c$$
$$\log(\sin x\cdot\sin y)=\log c\Rightarrow\sin x\cdot\sin y=c$$

6. $\sqrt{1+x^{2}}dx+\sqrt{1+y^{2}}dy=0$

Sol. $\sqrt{1+x^{2}}dx=-\sqrt{1+y^{2}}dy$

Integrating both sides $\int\sqrt{1+x^{2}}dx=-\int\sqrt{1+y^{2}}dy$

sakshieducation.com

$$\frac{x}{2}\times\sqrt{1+x^2}+\frac{1}{2}\sinh^{-1}x=$$

$$y\frac{\sqrt{1+y^2}}{2}=\frac{1}{2}\sinh^{-1}x+c$$

$$x\sqrt{1+x^2}+y\sqrt{1+y^2}+\log\left[(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})\right]=c$$

7. $y-x\frac{dy}{dx}=5\left(y^2+\frac{dy}{dx}\right)$

Sol. $y-5y^2=(x+5)\frac{dy}{dx}\Rightarrow\frac{dx}{x+5}=\frac{dy}{y(1-5y)}$

Integrating both sides

$$\int\frac{dx}{x+5}=\int\frac{dy}{y(1-5y)}=\int\left(\frac{1}{y}+\frac{5}{1-5y}\right)dy$$

$$\ln|x+5|=\ln y-\ln|1-5y|+\ln c$$

$$\ln|x+5|=\ln\left|\frac{cy}{1-5y}\right|\Rightarrow x+5=\left(\frac{cy}{1-5y}\right)$$

8. $\frac{dy}{dx}=\frac{xy+y}{xy+x}$

Sol. $\frac{dy}{dx}=\frac{y(x+1)}{x(y+1)}\Rightarrow\frac{y+1}{y}dy=\frac{x+1}{x}dx$

$$\int\left(1+\frac{1}{y}\right)dy=\int\left(1+\frac{1}{x}\right)dx$$

$$y+\log y=x+\log x+\log c$$

$$y-x=\log\left|\frac{cx}{y}\right|$$

Short Answer Questions

1. $\frac{dy}{dx} = \frac{1+y^2}{(1+x^2)xy}$

Sol. $\frac{dy}{dx} = \frac{1+y^2}{(1+x^2)xy}$

$\Rightarrow \frac{ydy}{1+y^2} = \frac{dx}{x(1+x^2)}$

$\frac{2ydy}{1+y^2} = \frac{2xdx}{x^2(1+x^2)}$

Integrating both sides

$\int \frac{2ydy}{1+y^2} = \int \left(\frac{1}{x^2} - \frac{1}{1+x^2}\right) 2x\ dx$

$\log(1+y^2) = \log x^2 - \log(1+x^2) + \log c$

$\log(1+x^2) + \log(1+y^2) = \log x^2 + \log c$

Solution is: $(1 + x^2)(1 + y^2) = cx^2$

2. $\frac{dy}{dx} + x^2 = x^2 \cdot e^{3y}$

Sol. $\frac{dy}{dx} + x^2 = x^2 \cdot e^{3y}$

$\Rightarrow \frac{dy}{dx} = x^2 \cdot e^{3y} - x^2 = x^2(e^{3y} - 1)$

Integrating both sides

$\int \frac{dy}{e^{3y} - 1} = \int x^2 dx \Rightarrow \int \frac{e^{-3y}}{1-e^{-3y}} = \int x^2 dx$

$\log \frac{(1-e^{-3y})}{3} = \frac{x^3}{3} + c$

$\log(1-e^{-3y}) = x^3 + c'\ (c' = 3c)$

Solution is: $1-e^{-3y} = e^{x^3} \cdot k\ \ (k = e^{c'})$

3. $(xy^2 + x)dx + (yx^2 + y)dy = 0$

Sol. $(xy^2 + x)dx + (yx^2 + y)dy = 0$

$x(y^2 + 1)dx + y(x^2 + 1)dy = 0$

Dividing with $(1 + x^2)(1 + y^2)$

$$\frac{xdx}{1+x^2}+\frac{ydy}{1+y^2}=0$$

Integrating both sides

$$\int\frac{xdx}{1+x^2}+\int\frac{ydy}{1+y^2}=0$$

$$\frac{1}{2}\left[(\log(1+x^2)+\log(1+y^2)\right]=\log c$$

$$\log(1+x^2)(1+y^2)=2\log c=\log c^2$$

$(1 + x^2)(1 + y^2) = k$ when $k = c^2$.

4. $\frac{dy}{dx}=2y\tanh x$

Sol. $\frac{dy}{dx}=2y\tanh x\Rightarrow\frac{dy}{y}=2\tanh xdx$

Integrating both sides

$$\int\frac{dy}{y}=2\int\tanh x\ dx$$

$$\log y=2\log\left|\cosh x\right|+\log c$$

$$\ln y=2\ln\cosh x+\ln c\Rightarrow y=c\cos^2 hx$$

5. $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$

$$\frac{dy}{dx}=\sin(x+y)\Rightarrow x+y=t$$

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\frac{dt}{dx}-1=\sin t \Rightarrow \frac{dt}{dx}=1+\sin t$$

$$\frac{dt}{1+\sin t}=dx$$

Integrating both sides

$$\int \frac{dt}{1+\sin t}=\int dx$$

$$\int \frac{1-\sin t}{\cos^2 t}dt=x+c$$

$$\int \sec^2 t dt-\int \tan t\cdot \sec t\ dt=x+c$$

$$\tan t-\sec t=x+c$$

$$\Rightarrow \tan(x+y)-\sec(x+y)=x+c$$

6. $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$

$$\frac{-dy}{y^2+y+1}=\frac{dx}{x^2+x+1}$$

Integrating both sides

$$-\int \frac{dy}{y^2+y+1}=\int \frac{dx}{x^2+x+1}$$

$$-\int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}=\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$-\frac{2}{\sqrt{3}}\tan^{-1}\frac{(y+1/2)}{\sqrt{3/2}}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{(x+1/2)}{\sqrt{3/2}}+c$$

$$\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1}\frac{2y+1}{\sqrt{3}}=c$$

7. $\frac{dy}{dx}=\tan^2(x+y)$

Sol. $\frac{dy}{dx}=\tan^2(x+y)$ put v = x + y

$$\frac{dv}{dx}=1+\frac{dy}{dx}=1+\tan^2 v=\sec^2 v$$

sakshieducation.com

$$\int \frac{dv}{\sec^2 v} = \int dx = \int \cos^2 v \cdot dv = x + c$$

$$\int \frac{(1+\cos 2v)}{2} dv = x + c$$

$$\Rightarrow \int (1+\cos 2v) dv = 2x + 2c$$

$$v + \frac{\sin 2v}{2} = 2x + 2c$$

$$2v + \sin 2v = 4x + c'$$

$$2(x+y) + \sin 2(x+y) = 4x + c'$$

$$x - y - \frac{1}{2}\sin[2(x+y)] = c$$

Homogeneous Equations

Homogeneous Differential Equations:

A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be a homogeneous differential equation in x, y if both f(x, y), g(x, y) are homogeneous functions of same degree in x and y.

To find the solution of the h .d.e put y = vx, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting these values in given differential equation, then it reduces to variable separable form. Then we find the solution of the d.e.

Very Short Answer Questions

1. Express $xdy - ydx = \sqrt{x^2 + y^2}dx$ **in the form** $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

Sol. $x \cdot dy - ydx = \sqrt{x^2 + y^2}dx$

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2} \Rightarrow x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2 + y^2}{x^2}} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Which is of the form $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$

2. Express $\left(x - y\text{Tan}^{-1}\frac{y}{x}\right)dx + x\tan^{-1}\frac{y}{x}dy = 0$ **in the form** $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

Sol. Given $\left(x - y\text{Tan}^{-1}\frac{y}{x}\right)dx + x\tan^{-1}\frac{y}{x}dy = 0$

$$x\tan^{-1}\left(\frac{y}{x}\right)dy = -\left(x - y\tan^{-1}\frac{y}{x}\right)dx$$

$$\tan^{-1}\left(\frac{y}{x}\right)\frac{dy}{dx} = -\left(1 - \frac{y}{x}\tan^{-1}\frac{y}{x}\right)$$

$$= \frac{y}{x}\tan^{-1}\left(\frac{y}{x}\right) - 1$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} \cdot \tan^{-1}\left(\frac{y}{x}\right) - 1}{\tan^{-1}\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$$

3. Express $x \cdot \frac{dy}{dx} = y(\log y - \log x + 1)$ **in the form** $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

Sol. $x \cdot \frac{dy}{dx} = y(\log y - \log x + 1)$

$$\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$$

sakshieducation.com

Short Answer Questions

Solve the following differential equations.

1. $\frac{dy}{dx}=\frac{x-y}{x+y}$

Sol. $\frac{dy}{dx}=\frac{x-y}{x+y}$ ----- (1)

(1) is a homogeneous d.e.

Put y = vx

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{x-vx}{x+vx}=\frac{x(1-v)}{x(1+v)}$$

$$x\cdot\frac{dv}{dx}=\frac{1-v}{1+v}-v=\frac{1-v-v-v^2}{1+v}=\frac{1-2v-v^2}{1+v}$$

$$\int\frac{(1+v)dv}{1-2v-v^2}=\int\frac{dx}{x}$$

$$-\frac{1}{2}\log(1-2v-v^2)=\log x+\log c$$

$$-\frac{1}{2}\log\left(1-2\cdot\frac{y}{x}-\frac{y^2}{x^2}\right)=\log cx$$

$$\log\frac{(x^2-2xy-y^2)}{x^2}=-2\log cx=\log(cx)^{-2}$$

$$\frac{x^2-2xy-y^2}{x^2}=(cx)^{-2}=\frac{1}{c^2x^2}$$

$$(x^2-2xy-y^2)=\frac{1}{c^2}=k$$

2. $(x^2 + y^2)dy = 2xy\,dx$

Sol. $\frac{dy}{dx}=\frac{2xy}{x^2+y^2}$ which is a homogeneous d.e.

Put y = vx

$$\frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

$$v+x\cdot\frac{dv}{dx}=\frac{2x(vx)}{x^2+v^2x^2}=\frac{2v}{1+v^2}$$

$$x\cdot\frac{dv}{dx}=\frac{2v}{1+v^2}-v=\frac{2v-v-v^3}{1+v^2}=\frac{v-v^3}{1+v^2}$$

$$\int\frac{1+v^2}{v(1-v^2)}dv=\int\frac{dx}{x}$$

Let $\frac{1+v^2}{v(1-v^2)}=\frac{A}{v}+\frac{B}{1+v}+\frac{C}{1-v}$

$1+v^2=A(1-v^2)+BV(1-v)+CV(1+v)$

$v=0\Rightarrow 1=A$

$v=1\Rightarrow 1+1=C(2)\Rightarrow c=1$

$v=-1\Rightarrow 1+1=B(-1)(2)\Rightarrow 2=-2B\Rightarrow B=-1$

$$\int\frac{1+v^2}{v(1-v^2)}dv=\int\frac{dv}{v}-\int\frac{dv}{1+v}+\int\frac{dv}{1-v}$$

$$=\log v-\log(1+v)-\log(1-v)=\log\frac{v}{1-v^2}$$

$$\therefore \log\frac{v}{1-v^2}=\log x+\log c=\log cx$$

$$\frac{v}{1-v^2}=cx\Rightarrow v=cx(1-v^2)$$

$$\frac{y}{x}=cx\left(1-\frac{y^2}{x^2}\right)\Rightarrow\frac{y}{x}=cx\frac{(x^2-y^2)}{x^2}$$

Solution is: $y=c(x^2-y^2)$

sakshieducation.com

3. $\frac{dy}{dx}=\frac{-(x^2+3y^2)}{(3x^2+y^2)}$

Sol. $\frac{dy}{dx}=\frac{-(x^2+3y^2)}{(3x^2+y^2)}$ which is a homogeneous d.e.

Put y = vx

$$\frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

$$v+x\cdot\frac{dv}{dx}=\frac{-(x^2+3v^2x^2)}{3x^2+v^2x^2}=\frac{-x^2(1+3v^2)}{x^2(3+v^2)}$$

$$x\cdot\frac{dv}{dx}=-v-\frac{1+3v^2}{3+v^2}$$

$$=\frac{-3v-v^3-1-3v^2}{3+v^2}=-\frac{(v+1)^3}{3+v^2}$$

$$\frac{3+v^2}{(v+1)^3}=\frac{-dx}{x}$$

$$\frac{3+v^2}{(v+1)^3}=\frac{A}{v+1}+\frac{B}{(v+1)^2}+\frac{C}{(v+1)^3}$$

Multiplying with $(v + 1)^3$

$3 + v^2 = A(v + 1)^2 + B(v + 1) + C$

$v = -1 \Rightarrow 3 + 1 = C \Rightarrow C = 4$

Equating the coefficients of v^2

A = 1

Equating the coefficients of V

0 = 2A + B

B = –2A = –2

$$\frac{v^2+3}{(v+1)^3}=\frac{1}{v+1}-\frac{2}{(v+1)^2}+\frac{4}{(v+1)^3}$$

$$\int\frac{v^2+3}{(v+1)^3}=-\int\frac{dx}{x}$$

$$\int\left(\frac{1}{v+1}-\frac{2}{(v+1)^2}+\frac{4}{(v+1)^3}\right)dv=-\log x+\log c\ \log(v+1)+\frac{2}{v+1}-\frac{4}{2(v+1)^2}=\log\frac{c}{x}$$

Solution is:

$$\log\left(\frac{y}{x}+1\right)+\frac{2}{\frac{y}{x}+1}-\frac{2}{\left(\frac{y}{x}+1\right)^2}=\log\frac{c}{x}$$

$$\frac{2x}{x+y}-\frac{2x^2}{(x+y)^2}=\log\frac{c}{x}-\log\frac{(x+y)}{x}$$

$$\frac{2x^2+2xy-2x^2}{(x+y)^2}=\log\frac{c}{x+y}$$

$$\frac{2xy}{(x+y)^2}=\log\frac{c}{x+y}$$

$$\log\left(\frac{x+y}{c}\right)c=-\log\left(\frac{c}{x+y}\right)=-\frac{2xy}{(x+y)^2}$$

4. y^2 dx + (x^2 – xy)dy = 0

Sol. y^2 dx + (x^2 – xy)dy = (xy – x^2)dy

$$\frac{dy}{dx}=\frac{y^2}{xy-x^2}$$ Which is a homogeneous d.e.

Let $y=vx\Rightarrow\frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{v^2x^2}{x^2(v-v^2)}$$

$$x\cdot\frac{dv}{dx}=\frac{v^2}{v-1}-v=\frac{v^2-v^2+v}{v-1}$$

$$\frac{v-1}{v}dv=\frac{dx}{x}\Rightarrow\int\left(1-\frac{1}{v}\right)dv=\int\frac{dx}{x}$$

$$v-\log v=\log x+\log k$$
$$v=\log v+\log x+\log k=\log k(vx)$$

$$\frac{y}{x}=\log ky\Rightarrow ky=e^{y/x}$$

5. $\frac{dy}{dx} = \frac{(x+y)^2}{2x^2}$

Sol,. $\frac{dy}{dx} = \frac{(x+y)^2}{2x^2}$ Which is a homogeneous d.e.

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{(x+vx)^2}{2x^2} = x^2\frac{(1+v)^2}{2x^2}$$

$$x\frac{dv}{dx} = \frac{(1+v^2)}{2} - v = \frac{1+v^2+2v-2v}{2}$$

$$2\int\frac{dv}{1+v^2} = \int\frac{dx}{x} \Rightarrow 2\tan^{-1}v = \log x + \log c$$

$$2\tan^{-1}\left(\frac{y}{x}\right) = \log cx$$

6. $\mathbf{(x^2 - y^2)dx - xy\ dy = 0}$

Sol. $(x^2 - y^2)dx - xy\ dy = 0$

$$(x^2 - y^2)dx = xy\ dy$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{vx^2} = \frac{x^2(1-v^2)}{vx^2}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-v^2-v^2}{v} = \frac{1-2v^2}{v}$$

$$\int\frac{vdv}{1-2v^2} = \int\frac{dx}{x}$$

$$-\frac{1}{4}\log(1-2v^2) = \log x + \log c$$

$$-\frac{1}{4}\log\left(1-\frac{2y^2}{x^2}\right)=\log x+\log c$$

$$-\frac{1}{4}\log\left(\frac{x^2-2y^2}{x^2}\right)=\log x+\log c$$

$$-\frac{1}{4}\left[\log(x^2-2y^2)-\log x^2\right]=\log x+\log c$$

$$-\frac{1}{4}\log(x^2-2y^2)+\frac{1}{4}\cdot 2\log x=\log x+\log c$$

$$-\frac{1}{4}\log(x^2-2y^2)=\frac{1}{2}\log x+\log c$$

$$-\log(x^2-2y^2)=-2\log x-4\log c$$

$$\log(x^2-2y^2)=-2\log x+\log k \text{ where } k=\frac{1}{c^4}=\log\frac{k}{x^2}\Rightarrow x^2-2y^2=\frac{k}{x^2}$$

Solution is: $x^2(x^2-2y^2)=k$

7. $(x^2y-2xy^2)dx=(x^3-3x^2y)dy$

Sol. $(x^2y-2xy^2)dx=(x^3-3x^2y)dy$

$\frac{dy}{dx}=\frac{x^2y-2xy^2}{x^3-3x^2y}$ Which is a homogeneous d.e.

Put $y=vx$ so that $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{x^3v-2v^2x^3}{x^3-3vx^3}$$

$$=\frac{(v-2v^2)x^3}{(1-3v)x^3}=\frac{v-2v^2}{1-3v}$$

$$x\frac{dv}{dx}=\frac{v-2v^2}{1-3v}-v$$

$$=\frac{v-2v^2-v(1-3v)}{1-3v}=\frac{-2v^2+3v^2}{1-3v}$$

$$x\frac{dv}{dx}=\frac{v^2}{1-3v}\Rightarrow\frac{1-3v}{v^2}dv=\frac{dx}{x}$$

$$\int\left(\frac{1}{v^2}-\frac{3}{v}\right)dv=\int\frac{dx}{x}$$

$$\frac{-1}{v}-3\log v=\log x+\log c$$

$$\frac{-x}{y}=3\log\left(\frac{y}{x}\right)=\log x+\log c$$

$$\frac{-x}{y}-\log\left(\frac{y}{x}\right)^3=\log xc$$

$$\frac{-x}{y}=\log xc+\log\frac{y^3}{x^3}$$

$$\frac{-x}{y}=\log\left(cx\cdot\frac{y^3}{x^3}\right)=\log\left(\frac{cy^3}{x^2}\right)$$

$$\frac{cy^3}{x^2}=e^{-x/y}\Rightarrow cy^3=\frac{x^2}{e^{x/y}}$$

$$cy^3\cdot e^{x/y}=x^2$$

8. $y^2\,dx+(x^2-xy+y^2)dy=0$

Sol. $y^2\,dx=-(x^2-xy+y^2)dy$

$$\frac{dy}{dx}=\frac{-y^2}{x^2-xy+y^2}$$

$$y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x^2-vx^2+v^2x^2}=\frac{-v^2x^2}{x^2(1-v+v^2)}$$

$$x\frac{dv}{dx}=\frac{-v^2}{1-v+v^2}-v$$

$$=\frac{-v^2-v+v^2-v^3}{1-v+v^2}=-\frac{v(1+v^2)}{1-v+v^2}$$

$$\frac{1-v+v^2}{v(1+v^2)}dv=-\frac{dx}{x}\qquad\ldots(1)$$

Let $\dfrac{1-v+v^2}{v(1+v^2)}=\dfrac{A}{v}+\dfrac{Bv+C}{1+v^2}$

$$1-v+v^2 = A(1+v^2)+(Bv+C)v$$

$v = 0 \Rightarrow 1 = A$

Equating the coefficients of v^2

$1 = A + B \Rightarrow B = 0$

Equating the coefficients of v

$$-1 = C \Rightarrow \frac{1-v+v^2}{v(1+v^2)} = \frac{1}{v} - \frac{1}{1+v^2}$$

$$\int \frac{1-v+v^2}{v(1+v^2)}dv = \int \frac{dv}{v} - \int \frac{dv}{1+v^2} = \log v - \tan^{-1} v$$ From (1) we get

$$\log v - \tan^{-1} v = -\log x + \log c$$

$$\tan^{-1} v = \log v + \log x - \log c = \log \frac{vx}{c} = \log \frac{y}{c}$$

$$\frac{y}{c} = e^{\tan^{-1} v} = e^{\tan^{-1}(y/x)}$$

Solution is: $y = c \cdot e^{\tan^{-1}(y/x)}$

9. $(y^2 - 2xy)dx + (2xy - x^2)dy = 0$

Sol. $(y^2 - 2xy)dx + (2xy - x^2)dy = 0$

$(2xy - x^2)dy = -(y^2 - 2xy)dx$

$$\frac{dy}{dx} = \frac{2xy - y^2}{2xy - x^2}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{2vx^2 - v^2x^2}{2vx^2 - x^2} = \frac{x^2(2v - v^2)}{x^2(2v-1)}$$

$$x\frac{dv}{dx} = \frac{2v - v^2}{2v-1} - v$$

$$= \frac{2v - v^2 - 2v^2 + v}{2v-1} = \frac{3v(1-v)}{2v-1}$$

$$\int \frac{2v-1}{v(1-v)}dv = 3\int \frac{dx}{x} \qquad \ldots (1)$$

Let $\frac{2v-1}{v(1-v)} = \frac{A}{v} + \frac{B}{1-v}$

$2v-1 = A(1-v) + Bv$

$v = 0 \Rightarrow -1 = A \Rightarrow A = -1$

$v = 1 \Rightarrow 1 = B \Rightarrow B = 1$

$$\int\left(-\frac{1}{v} + \frac{1}{1+v}\right)dv = 3\int \frac{dx}{x}$$

$-\log v - \log(1-v) = 3\log x + \log c$

$\log \frac{1}{v(1-v)} = \log cx^3$

$\frac{1}{v(1-v)} = cx^3 \Rightarrow v(1-v) = \frac{1}{cx^3}$

$\frac{y}{x}\left(1-\frac{y}{x}\right) = \frac{1}{cx^3} \Rightarrow \frac{y}{x}\left(\frac{x-y}{x}\right) = \frac{1}{cx^3}$

$xy(x-y) = \frac{1}{c} = k \Rightarrow xy(y-x) = -\frac{1}{c} = k$

10. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Sol. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ Which is a homogeneous d.e.

Put y = vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} + v = \frac{v^2x^2}{x^2} \Rightarrow x\frac{dv}{dx} = v^2 - 2v$

$\frac{dv}{v^2-2v} = \frac{dy}{x}$

Let $\frac{1}{v^2-2v} = \frac{A}{v} + \frac{B}{v-2}$

$1 = A(v-2) + Bv$

$$v=0\Rightarrow 1=A(-2)\Rightarrow -\frac{1}{2}$$

$$v=2\Rightarrow 1=2B\Rightarrow B=\frac{1}{2}$$

$$-\frac{1}{2}\int\left(\frac{1}{v}-\frac{1}{v-2}\right)dv=\int\frac{dx}{x}$$

$$-\frac{1}{2}[\log v-\log(v-2)]=\log x+\log c$$

$$-\frac{1}{2}\left[\log\frac{v}{v-2}\right]=\log cx$$

$$\log\frac{v}{v-2}=-\log cx=\log(cx)^{-2}$$

$$\frac{v}{v-2}=(cx)^{-2}\Rightarrow\frac{(y/x)}{(y/x)-2}=\frac{1}{c^2x^2}$$

$$\frac{y}{y-2x}=\frac{1}{c^2x^2}\Rightarrow x^2y=\frac{1}{c^2}(y-2x)$$

Solution is:

$y-2x=c^2x^2y=kx^2y$ Where $k = c^2$

11. $xdy-ydx=\sqrt{x^2+y^2}\,dx$

Sol. $xdy-ydx=\sqrt{x^2+y^2}\,dx$

$$x\frac{dy}{dx}-y=\sqrt{x^2+y^2}$$

$$\frac{dy}{dx}-\frac{y}{x}=\frac{\sqrt{x^2+y^2}}{x}$$

Put $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore x\frac{dv}{dx}=\frac{\sqrt{x^2+y^2}}{x}=\frac{x\sqrt{1+v^2}}{x}$$

$$\int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x}\Rightarrow\sinh^{-1}v=\log x+\log c$$

$$\log\left[v+\sqrt{1+v^2}\right]=\log cx \Rightarrow v+\sqrt{1+v^2}=cx$$

$$\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}=cx \Rightarrow y+\sqrt{x^2+y^2}=cx^2$$

12. (2x – y)dy = (2y – x)dx

Sol. $\frac{dy}{dx}=\frac{2y-x}{2x-y}$

Put y = vx $\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{x(2v-1)}{x(2-v)}$$

$$x\frac{dv}{dx}=\frac{2v-1}{2-v}-v=\frac{2v-1-2v+v^2}{2-v}$$

$$\frac{2-v}{v^2-1}dv=\frac{dx}{x} \Rightarrow 2\int\frac{dv}{v^2-1}-\int\frac{vdv}{v^2-1}=\int\frac{dx}{x}$$

$$2\cdot\frac{1}{2}\log\frac{v-1}{v+1}-\frac{1}{2}\log(v^2-1)=\log x+\log c$$

$$\frac{1}{2}\left(2\log\frac{v-1}{v+1}-\log(v^2-1)\right)=\log cx$$

$$\frac{1}{2}\log\frac{(v-1)^2}{(v+1)^2}\cdot\frac{1}{(v-1)(v+1)}=\log cx$$

$$\log\frac{v-1}{(v+1)^2}=2\log cx=\log c^2x^2$$

$$\therefore \frac{v-1}{(v+1)^3}=c^2x^2 \Rightarrow \frac{\frac{y}{x}-1}{\left(\frac{y}{x}+1\right)^3}=c^2x^2$$

$$\frac{\frac{y-x}{x}}{\frac{(y-x)^3}{x^3}}=c^2x^2 \Rightarrow \frac{x^2(y-x)}{(x+y)^3}=c^2x^2$$

$(y-x)=c^2(x+y)^3$.

13. $(x^2 - y^2)\frac{dy}{dx} = xy$

Sol. $\frac{dy}{dx} = \frac{xy}{(x^2 - y^2)}$

Put y = vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x(vx)}{x^2 - v^2x^2} = \frac{v}{1 - v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1 - v^2} - v = \frac{v - v + v^3}{1 - v^2} = \frac{v^3}{1 - v^2}$$

$$\frac{1 - v^2}{v^3}dv = \frac{dx}{x} \Rightarrow \int\frac{dv}{v^3} - \int\frac{dv}{v} = \int\frac{dx}{x}$$

$$-\frac{1}{2v^2} - \log v = \log x + c$$

$$-\frac{1}{2}\frac{x^2}{y^2} = \log vx + c = \log y + c$$

$$\frac{-x^2}{2y^2} = (\log y + c) \Rightarrow -x^2 = 2y^2(c + \log y)$$

$x^2 + 2y^2$ (c + log y) = 0.

14. **Solve** $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Sol. Put y = vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$2v + 2x\frac{dv}{dx} = v + v^2 \Rightarrow 2x\frac{dv}{dx} = v^2 - v$$

$$\frac{dv}{v(v-1)} = 2\frac{dx}{x} \Rightarrow \int\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = 2\int\frac{dx}{x}$$

log(v – 1) – log v = 2 logx + log c

$$\log\frac{v-1}{v} = \log cx^2 \Rightarrow \frac{v-1}{v} = cx^2$$

$$\frac{\frac{y}{x}-1}{\frac{y}{x}} = cx^2 \Rightarrow \frac{y-x}{y} = cx^2$$

Solution is : $(y - x) = cx^2y$

Long Answer Questions

1. Solve $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$.

Sol. $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$\Rightarrow \frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{x/y}\right)}$ Which is a homogeneous d.e.

Put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$(1+e^v)\frac{dx}{dy} + e^v(1-v) = 0$$

$$(1+e^v)\left(v + y\frac{dv}{dy}\right) + e^v(1-v) = 0$$

$$v + ve^v + y(1+e^v)\frac{dv}{dy} + e^v - ve^v = 0$$

$$y(1+e^v)dv = -(v+e^v)dy$$

$$\int \frac{1+e^v}{v+e^v}dv = -\int \frac{dy}{y}$$

$$\log(v+e^v) = -\log y + \log c \Rightarrow v + e^v = \frac{c}{y}$$

$$\frac{x}{y} + e^{x/y} = \frac{c}{y} \Rightarrow x + y\cdot e^{x/y} = c$$

2. Solve $x\sin\frac{y}{x}\cdot\frac{dy}{dx}=y\sin\frac{y}{x}-x$

Sol. $x\sin\frac{y}{x}\cdot\frac{dy}{dx}=y\sin\frac{y}{x}-x$

$$\Rightarrow\frac{dy}{dx}=\frac{\frac{y}{x}\left(\sin\left(\frac{y}{x}\right)-\frac{x}{y}\right)}{\sin\left(\frac{y}{x}\right)}$$

Put $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{v\left(\sin v-\frac{1}{v}\right)}{\sin v}$$

$$x\frac{dv}{dx}=\frac{v\sin v-1-v\sin v}{\sin v}$$

$$-\sin v dv=\frac{1}{x}dx$$

$$\Rightarrow\int-\sin v\cdot dv=+\int\frac{dx}{x}$$

$$\Rightarrow\cos v=\log x+\log c=\log cx$$

$$\Rightarrow cx=e^{\cos v}=e^{\cos(y/x)}.$$

3. Solve $xdy=\left(y+x\cos^2\frac{y}{x}\right)dx$**.**

Sol. $x\frac{dy}{dx}=y+x\cdot\cos^2\frac{y}{x}\Rightarrow\frac{dy}{dx}=\frac{y}{x}+\cos^2\frac{y}{x}$

Put $y=vx$,

$$\frac{dy}{dx}=v+x\frac{dv}{dx}\Rightarrow v+x\frac{dv}{dx}=v+\cos^2 v$$

$$\frac{dv}{\cos^2 v}=\frac{dx}{x}\Rightarrow\int\sec^2 v\cdot dv=\int\frac{dx}{x}$$

$\tan v=\log x+c$

i.e. $\tan\left(\frac{y}{x}\right)=\log x+c$.

sakshieducation.com

4. Solve: (x – y log y + y log x)dx + x(log y – log x)dy = 0.

Sol. Dividing with x. dx we get

$$1-\frac{y}{x}\log y+\frac{y}{x}\log x+\log\left(\frac{y}{x}\right)\frac{dy}{dx}=0$$

$$1-\frac{y}{x}\left(\log y-\log\frac{y}{x}\right)+\log\left(\frac{y}{x}\right)\frac{dy}{dx}=0$$

Put y = vx, $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$1-v[\log v]+\log v\left(v+x\frac{dv}{dx}\right)=0$$

$$1-v\log v+v\log v+x\log v\frac{dv}{dx}=0$$

$$x\log v\frac{dv}{dx}=-1\Rightarrow\int\log v\,dv=-\int\frac{dx}{x}$$

$$v\log v-\int dv=c-\log x$$

$$v\log v-v=c-\log x$$

$$v+c=v\log v+\log x\Rightarrow v+c=v\log v+\log x$$

$$\frac{y}{x}+c=\frac{y}{x}\log\left(\frac{y}{x}\right)+\log x$$

$$y+cx=y\log\left(\frac{y}{x}\right)+x\log x$$

$$=y\log y-y\log x+x\log x$$

$$=(x-y)\log x+y\log y$$

5. Solve: $(ydx+xdy)x\cos\frac{y}{x}=(xdy-ydx)y\sin\frac{y}{x}$

Sol. $(ydx+xdy)x\cos\frac{y}{x}=(xdy-ydx)y\sin\frac{y}{x}$

$$\left(xy\cdot\cos\frac{y}{x}+y^2\sin\frac{y}{x}\right)-\left(xy\cdot\sin\frac{y}{x}-x^2\cos\frac{y}{x}\right)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{xy\cdot\cos\left(\frac{y}{x}\right)+y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right)-x^2\cos\left(\frac{y}{x}\right)}$$

$$=\frac{\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\sin\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)-\cos\left(\frac{y}{x}\right)}=F\left(\frac{y}{x}\right)$$

$\therefore$ This is a homogeneous equation

$$y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}$$

$$x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}-v$$

$$=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v}$$

$$=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\frac{v\sin v-\cos v}{v\cos v}=\frac{2}{x}dv$$

$$\int\left(\tan v-\frac{1}{v}\right)dv=\int\frac{2}{x}dv$$

$$\log\sec v-\log v=2\log|x|$$

$$\therefore\log x^2=\log\left|\frac{c}{v\cos v}\right|\Rightarrow x^2=\frac{c}{v\cos v}$$

But $\frac{y}{x}=v$

Solution is:

$$x^2=\frac{c}{\frac{y}{x}\cdot\cos\left(\frac{y}{x}\right)}\Rightarrow xy\cos\left(\frac{y}{x}\right)=c$$

6. Find the equation of a curve whose gradient is $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$, where x > 0, y > 0 and which passes through the point (1, π/4).

Sol. $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ which is homogeneous d.eq.

Put y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v - \cos^2 v \Rightarrow \int\frac{dv}{\cos^2 v} = -\int\frac{dx}{x}$$

$$\int \sec^2 v = -\int\frac{dx}{x} \Rightarrow \tan v = -\log|x| + c$$

This curve passes through (1, π/4)

$$\tan\left(\frac{\pi}{4}\right) = c - \log 1 \Rightarrow c = 1$$

Equation of the curve is :

$$\tan v = 1 - \log|x| \Rightarrow \tan\left(\frac{y}{x}\right)1 - \log|x|$$

7. Solve $(1+2e^{x/y})dx + 2e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$.

Sol. $(1+2e^{x/y})dx = -2e^{x/y}\left(1-\frac{x}{y}\right)dy$

$$= 2e^{x/y}\left(\frac{x}{y}-1\right)dy$$

$$\frac{dy}{dx} = \frac{2e^{x/y}\left(\frac{x}{y}-1\right)}{1+2e^{x/y}}$$

Put x = vy

$$\frac{dx}{dy}=v+y\frac{dv}{dy}\Rightarrow v+y\frac{dv}{dy}=\frac{2e^{v}(v-1)}{1+2e^{v}}$$

$$y\frac{dv}{dy}=\frac{2e^{v}(v-1)}{1+2e^{v}}-v$$

$$=\frac{2ve^{v}-2e^{v}-v-2v}{1+2e^{v}}e^{v}=\frac{-(2e^{v}+v)}{1+2e^{v}}$$

$$\int\frac{1+2e^{v}}{v+2e^{v}}dv=-\int\frac{dy}{y}$$

$$\log(v+2e^{v})=-\log y+\log c=\log\frac{c}{y}$$

$$v+2e^{v}=\frac{c}{y}$$

Solution is: $\frac{x}{y}+2e^{x/y}=\frac{c}{y}\Rightarrow x+2y\cdot e^{x/y}=c$

8. **Solve** $x\sec\left(\frac{y}{x}\right)(ydx+xdy)=y\csc\left(\frac{y}{x}\right)(xdy-ydx)$.

Sol. $x\sec\left(\frac{y}{x}\right)(ydx+xdy)=y\csc\left(\frac{y}{x}\right)(xdy-ydx)\Rightarrow x\sec\left(\frac{y}{x}\right)\left(y+x\frac{dy}{dx}\right)=y\csc\left(\frac{y}{x}\right)\left(x\frac{dy}{dx}-y\right)$

$$x\frac{dy}{dx}\left(x\cdot\sec\left(\frac{y}{x}\right)-y\cdot\csc\left(\frac{y}{x}\right)\right)$$

$$=-y\left[y\csc\left(\frac{y}{x}\right)+x\sec\left(\frac{y}{x}\right)\right]$$

$$\frac{dy}{dx}=\frac{-y\left(y\csc\left(\frac{y}{x}\right)+x\sec\left(\frac{y}{x}\right)\right)}{x\left(x\sec\left(\frac{y}{x}\right)-y\csc\left(\frac{y}{x}\right)\right)}$$

This is a homogeneous equation.

Put $y = vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=v\left(\frac{v\csc v+\sec v}{v\csc v-\sec v}\right)$$

$$= \frac{v\left(\frac{v}{\sin v} + \frac{1}{\cos v}\right)}{\left(v\frac{1}{\sin v} - \frac{1}{\cos v}\right)} = \frac{v(v\cos v + \sin v)}{v\cos v - \sin v}$$

$$x\frac{dv}{dx} = \frac{v(v\cos v + \sin v)}{v\cos v - \sin v} - v$$

$$= \frac{v(v\cos v + \sin v - v\cos v + \sin v)}{v\cos v - \sin v}$$

$$= \frac{2v\sin v}{v\cos v - \sin v}$$

$$\int \frac{v\cos v - \sin v}{v\sin v} dv = 2\int \frac{dx}{x}$$

$$\int \frac{\cos v}{\sin v} dv - \int \frac{1}{v} dv = 2\int \frac{dx}{x}$$

$$\log \sin v - \log v = 2\log x + \log c$$

$$\log\left(\frac{\sin v}{v}\right) = \log cx^2 \Rightarrow \frac{\sin v}{v} = cx^2$$

$$\frac{x}{y}\sin\left(\frac{y}{x}\right) = cx^2 \Rightarrow \sin\left(\frac{y}{x}\right) = cxy$$

Non- Homogeneous Equations

Equations Reducible to Homogeneous Form:

The differential equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ is called non homogeneous differential equation.

Very Short Answer Questions

I. Solve the following differential equations.

1. $\frac{dy}{dx} = -\frac{12x+5y-9}{5x+2y-4}$

Sol. from given d.e.

$b = -5, a = 5 \Rightarrow b = -a$

$(5x + 2y - 4)dy = -(12x + 5y - 9)dx$

$(5x + 2y - 4)dy + (12x + 5y - 9)dx = 0$

$5(x\,dy + y\,dx) + 2y\,dy - 4\,dy + 12x\,dx - 9\,dx = 0$

Integrating $5xy + y^2 - 4y + 6x^2 - 9x = c.$

2. $\frac{dy}{dx} = \frac{-3x-2y+5}{2x+3y+5}$

Sol. From given d.e.

$b = -2, a = 2 \Rightarrow b = -a$

$(2x + 3y + 5)dy = (-3x - 2y + 5)dx$

$2x\,dy + 3y\,dy + 5dy = -3x\,dx - 2y\,dx + 5\,dx$

$2x\,dy + 3y\,dy + 5dy + 3x\,dx - 2y\,dx + 5\,dx = 0$

Integrating

$$2xy + \frac{3}{2}y^2 + \frac{3}{2}x^2 + 5y - 5x = c$$

$$4xy + 3y^2 + 3x^2 - 10x + 10y = 2c = c'$$

Solution is

$4xy + 3(x^2 + y^2) - 10(x - y) = c.$

3. $\frac{dy}{dx}=\frac{-3x-2y+5}{2x+3y-5}$

Sol. $\frac{dy}{dx}=\frac{-3x-2y+5}{2x+3y-5}$

Here b = –2, a′ = 2, b = –a′

(2x + 3y – 5)dy = (–3x – 2y + 5)dx

⇒ 2(x dy+y dx) + (3y – 5)dy + (3x – 5)dx = 0

⇒ 2d xy + (3y – 5)dy + (3x – 5)dx = 0

Now integrating term by term, we get

$$2\int d(xy)+\int(3y-5)dy+\int(3x-5)dx=0$$

$$\Rightarrow 2xy+3\frac{y^2}{2}-5y+3\frac{x^2}{2}-5x=\frac{c}{2}$$

$$(or) 3x^2+4xy+3y^2-10x-10y=c$$

Which is the required solution.

4. $2(x-3y+1)\frac{dy}{dx}=4x-2y+1$

Sol. (2x – 6y + 2)dy = (4x – 2y + 1)dx

(2x – 6y + 2)dy – (4x – 2y + 1)dx = 0

2(x dy + y dx) – 6y dy + 2 dy – 4x dx – dx = 0

Integrating

$2xy-3y^2-2x^2+2y-x=c.$

5. $\frac{dy}{dx}=\frac{x-y+2}{x+y-1}$

Sol. b = –1, a′ = a ⇒ b = –a′

(x + y – 1)dy = (x – y + 2)dx

(x + y – 1)dy – (x – y + 2)dx = 0

(x dy + y dx) + y dy – dy – x dx – 2 dx = 0

sakshieducation.com

Integrating

$$xy + \frac{y^2}{2} - \frac{x^2}{2} - y - 2x = c$$

$$2xy + y^2 - x^2 - 2y - 4x = 2c = c'$$

6. $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$

Sol. $b = -1,\ a' = 1 \Rightarrow b = -a'$

$(x + 2y - 3)dy = (2x - y + 1)dx$

$(x + 2y - 3)dy - (2x - y + 1)dx = 0$

$(x\,dy + y\,dx) + 2y\,dy - 3\,dy - 2x\,dx - dx = 0$

Integrating:

$xy + y^2 - x^2 - 3x - x = c$

sakshieducation.com

Short Answer Questions

Solve the following differential equations.

1. $(2x + 2y + 3)\dfrac{dy}{dx} = x + y + 1$

Sol. $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3} = \dfrac{(x+y)+1}{2(x+y)+3}$

Let v = x + y so that $\dfrac{dv}{dx} = 1 + \dfrac{dy}{dx}$

$$\frac{dv}{dx} = 1 + \frac{v+1}{2v+3} = \frac{2v+3+v+1}{2v+3} = \frac{3v+4}{2v+3}$$

$$\frac{2v+3}{3v+4}dv = dx$$

$$\frac{2}{3}\int dv + \frac{1}{9}\int\frac{3\cdot dv}{3v+4} = \int dx$$

$$\frac{2}{3}v + \frac{1}{9}\log(3v+4) = x + c$$

Multiplying with 9

$6v + \log(3v + 4) = 9x + 9c$

$6(x + y) + \log[3(x + y) + 4] = 9x + c$

i.e. $\log(3x + 3y + 4) = 3x - 6y + c$

2. $\dfrac{dy}{dx} = \dfrac{4x+6y+5}{2x+3y+4}$

Sol. $\dfrac{dy}{dx} = \dfrac{4x+6y+5}{2x+3y+4} = \dfrac{2(2x+3y)+5}{2x+3y+4}$

Let v = 2x + 3y

$$\frac{dv}{dx} = 2 + 3\frac{dy}{dx} \Rightarrow \frac{dv}{dx} = 2 + \frac{3(2v+5)}{v+4}$$

$$= \frac{2v+8+6v+15}{v+4} = \frac{8v+23}{v+4}$$

$$\frac{v+4}{8v+23}dv = dx$$

$$\frac{1}{8}\int dv + \frac{9}{8}\int \frac{dv}{8v+23} = \int dx$$

$$\frac{1}{8}v + \frac{9}{64}\log(8v+23) = x + c$$

Multiplying with 64

8v + 9 log(8v + 23) = 64x + 64c

8(2x + 3y) – 64x + 9log(16x + 24y + 23) = c′

Dividing with 8

$$2x + 3y - 8x + \frac{9}{8}\log(16x+24y+23) = c''$$

$$3x - 6x + \frac{9}{8}\log(16x+24y+23) = c''$$

Dividing with 3, solution is :

$$y - 2x + \frac{3}{8}\log(16x+24y+23) = k$$

3. (2x + y + 1)dx + (4x + 2y – 1)dy = 0

Sol. $\frac{dy}{dx} = -\frac{2x+y+1}{4x+2y-1}$

$$\Rightarrow a_1 = 2, b_1 = 1, a_2 = 4, b_2 = 2$$

$$\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2} = \frac{b_1}{b_2}$$

Let 2x + y=v so that $\frac{dv}{dx} = 2 + \frac{dy}{dx}$

$$\frac{dv}{dx} = 2 - \frac{v+1}{2v-1} = \frac{4v-2-v-1}{2v-1} = \frac{3(v-1)}{2v-1}$$

$$\frac{2v-1}{3(v-1)}dv = dx \Rightarrow \frac{2v-1}{v-1}dv = 3dx$$

$$\int\left(2+\frac{1}{v-1}\right)dv = 3\int dx$$

$$2v+\log(v-1) = 3x+c$$
$$2v-3x+\log(v-1) = c$$
$$2(2x+y)-3x+\log(2x+y-1) = c$$
$$4x+2y-3x+\log(2x+y-1) = c$$

Solution is x + 2y + log(2x + y – 1) = c

4. $\frac{dy}{dx} = \frac{2y+x+1}{2x+4y+3}$

Sol. $\frac{dy}{dx} = \frac{2y+x+1}{2x+4y+3}$

Let v = x + 2y so that $\frac{dv}{dx} = 1+\frac{2dy}{dx}$

$$\frac{dv}{dx} = 1+\frac{2(v+1)}{2v+3} = \frac{2v+3+2v+2}{2v+3} = \frac{4v+5}{2v+3}$$

$$\frac{2v+3}{4v+5}dv = dx \Rightarrow \int\left(\frac{1}{2}+\frac{1}{2(4v+5)}\right)dv = \int dx$$

$$\frac{1}{2}v+\frac{1}{2}\cdot\frac{1}{4}\log(4v+5) = x+c$$

Multiplying with 8

$$4v+\log(4v+5) = 8x+8c$$
$$4(x+2y)-8x+\log[4(x+2y)+5] = c'$$

Solution is:

4x + 8y – 8x + log(4x + 8y + 5) = c′

8y – 4x + log(4x + 8y + 5) = c′

5. (x + y – 1)dy = (x + y + 1)dx

Sol. $\frac{dy}{dx}=\frac{x+y+1}{x+y-1}$

$v=x+y\Rightarrow\frac{dv}{dx}=1+\frac{dy}{dx}$

$\frac{dv}{dx}=1+\frac{v+1}{v-1}=\frac{v-1+v+1}{v-1}=\frac{2v}{v-1}$

$\int\frac{v-1}{v}dv=2\int dx\Rightarrow\int\left(1-\frac{1}{v}\right)dv=2x+c$

$v-\log v=2x+c$

$x+y-\log(x+y)=2x-c$

(x – y) + log (x + y) = c is the required solution.

sakshieducation.com

Long Answer Questions

Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{3y - 7x + 7}{3x - 7y - 3}$

Sol.

$\Rightarrow a_1 = -7, b_1 = 3, a_2 = 3, b_2 = -7$

$\frac{a_1}{a_2} = \frac{-7}{3}, \frac{b_1}{b_2} = \frac{3}{-7} \quad \therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Let x = x + h, y = y + k

Where $3k - 7h + 7 = 0$ and $3h - 7k - 3 = 0$ and $\frac{dy}{dx} = \frac{dY}{dX}$

Solving these equations,

h = 0 and k = 1

$\frac{dy}{dx} = \frac{3y - 7x}{3x - 7y}$ which is a homogeneous d.e.

Put y = vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{x(3v - 7)}{x(3 - 7v)}$

$x\frac{dv}{dx} = \frac{3v - 7}{3 - 7v} - v = \frac{3v - 7 - 3v + 7v^2}{3 - 7v}$

$= \frac{7v^2 - 7}{3 - 7v} = \frac{7v^2 - 7}{3 - 7v}$

$\frac{3 - 7v}{7v^2 - 7} = \frac{dx}{x}$

$\int \frac{3}{7v^2 - 7} dv - \int \frac{7v dv}{7v^2 - 7} = \int \frac{dx}{x}$

$\ln x = \frac{3}{14}\ln\left|\frac{v - 1}{v + 1}\right| - \frac{1}{2}\ln|v^2 - 1|$14 log x–log c

$x = 3\log\left|\frac{v - 1}{v + 1}\right| - 7\log|v^2 - 1| \Rightarrow 14\ln x - \ln c$

$$= 3\ln(v-1) - 3\ln(v+1) - 7\ln(v+1) - 7\ln(v-1)$$

$$14\ln x - \ln c = -10\ln(v+1) - 4\ln(v-1)$$

$$\ln(v+1)^5 + \ln(v-1)^2 + \ln x^7 = \ln c$$

$$(v+1)^5 \cdot (v-1)^2 \cdot x^7 = c$$

$$\left(\frac{y}{x}+1\right)^5 \left(\frac{y}{x}-1\right)^2 x^7 = c$$

$$(y-x)^2 (y+x)^5 = c$$

$$[y-(x-1)]^2 (y+x-1)^5 = c$$

Solution is $[y-x+1)]^2 (y+x-1)^5 = c$

2. $\dfrac{dy}{dx} = \dfrac{6x+5y-7}{2x+18y-14}$

Sol. $\dfrac{dy}{dx} = \dfrac{6x+5y-7}{2x+18y-14}$

x = X + h, y = Y + k

$$\frac{dY}{dX} = \frac{dy}{dx} = \frac{6(X+h)+5(Y+k)-7}{2(X+h)+18(Y+k)-14}$$

$$= \frac{(6X+5Y)+(6h+5k-7)}{(2X+18Y)+(2h+18k-14)}$$

h k 1

+5 −7 +6 5

18 −14 +2 18

$$\frac{h}{-70+126} = \frac{k}{-14+84} = \frac{1}{108-10}$$

$$h = \frac{56}{98} = \frac{4}{7}, k = \frac{70}{98} = \frac{5}{7}$$

$$\frac{dY}{dX} = \frac{6X+5Y}{2X+18Y}$$

$$Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{6X+5VX}{2X+18VX} = \frac{X(6+5V)}{X(2+18V)}$$

$$X\frac{dV}{dX} = \frac{6+5V}{2+18V} - V = \frac{6+5V-2V-18V^2}{2+18V}$$

sakshieducation.com

$$= \frac{6+3V-18V^2}{2+18V} = \frac{3(2+V-6V^2)}{2+18V}$$

$$\int \frac{2+18V}{6V^2-V-2} dV = -3\int \frac{dX}{X}$$

Let $\frac{2+18V}{6V^2-V-2} = \frac{A}{3V-2} + \frac{B}{2V+1}$

Multiplying with (3V – 2)(2V + 1)

2 + 18V = A(2V + 1) + B(3V – 2)

$$V = \frac{2}{3} \Rightarrow 2+12 = A\left(\frac{4}{3}+1\right)$$

$$14 = A - \frac{7}{3} \Rightarrow A = 6$$

$$V = -\frac{1}{2} \Rightarrow 2-9 = B\left(-\frac{3}{2}-2\right)$$

$$-7 = -\frac{7}{2}B \Rightarrow B = 2$$

$$\int \left(\frac{6}{3V-2} + \frac{2}{2V+1}\right) dV = -3\int \frac{dX}{X}$$

2 log(3V – 2) + log(2V + 1) = –3 log X + log c

log $(3V-2)^2$ (2V + 1) + log X^3 = log c

log $X^3 (3V-2)^2$ (2V + 1) = log c

$X^3 (3V-2)^2 (2V+1) = c$

$$X^3\left(\frac{3Y}{X}-2\right)^2\left(\frac{2Y}{X}+1\right) = c$$

$$X^3 \frac{(3Y-2X)^2}{X^2} \frac{(2Y+X)}{X} = c$$

$$\left(3\left(y-\frac{5}{7}\right)-2\left(x-\frac{4}{7}\right)\right)^2 \left(2\left(y-\frac{5}{7}\right)+\left(x-\frac{4}{7}\right)\right) = c$$

$$\frac{(3y-2x-1)^2}{7^2} \frac{(2y+x-2)}{7} = c$$

Solution is:

$(3y - 2x - 1)^2 (x + 2y - 2) = 343c = c''$

3. $\frac{dy}{dx} + \frac{10x + 8y - 12}{7x + 5y - 9} = 0$

Sol. $\frac{dy}{dx} + \frac{10x + 8y - 12}{7x + 5y - 9} = 0$

$x = X + h,\ y = Y + k$

$\Rightarrow \frac{dY}{dX} = \frac{dy}{dx}$

$$\frac{dY}{dX} + \frac{10(X+h) + 8(Y+k) - 12}{7(X+h) + 5(Y+k) - 9} = 0$$

$$\frac{dY}{dX} + \frac{(10X + 8Y) + (10h + 8k - 12)}{(7X + 5Y) + (7h + 5k - 9)}$$

Choose h and k so that:

10h + 8k – 12 = 0, 7h + 5k – 9 = 0

h		k		1
+8	–12	10	8	
+5	–9	+7	5	

$$\frac{h}{-72 + 60} = \frac{k}{-84 + 90} = \frac{1}{50 - 56}$$

$$h = \frac{-12}{-6} = 2, k = \frac{6}{-6} = -1$$

$$\frac{dY}{dX} = -\frac{10X + 8Y}{7X + 5Y}$$

$$Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = -\frac{10X + 8VX}{7X + 5VX} = -\frac{X(10 + 8V)}{X(7 + 5V)}$$

$$X\frac{dV}{dX} = -\frac{10 + 8V}{7 + 5V} - V = \frac{-10 - 8V - 7V - 5V^2}{7 + 5V}$$

$$X\frac{dV}{dX} = -\frac{5(V^2 + 3V + 2)}{7 + 5V}$$

sakshieducation.com

$$\int \frac{5V+7}{(V+1)(V+2)}dV = -5\int \frac{dX}{X}$$

$$\frac{5V+7}{(V+1)(V+2)} = \frac{A}{V+1} + \frac{B}{V+2}$$

$$5V+7 = A(V+2) + B(V+1)$$
$$V = -1 \Rightarrow 2 = A(-1+2) = A \Rightarrow A = 2V$$
$$V = -2 \Rightarrow -3 = B(-2+1) = -B, B = 3$$

$$\int\left(\frac{2}{V+1} + \frac{3}{V+2}\right)dV = -5\int \frac{dX}{X}$$

$2\log(V + 1) + 3\log(V + 2) = -5\log X + c$

$c = 2\log(V + 1) + 3\log(V + 2) + 5\log X = \log(V + 1)^2 (V + 2)^3 x^5$

$$\log\left(\frac{Y}{X}+1\right)^2\left(\frac{Y}{X}+2\right)^3 X^5 = \log\frac{(Y+X)^2}{X^2}\frac{(Y+2X)^3}{X^3}X^5$$

$$\Rightarrow (Y+X)^2(Y+2X)^3 = e^c = c'$$
$$(Y+1-X-2)^2(Y+1-2x-4)^3 = c$$

Solution is : $(x + y - 1)^2 (2x + y - 3)^3 = c$

4. $(x - y - 2)dx + (x - 2y - 3)dy = 0$

Sol. Given equation is $\frac{dy}{dx} = \frac{-x+y+2}{x-2y-3}$

Let $x = X + h$, $y = Y + k$

$$\frac{dy}{dx} = \frac{-(X+h)+(Y+k)+2}{(X+h)-2(Y+k)-3} = \frac{-X+y+(k-h+2)}{(X-2Y)+(h-2k-3)}$$

Choose h and k so that:

$-h + k + 2 = 0$, $h - 2k - 3 = 0$

h k I

−1 −2 1 −1
−2 −3 1 −2

$$\frac{h}{+3-4} = \frac{k}{-2+3} = \frac{1}{-2+1}$$

$h = 1$, $k = -1$

$$\frac{dY}{dX}=\frac{-X+Y}{X-2Y}$$

Put Y = VX so that $\frac{dY}{dX}=V+X\frac{dV}{dX}$

$$V+X\frac{dV}{dX}=\frac{X(-1+V)}{X(1-2V)}=\frac{-1+V}{1-2V}$$

$$X\frac{dV}{dX}=\frac{-1+V}{1-2V}-V=\frac{-1+V-V+2V^2}{1-2V}=\frac{2V^2-1}{1-2V}$$

$$\int\frac{(1-2V)dV}{2V^2-1}=\int\frac{dX}{X}$$

$$\int\frac{dX}{X}=\int\frac{-\frac{1}{2}(-4V)-1}{1-2V^2}dV$$

$$=\frac{1}{2}\int\frac{(-4VdV)}{1-2V^2}-\int\frac{dV}{1-2V^2}$$

$$\log|x|=-\frac{1}{2}\log|1-2V^2|-\frac{1}{2}\int\frac{dV}{\left(\frac{1}{\sqrt{2}}\right)^2-V^2}$$

$$=-\frac{1}{2}\log|1-2V^2|-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+V}{\frac{1}{\sqrt{2}}-V}\right|+\log c\ \ 2\log|x|=-\log|1-2V^2|$$

$$-\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\log\left|\frac{\frac{1}{\sqrt{2}}+V}{\frac{1}{\sqrt{2}}-V}\right|+\log C$$

$$2\log|x|+\log|1-2V^2|=-\frac{1}{\sqrt{2}}\log\left|\frac{1+V\sqrt{2}}{1-V\sqrt{2}}\right|+\log c$$

$$\log X^2(1-2V^2)=-\frac{1}{\sqrt{2}}\log\left|\frac{X+Y\sqrt{2}}{X-Y\sqrt{2}}\right|+\log c$$

$$\log|X^2-2Y^2|=\log c\left(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right)^{1/\sqrt{2}}$$

$$\therefore X^2-2Y^2=c\left(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right)^{1/\sqrt{2}}$$

sakshieducation.com

Substituting X = x–h = x – 1 , Y = y–k = y + 1

$$(x-1)^2 - 2(y+1)^2 = c\left(\frac{x-1-(y+1)\sqrt{2}}{x-1+(y+1)\sqrt{2}}\right)^{1/\sqrt{2}}$$

$$(x^2 - 2y^2 - 2x - 4y - 1) = c\left(\frac{x - y\sqrt{2} - 1 - \sqrt{2}}{x + y\sqrt{2} - 1 + \sqrt{2}}\right)^{1/\sqrt{2}}$$

5. (x – y)dy = (x + y + 1) dx

Sol. $\frac{dy}{dx} = \frac{x+y+1}{x-y}$

x = X + h, y = Y + k

$$\frac{dy}{dx} = \frac{X+h+Y+k+1}{(X+h)-(Y+k)} = \frac{(X+Y)+(h+k+1)}{(X-Y)+(h-k)}$$

Choose h and k so that h + k + 1 = 0, h – k = 0

Solving $h = -\frac{1}{2}, k = -\frac{1}{2}$

$$\therefore \frac{dY}{dX} = \frac{X+Y}{X-Y}$$

Put Y = VX ⇒ $\frac{dY}{dX} = V + X\frac{dV}{dX}$

$$V + X\frac{dV}{dX} = \frac{X(1+V)}{X(1-V)}$$

$$X\frac{dV}{dX} = \frac{1+V}{1-V} - V = \frac{1+V-V+V^2}{1-V} = \frac{1+V^2}{1-V}$$

$$\therefore \frac{(1-V)dV}{1+V^2} = \frac{dX}{X}$$

$$\int \frac{dV}{1+V^2} - \frac{1}{2}\int \frac{2VdV}{1+V^2} = \int \frac{dX}{X}$$

$$\tan^{-1} V - \frac{1}{2}\log(1+V^2) = \log x + \log c$$

$$2\tan^{-1} V = \log(1+V)^2 + 2\log x + 2\log c$$

$$= \log c^2 x^2 (1+V^2)$$

sakshieducation.com

$$2\tan^{-1}\left(\frac{Y}{X}\right) = \log c^2x^2\left(1+\frac{y^2}{x^2}\right)$$

$$2\tan^{-1}\left(\frac{y+\frac{1}{2}}{x+\frac{1}{2}}\right) = \log c^2(Y^2+X^2)$$

$$2\tan^{-1}\left(\frac{2y+1}{2x+1}\right) = \log c^2\left[\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2\right] = \log c^2\left(x^2+y^2+x+y+\frac{1}{2}\right)$$

6. **(2x + 3y – 8)dx = (x + y – 3)dy**

Sol. $\frac{dy}{dx} = \frac{2x+3y-8}{x+y-3}$

$x = X + h$, $y = Y + k$ so that $\frac{dY}{dX} = \frac{dy}{dx}$

$$\frac{dY}{dX} = \frac{2(X+h)+3(Y+k)-8}{(X+h)+(Y+k)-3}$$

$$= \frac{(2X+3Y)+(2h+3k-8)}{(X+Y)+(h+k-3)}$$

Choose h and k so that:

$2h + 3k - 8 = 0$, $h + k - 3 = 0$

h   k   I

3   –8   2   3

1   –3   1   1

$$\frac{h}{-9+8} = \frac{k}{-8+6} = \frac{1}{2-3}$$

$$h = 1, k = 2$$

$$\therefore \frac{dY}{dX} = \frac{2X+3Y}{X+Y}$$

Put $Y = VX$ so that $\frac{dY}{dX} = V + X\frac{dV}{dX}$

$$V + X\frac{dV}{dX} = \frac{X(2+3V)}{X(1+V)} \Rightarrow X\frac{dV}{dX} = \frac{2+3V}{1+V} - V$$

$$= \frac{2+3V-V-V^2}{1+V} = \frac{2+2V-V^2}{1+V}$$

sakshieducation.com

$$\int \frac{(1+V)dV}{2+2V-V^2} = \int \frac{dX}{X}$$

Consider $\int \frac{(1+V)dV}{2+2V-V^2}$

Let (1 + V) = A(2 – 2V) + B

Equating the coefficients of V

$1 = -2A \Rightarrow A = -1/2$

Equating constants:

$1 = 2A + B = -1 + B \Rightarrow B = 2$

$$\int \frac{(1+V)dV}{2+2V-V^2} = -\frac{1}{2}\int \frac{(2-2V)dV}{2+2V-V^2} + 2\int \frac{dV}{2+2V-V^2}$$

$$= -\frac{1}{2}\log(2+2V-V^2) + 2\int \frac{dV}{(\sqrt{3})^2-(V-1)^2}$$

$$= -\frac{1}{2}\log(2+2V-V^2) + 2\frac{1}{2\sqrt{3}}\log\frac{\sqrt{3}+V-1}{\sqrt{3}-V+1}$$

$$= -\frac{1}{2}\log(2+2V-V^2) + \frac{1}{\sqrt{3}}\log\frac{V+(\sqrt{3}-1)}{-V+(\sqrt{3}+1)}$$

$$= -\frac{1}{2}\log\left(2+\frac{2Y}{X}-\frac{Y^2}{X^2}\right) + \frac{1}{\sqrt{3}}\log\frac{\frac{Y}{X}+(\sqrt{3}-1)}{-\frac{Y}{X}+\sqrt{3}+1}$$

$$= -\frac{1}{2}\log(2X^2+2XY-Y^2) + \frac{1}{\sqrt{3}}\log\frac{Y+(\sqrt{3}-1)X}{Y-(\sqrt{3}+1)Y}$$

$$\therefore \log X + c = -\frac{1}{2}\log(2X^2+2XY-Y^2) + \frac{1}{\sqrt{3}}\log\frac{[Y+\sqrt{3}-1)X]}{Y-(\sqrt{3}+1)X}$$

7. $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Sol. Let x = X + h, y = Y + k so that $\frac{dY}{dX} = \frac{dy}{dx}$

$$\frac{dY}{dX} = \frac{(X+h)+2(Y+k)+3}{2(X+h)+3(Y+k)+4}$$

$$= \frac{(X+2Y)+(h+2k+3)}{(2X+3Y)+(2h+3k+4)}$$

Choose h and k so that:

h + 2k + 3 = 0, 2h + 3k + 4 = 0

$$\begin{array}{cccc} h & k & I & \\ 2 \; 3 & 1 & 2 & \\ 3 \; 4 & 2 & 3 & \end{array}$$

$$\frac{h}{8-9}=\frac{k}{6-4}=\frac{1}{3-4}$$

$$h=\frac{-1}{-1}=1, k=\frac{2}{-1}=-2$$

$$\frac{dY}{dX}=\frac{X+2Y}{2X+3Y}$$

This is a homogeneous equation

$$Y=VX \Rightarrow \frac{dY}{dX}=V+X\frac{dV}{dX}$$

$$V+X\frac{dV}{dX}=\frac{X(1+2V)}{X(2+3V)}$$

$$X\frac{dV}{dX}=\frac{1+2V}{2+3V}-V=\frac{1+2V-2V-3V^2}{2+3V}$$

$$\frac{(2-3V)dV}{1-3V^2}=\frac{dX}{X}$$

$$2\int\frac{dV}{1-3V^2}-\frac{1}{2}\int\frac{-6VdV}{1-3V^2}=\int\frac{dX}{X}$$

$$\frac{2}{3}\int\frac{dV}{\left(\frac{1}{\sqrt{3}}\right)^2-V^2}-\frac{1}{2}\log|1-3V|^2=\log X+\log c \quad \frac{2}{3}\frac{1}{2\cdot\frac{1}{\sqrt{3}}}\log\left|\frac{\frac{1}{\sqrt{3}}+V}{\frac{1}{\sqrt{3}}-V}\right|-\frac{1}{2}\log|1-3V^2|=\log cx$$

$$\frac{1}{3\sqrt{3}}\log\left|\frac{1+\sqrt{3}V}{1-\sqrt{3}V}\right|\log|1-3V^2|=\log cy$$

$$\frac{1}{3\sqrt{3}}\log\left|\frac{1+\frac{\sqrt{3}Y}{X}}{1-\frac{\sqrt{3}Y}{X}}\right|-\frac{1}{2}\log\left|1-\frac{3y^2}{x^2}\right|=\log cy$$

$$\frac{1}{\sqrt{3}}\log\left|\frac{X+\sqrt{3}Y}{X-\sqrt{3}Y}\right|-\frac{1}{2}\log\left|\frac{X^2-3Y^2}{X^2}\right|=\log cx$$

Where X = x – 1, Y = y + 2

sakshieducation.com

$$\frac{1}{\sqrt{3}}\left[\log(X+\sqrt{3}Y)-\log(X-\sqrt{3})\right]$$

$$-\frac{1}{2}\left(\log(X+\sqrt{3}Y)+\log(X-\sqrt{3}Y)-2\log X\right)$$

$$=\log CX$$

$$\log(X+\sqrt{3}Y)\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}\right)-\log(X-\sqrt{3}Y)$$

$$\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)+\log X=\log c+\log X$$

$$\frac{2-\sqrt{3}}{2\sqrt{3}}\log(X+\sqrt{3}Y)\frac{2+\sqrt{3}}{2\sqrt{3}}\log(X-\sqrt{3}Y)$$

$$=\log c$$

$$\text{i.e.}\frac{(2-\sqrt{3})}{2}\log(X+\sqrt{3}Y)\frac{(2+\sqrt{3})}{2}$$

$$\log(X-\sqrt{3}Y)=\sqrt{3}c' \text{ where } c'=\log c$$

8. $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$

Sol. Given equation is $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$

Let x = X + h, y = Y + k so that $\frac{dY}{dX}=\frac{dy}{dx}$

$$\frac{dY}{dX}=\frac{2(X+h)+9(Y+k)-20}{6(X+h)+2(Y+k)-10}$$

$$=\frac{(2X+9Y)+(2h+9k-20)}{(6X+2Y)+(6h+2k-10)}$$

Choose h and k so that:

2h + 9k – 20 = 0, 6h + 2k – 10 = 0

h k I

9 ⨯ –20 ⨯ 2 ⨯ 9

2 ⨯ –10 ⨯ 6 ⨯ 2

$$\frac{h}{-90+40}=\frac{k}{-120+20}=\frac{1}{4-54}$$

sakshieducation.com

$$h=\frac{-50}{-50}=1,\ k=\frac{-100}{-50}=2$$

$$\therefore \frac{dY}{dX}=\frac{2X+9Y}{6X+2Y}$$

This is homogeneous equation

$$y=VX\Rightarrow\frac{dY}{dX}=V+X\frac{dV}{dX}$$

$$V+X\frac{dV}{dX}=\frac{x(2+9V)}{x(6+2V)}$$

$$X\frac{dV}{dX}=\frac{2+9V}{6+2V}-V=\frac{2+9V-6V-2V^2}{6+2V}$$

$$\frac{6+2V\cdot dY}{2+3V-2V^2}=\frac{dX}{X}$$

$$\int\frac{(6+2V)dV}{(1+2V)(2-V)}=\int\frac{dx}{x} \qquad \ldots (1)$$

Let $\frac{6+2V}{(1+2V)(2-V)}=\frac{A}{1+2V}+\frac{B}{2-V}$

6 + 2V = A(2 – V) + B(1 + 2V)

V = 2 ⇒ 10 = B(5) ⇒ B = 2

$$V=-\frac{1}{2}\Rightarrow 5=A\left(\frac{5}{2}\right)\Rightarrow A=2$$

$$\frac{6+2V}{(1+2V)(2-V)}=\frac{2}{1+2V}+\frac{2}{2-V}$$

$$\int\frac{(6+2V)dV}{(1+2V)(2-V)}=\int\frac{2dV}{1+2V}-2\int\frac{dV}{V-2}$$
$$=\log(1+2V)-2\log(V-2)$$

From (1) we get

$$\log(1+2V)-\log(V-2)^2=\log X-\log c$$

$$\log\frac{1+2V}{(V-2)^2}=\log\frac{X}{c}\Rightarrow\frac{1+2V}{(V-2)^2}=\frac{X}{c}$$

$$1+2V=\frac{X}{c}(V-2)^2\Rightarrow 1+\frac{2V}{X}=\frac{x}{c}\left(\frac{Y}{X}-2\right)^2$$

$$\frac{X+2Y}{X}=\frac{x}{c}\frac{(Y-2X)^2}{X^2}$$

Solution is $(X + 2Y) = (Y - 2X)^2$

Where $X = x - 1$, $Y = y - 2$

$c(x - 1 + 2y - 4) = (y - 2 - 2x + 2)^2$

$c(x + 2y - 5) = (y - 2x)^2 = (2x - y)^2$

Linear Differential Equations

Linear Equations: A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only is called a linear differential equation of the first order in y.

Bernoulli's Equation: An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only, is called a Bernoulli's equation

Very Short Answer Questions

I. Find the I.F. of the following differential equations by transforming them into linear form.

1. $x\frac{dy}{dx}-y=2x^2\sec^2 2x$

Sol. $x\frac{dy}{dx}-y=2x^2\sec^2 2x$

$\frac{dx}{dy}-\frac{1}{x}y=2x\sec^2 2x$ which is linear in y .

$$\text{I.F.}=e^{\int p\,dx}=\int_e -\frac{1}{x}\log x=e^{-\log x}=e^{\log(1/x)}=\frac{1}{x}$$

2. $y\frac{dx}{dy}-x=2y^3$

Sol. $y\frac{dx}{dy}-x=2y^3 \Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2$ which is linear equation in x.

$$\text{I.F.}=e^{\int p\,dy}=e^{\int -\frac{1}{y}dy}=e^{-\log y}=e^{\log(1/y)}=\frac{1}{y}$$

Short Answer Questions

Solve the following differential equations.

1. $\frac{dy}{dx} + y\tan x = \cos^3 x$

Sol. $\frac{dy}{dx} + y\tan x = \cos^3 x$ which is linear d.eq. in y.

$\text{I.F.} = e^{\int p\,dx} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$

Solution of the equation is

y. I.F. $= \text{y.I.F} = \int Q.\ \text{I.F.}\ dx$

$\Rightarrow y\sec x = \int \sec x \cos^3 x\ dx = \int \cos^2 x\ dx$

$= \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$

$\frac{2y}{\cos x} = x + \sin x \cdot \cos x + c$

Solution is:

$2y = x\cos x + \sin x \cdot \cos^2 x + c \cdot \cos x$

2. $\frac{dy}{dx} + y\sec x = \tan x$

Sol. $\frac{dy}{dx} + y\sec x = \tan x$ which is l.d.e. in y

$\text{I.F.} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$

Sol is y.I.F. $= \text{y.I.F} = \int Q.\ \text{I.F.}\ dx$

$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx$

$= \int (\sec x \cdot \tan x + \tan^2 x)dx$

$= \int (\sec x \cdot \tan x + \sec^2 x - 1)dx$

Solution is

$$y(\sec x+\tan x)=\sec x+\tan x-x+c$$

3. $\frac{dy}{dx}-y\tan x=e^{x}\sec x$ **Which is l. d.e. in y.**

Sol. $\text{I.F.}=e^{-\int \tan x\,dx}=e^{\log\cos x}=\cos x$

Sol is $y.\text{I.F.}=y.\text{I.F}=\int Q.\ \text{I.F.}\ dx$

$$y\cos x=\int e^{x}\sec x\cos x dx=\int e^{x}dx=e^{x}+c$$

4. $x\frac{dy}{dx}+2y=\log x$

Sol. $\text{I.F.}=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^{2}}=x^{2}$

Solution is:

$$\therefore y\cdot x^{2}=\int x^{2}\frac{\log x}{x}dx=\int x\log x dx$$

$$=\log x\left(\frac{x^{2}}{2}\right)-\frac{1}{2}\int x^{2}\frac{1}{x}dx=\frac{x^{2}}{2}\log x-\frac{1}{2}\int x\,dx=\frac{x^{2}}{2}\log x-\frac{x^{2}}{4}+c$$

5. $(1+x^{2})\frac{dy}{dx}+y=e^{\tan^{-1}x}$

Sol. $\frac{dy}{dx}+\frac{1}{1+x^{2}}\cdot y=\frac{e^{\tan^{-1}x}}{1+x^{2}}$ which linear differential equation in y.

$$\text{I.F.}=e^{\int pdx}=e^{\int\frac{dx}{1+x^{2}}}=e^{\tan^{-1}x}$$

Sol is $y.\text{I.F.}=y.\text{I.F}=\int Q.\ \text{I.F.}\ dx$

$$y\cdot e^{\tan^{-1}x}=\int\frac{(e^{\tan^{-1}x})^{2}}{1+x^{2}}dx \qquad \ldots (1)$$

Consider $\int\frac{(e^{\tan^{-1}x})^{2}}{1+x^{2}}dx$ put $\tan^{-1}x=t\Rightarrow\frac{dx}{1+x^{2}}=dt$

$$= \int (e^t)^2 dt = \int e^{2t} dt = \frac{e^{2t}}{2} = \frac{e^{2\tan^{-1}x}}{2}$$

Solution is $y \cdot e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + \frac{c}{2}$

$$2y \cdot e^{\tan^{-1}x} = e^{2\tan^{-1}x} + c$$

6. $\frac{dy}{dx} + \frac{2y}{x} = 2x^2$

Sol. $\frac{dy}{dx} + \frac{2y}{x} = 2x^2$ which is l.d.e. in y.

$$\text{I.F.} = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Sol is y.I.F. $=$ y.I.F $= \int$ Q. I.F. dx

$$y \cdot x^2 = \int 2x^4 dx = \frac{2x^5}{5} + c$$

7. $\frac{dy}{dx} + \frac{4x}{1+x^2} y = \frac{1}{(1+x^2)^2}$

Sol. $\text{I.F.} = e^{\int \frac{4x}{1+x^2} dx} = e^{2\log(1+x^2)}$

$$= e^{\log(1+x^2)^2} = (1+x^2)^2$$

$\therefore$ Solution is $y(1+x^2)^2 = \int dx = x + c$

8. $x\frac{dy}{dx} + y = (1+x)e^x$

Sol. $\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{(1+x)e^x}{x}$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$y \cdot x = \int (1+x)e^x dx = x \cdot e^x + c$$

sakshieducation.com

9. $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$

Sol. $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$ which is linear differential equation in y .

$$\text{I.F.}=e^{\int\frac{3x^2}{1+x^3}dx}=e^{\log(1+x^3)}=1+x^3$$

Sol is $\text{y.I.F.}=\text{y.I.F}=\int \text{Q. I.F. dx}$

$$y(1+x^3)=\int(1+x^2)dx=x+\frac{x^3}{3}+c$$

10. $\frac{dy}{dx}-y=-2e^{-x}$

Sol. $\text{I.F.}=e^{\int-dx}=e^{-x}$

$$y\cdot e^{-x}=-2\int e^{-2x}dx=e^{-2x}+c$$

$$y=e^{-x}+ce^{x}$$

11. $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$.

Sol. $\frac{dy}{dx}+\frac{1}{1+x^2}\cdot y=\frac{\tan^{-1}x}{1+x^2}$ which is linear differential equation in y.

$$\text{I.F. } e^{\int\frac{dx}{1+x^2}}=e^{\tan^{-1}x}$$

Sol is $\text{y.I.F.}=\text{y.I.F}=\int \text{Q. I.F. dx}$

$$y\cdot e^{\tan^{-1}x}=\int\tan^{-1}x\frac{e^{\tan^{-1}x}}{1+x^2}dx$$

Put $t=\tan^{-1}x$ so that $dt=\frac{dx}{1+x^2}$

$$\text{R.H.S.}=\int t\cdot e^t dt=t\cdot e^t-\int e^t dt=t\cdot e^t-e^t$$

Solution is: $y.e^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+c$

$y = \tan^{-1} x - 1 + c \cdot e^{-\tan^{-1} x}$

12. $\dfrac{dy}{dx} + y \tan x = \sin x$.

Sol. I.F. $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

$y \sec x = \int \sin x \cdot \sec x\, dx$

$= \int \tan x\, dx = \log \sec x + c$

Long Answer Questions

Solve the following differential equations.

1. $\cos x \dfrac{dy}{dx} + y \sin x = \sec^2 x$

Sol. $\dfrac{dy}{dx} + \tan x \cdot y = \sec^3 x$ which is l.d.e in y

I.F. $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

Sol is $y.I.F. = y.I.F = \int Q.\ I.F.\ dx$

$y \cdot \sec x = \int \sec^4 x dx = \int (1 + \tan^2 x) \sec^2 x dx$

$= \tan x + \dfrac{\tan^3 x}{3} + c$

2. $\sec x \cdot dy = (y + \sin x) dx$

Sol. $\dfrac{dy}{dx} = \dfrac{y + \sin x}{\sec x} = y \cos x + \sin x \cdot \cos x$

$\dfrac{dy}{dx} - y \cos x = \sin x \cdot \cos x$ which is l..d.e in y

I.F. $= e^{-\int \cos x dx} = e^{-\sin x}$

Sol is $y.I.F. = y.I.F = \int Q.\ I.F.\ dx$

$y \cdot e^{-\sin x} = \int e^{-\sin x} \cdot \sin x \cdot \cos x \cdot dx$

Consider $\int e^{-\sin x} \cdot \sin x \cdot \cos x \cdot dx$

$t = -\sin x \Rightarrow dt = -\cos x\, dx$

$$\int e^{-\sin x} \cdot \sin x \cdot \cos x\, dx = +\int e^{t} \cdot t\, dt$$
$$= t \cdot e^{t} - e^{t} + c = e^{-\sin x}(-\sin x - 1) + c$$

$$y \cdot e^{-\sin x} = -e^{-\sin x}(\sin x + 1) + c$$
or $y = -(\sin x + 1) + c \cdot e^{\sin x}$

3. $x \log x \cdot \dfrac{dy}{dx} + y = 2 \log x$

Sol. $\dfrac{dy}{dx} + \dfrac{1}{x \log x} y = \dfrac{2}{x}$

$$\text{I.F.} = e^{\int \frac{dx}{x \log x}} = e^{\log(\log x)} = \log x$$

$$y \log x = 2\int \frac{\log x}{x} dx = (\log x)^2 + c$$

4. $(x + y + 1)\dfrac{dy}{dx} = 1$

Sol. $(x + y + 1)\dfrac{dy}{dx} = 1$

$\dfrac{dx}{dy} = x + y + 1 \Rightarrow \dfrac{dx}{dy} - x = y + 1$ which is l.d.e in x.

$$\text{I.F.} = e^{\int p dy} = e^{\int -dy} = e^{-y}$$

sol is x.I.F.$= \int$ Q.IFdy

$$x \cdot e^{-y} = \int e^{-y}(y+1)dy = -(y+1)e^{-y} + \int e^{-y} dy$$

$$= -(y+1)e^{-y} - e^{-y} = -(y+2)e^{-y} + c$$

$$x = -(y+2) + c \cdot e^{y}$$

5. Solve $x(x-1)\frac{dy}{dx}-y=x^3(x-1)^3$.

Sol. $\frac{dy}{dx}-\frac{1}{x(x-1)}y=x^2(x-1)^2$ which is l.d.e in y

$$I.F.=e^{\int pdx}=e^{\int -\frac{dx}{x(x-1)}}=e^{\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx}$$

$$=e^{\log x-\log(x-1)}=e^{\log\frac{x}{x-1}}=\frac{x}{x-1}$$

Sol is $y.I.F.=y.I.F=\int Q.\ I.F.\ dx$

$$y\cdot\frac{x}{x-1}=\int x^2(x-1)^2\frac{x}{(x-1)}dx=\int x^3(x-1)dx$$

Hence solution is $\frac{xy}{x-1}=\frac{x^5}{5}-\frac{x^4}{4}+c$

6. $(x+2y^3)\frac{dy}{dx}=y$

Sol. $\frac{dy}{dx}=\frac{x+2y^3}{y}=\frac{x}{y}+2y^2$

$$\frac{dx}{dy}-\frac{1}{y}x=2y^2$$

$$I.F.=e^{\int -\frac{1}{y}dy}=e^{-\log y}=e^{\log 1/y}=\frac{1}{y}$$

$$x\cdot\frac{1}{y}=\int 2ydy=y^2+c$$

Solution is: $x=y(y^2+c)$

7. Solve $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$.

Sol. $\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{1-x^2}}$

$$I.F.=e^{\int\frac{2x}{1-x^2}dx}=e^{-\log(1-x^2)}$$

sakshieducation.com

$$= e^{\log(1-x^2)^{-1}} = \frac{1}{1-x^2}$$

$$\frac{y}{1-x^2} = \int \frac{x dx}{(1-x^2)^{3/2}} = (1-x^2)^{-1/2} + c$$

$$(or) y = \sqrt{1-x^2} + c(1-x^2)$$

8. $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

Sol. $\frac{dy}{dx} - \frac{x-2}{x(x-1)} y = \frac{x^3(2x-1)}{x(x-1)}$

$$I.F. = e^{\int \frac{2-x}{x(x-1)} dx} \Rightarrow \frac{2-x}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$2 - x = A(x-1) + Bx$$

$$x = 0 \Rightarrow 2 = -A \Rightarrow A = -2$$

$$x = 1 \Rightarrow 1 = B \Rightarrow B = 1$$

$$\frac{2-x}{x(x-1)} = \frac{-2}{x} + \frac{1}{x+1}$$

$$\int \frac{2-x}{x(x-1)} dx = -2\int \frac{dx}{x} + \int \frac{dx}{x-1}$$

$$= -\log x + \log(x-1) = \log \frac{x-1}{x^2}$$

$$I.F. = e^{\log \frac{x-1}{x^2}} = \frac{x-1}{x^2}$$

$$y \frac{x-1}{x^2} = \int \frac{x^3(2x-1)}{x(x-1)} \cdot \frac{x-1}{x^2} dx$$

$$= \int (2x-1) dx = x^2 - x + c$$

Solution is $y(x - 1) = x^2(x^2 - x + c)$

9. $\frac{dy}{dx}(x^2y^3+xy)=1$

Sol. $\frac{dy}{dx}(x^2y^3+xy)=1$

$$\frac{dx}{dy}=xy+x^2y^3$$

$$\Rightarrow \frac{dx}{dy}-xy=x^2y^3 \text{---- (1)}$$

Which is Bernoulli's equation

Dividing with x^2,

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}y=y^3$$

Put $z=-\frac{1}{x}$ so that $\frac{dz}{dy}=\frac{1}{x^2}\frac{dx}{dy}$

$$\Rightarrow \frac{dz}{dy}+z\cdot y=y^3 \text{----2)}$$

Which is linear diff.eq. in z

$$\text{I.F.}=e^{\int y dy}=e^{y^2/2}$$

Sol is $z.\text{I.F}=\int Q.\ \text{I.F. } dy$

$$z\cdot e^{y^2/2}=\int y^3 e^{y^2/2}\cdot dy$$

put $\frac{y^2}{2}=t \Rightarrow y\,dy=dt$

$$=\int t\cdot dt\cdot e^t=e^t(t-1)=e^{y^2/2}\left(\frac{y^2}{2}-1\right)$$

$$z\cdot e^{y^2/2}=e^{y^2/2}\left(\frac{y^2}{2}-1\right)+c$$

$$z=\frac{y^2}{2}-1+c\cdot e^{-y^2/2}\Rightarrow -\frac{1}{x}=\frac{y^2}{2}-1+c\cdot e^{-y^2/2}$$

$$-1=x\left(\frac{y^2}{2}-1+c\cdot e^{-y^2/2}\right)$$

Hence solution is $1+x\left(\frac{y^2}{2}-1+c\cdot e^{-y^2/2}\right)=0$

10. $\frac{dy}{dx}+x\cdot\sin 2y=x^3\cos^2 y$

Sol. $\frac{1}{\cos^2 y}\frac{dy}{dx}+\frac{2\sin y\cos y}{\cos^2 y}x=x^3$

$\sec^2 y\frac{dy}{dx}+2\tan y.x=x^3$

$z=\tan y\Rightarrow\frac{dz}{dx}=\sec^2 y\frac{dy}{dx}$

$\frac{dz}{dx}+2zx=x^3$

$I.F.=e^{\int 2xdx}=e^{x^2}$

$z\cdot e^{x^2}=\int x^3\cdot e^{x^2}dx$ ...(1)

Consider $\int x^3\cdot e^{x^2}dx$

$t=x^2\Rightarrow dt=2x\cdot dx$

$\int x^3\cdot e^{x^2}dx=\int x\cdot x^2\cdot e^{x^2}dx$

$=\frac{1}{2}\int t\cdot e^t dt=\frac{1}{2}e^t(t-1)$

$=z\cdot e^{x^2}=\frac{1}{2}e^{x^2}(x^2-1)+c$

$z=\frac{x^2-1}{2}+c\cdot e^{-x^2}$

$\tan y=\frac{x^2-1}{2}+c\cdot e^{-x^2}$

sakshieducation.com

11. $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

Sol.

$$y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$$

$$\left(x - \frac{1}{y}\right)\frac{dy}{dx} = -y^2$$

$$\frac{dx}{dy} = \frac{x - 1/y}{-y^2} = -\frac{x}{y^2} + \frac{1}{y^3}$$

$$\frac{dx}{dy} + \frac{1}{y^2}\cdot x = \frac{1}{y^3}$$ Which is l.d.e in x

$$\text{I.F.} = e^{\int \frac{1}{y^2}dy} = e^{-1/y}$$

Sol is $x.\text{I.F} = \int Q.\ \text{I.F.}\ dy$

$$x\cdot e^{-1/y} = \int \frac{e^{-1/y}}{y^3}dy \qquad \dots (1)$$

$$\text{put} -\frac{1}{y} = z \Rightarrow \frac{1}{y^2}dy = dz$$

$$= \int z\cdot e^z dz = e^z(z-1)$$

$$x\cdot e^{-1/y} = -e^{-1/y}\left(-\frac{1}{y} - 1\right) + c$$

$$\frac{x}{e^{1/y}} = \frac{1+y}{y\cdot e^{1/y}} + c$$

Hence solution is $xy = 1 + y + cy\,e^{1/y}$.

sakshieducation.com

Problems for Practice

1. Form the differential equation corresponding to the family of circles passing through the origin and having centers on Y-axis.

Ans. $(x^2 - y^2)\frac{dy}{dx} - 2xy = 0$

2. Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Ans. $x = \log\left(1 + \tan\frac{x+y}{2}\right) + c$

3. Give the solution of $x\sin^2\frac{y}{x}dx = ydx - xdy$ **which passes through the point (1, π/4).**

Ans. $\cot\frac{y}{x} = \log x + 1$

4. Solve $\left(2x - 10y^3\right)\frac{dy}{dx} + y = 0$

Ans. $\text{I.F.} = e^{\int\frac{2}{y}dy} = e^{2\log y} = e^{\log y^2} = y^2$

5. Solve $\left(1+x^2\right)\frac{dy}{dx} + 2xy - 4x^2 = 0$

Ans. $y(1+x^2) = \int 4x^2\,dx = \frac{4x^3}{3} + c$

6. Solve $\frac{1}{x}\cdot\frac{dy}{dx} + y\cdot e^x = e^{(1-x)e^2}$**.**

Ans. $2y\cdot e^{(x-1)e^x} = x^2 + 2c$

7. **Solve** $\sin^2 x \cdot \frac{dy}{dx} + y = \cot x$.

Ans. $y \cdot e^{-\cot x} = (\cot x + 1)e^{-\cot x} + c$

8. **Find the equation of the equation** $x(x-2)\frac{dy}{dx} - 2(x-1)y = x^3(x-2)$ **which satisfies the condition that y = 9 when x = 3.**

Ans. $\frac{y}{x(x-2)} = x + 2\log(x-2)$

9. **Solve** $(1+y^2)dx = (\tan^{-1} y - x)dy$.

Ans. $x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$.

10. **Solve** $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0$.

Ans. $\frac{\sqrt{\frac{y^2}{x^2} - 1}}{\frac{y^2}{x^2} + 1} = cx$

sakshieducation.com