CIRCLES PART - III

Theorem:

If a line passing through a point $P(x_1, y_1)$ intersects the circle $S = 0$ at the points A and B then $PA.PB = |S_{11}|$.

Corollary:

If the two lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct points then those points are concyclic $\Leftrightarrow a_1a_2 = b_1b_2$.

Corollary:

If the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct concyclic points then the equation of the circle passing through these concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - (a_1b_2 + a_2b_1)xy = 0$.

Theorem:

Two tangents can be drawn to a circle from an external point.

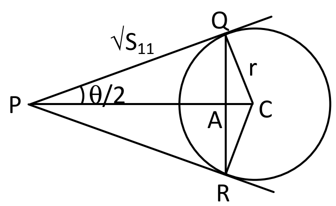
Note:

If m_1, m_2 are the slopes of tangents drawn to the circle $x^2 + y^2 = a^2$ from an external point (x_1, y_1) then $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$, $m_1m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$.

Theorem:

If θ is the angle between the tangents through a point P to the circle S = 0 then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$ where r is the radius of the circle.

Proof:



Let the two tangents from P to the circle S = 0 touch the circle at Q, R and θbe the angle between these two tangents. Let C be the centre of the circle. Now QC = r, PQ = $\sqrt{S_{11}}$ and ΔPQC is a right angled triangle at Q.

$$\therefore \tan\frac{\theta}{2} = \frac{QC}{PQ} = \frac{r}{\sqrt{S_{11}}}$$

Theorem: The equation to the chord of contact of $P(x_1, y_1)$ with respect to the circle S = 0 is $S_1 = 0$.

Theorem: The equation of the polar of the point $P(x_1, y_1)$ with respect to the circle S = 0 is $S_1 = 0$.

Theorem: The pole of the line lx + my + n = 0 ($n \neq 0$) with respect to $x^2 + y^2 = a^2$ is $\left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$.

Proof:

Let $P(x_1, y_1)$ be the pole of lx + my + n = 0 … (1)

The polar of P with respect to the circle is:

$xx_1 + yy_1 - a^2 = 0$ …(2)

Now (1) and (2) represent the same line

$$\therefore \frac{x_1}{\ell} = \frac{y_1}{m} = \frac{-a^2}{n} \Rightarrow x_1 = \frac{-la^2}{n}, y = \frac{-ma^2}{n}$$

$$\therefore \text{Pole } P = \left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$$

Theorem: If the pole of the line lx + my + n = 0 with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is (x_1, y_1) then $\frac{x_1+g}{\ell} = \frac{y_1+f}{m} = \frac{r^2}{lg + mf - n}$ where r is the radius of the circle.

Proof:

Let $P(x_1, y_1)$ be the pole of the line lx + my + n = 0 … (1)

The polar of P with respect to S = 0 is S_1=0

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow (x_1 + g)x + (y_1 + f)y + gx_1 + fy_1 + c = 0$ …(2)

Now (1) and (2) represent the same line.

$$\therefore \frac{x_1+g}{\ell} = \frac{y_1+f}{m} = \frac{gx_1+gy_1+c}{n} = k(\text{say})$$

$$\frac{x_1+g}{\ell} = k \Rightarrow x_1+g = \ell k \Rightarrow x_1 = \ell k - g$$

$$\frac{y_1+f}{m} = k \Rightarrow y_1+f = mk \Rightarrow y_1 = mk - f$$

$$\frac{gx_1+gy_1+c}{n} = k \Rightarrow gx_1+gy_1+c = nk$$

$$\Rightarrow g(lk-g)+f(mk-f)+c = nk$$

$\Rightarrow k(lg+mf-n) = g^2+f^2-c = r^2$ Where r is the radius of the circle $\Rightarrow k = \dfrac{r^2}{lg+mf-n}$.

$$\therefore \frac{x_1+g}{\ell} = \frac{y_1+f}{m} = \frac{r^2}{lg+mf-n}.$$

Theorem: The lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ iff $r^2(l_1l_2 + m_1m_2) = (l_1g + m_1f - n_1)(l_2g + m_2f - n_2)$.

Theorem: The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the circle $x^2 + y^2 = a^2$ is $a^2(l_1l_2 + m_1m_2) = n_1n_2$.

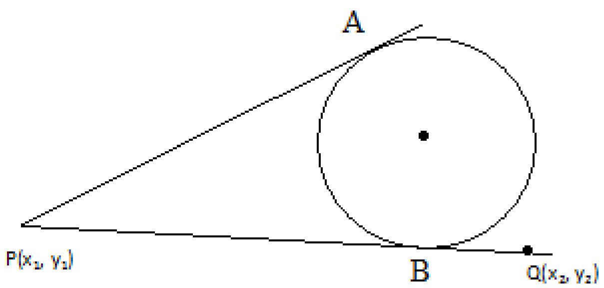
Theorem: The equation of the chord of the circle S = 0 having $P(x_1, y_1)$ as its midpoint is $S_1 = S_{11}$.

Theorem: The length of the chord of the circle S = 0 having $P(x_1, y_1)$ as its midpoint is $2\sqrt{|S_{11}|}$.

Theorem: The equation to the pair of tangents to the circle

S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

Proof:



Let the tangents from P to the circle S=0 touch the circle at A and B.

Equation of AB is $S_1 = 0$.

i.e., $x_1 x + y_1 y + g(x + x_1) + f(y + y_1) + c = 0$ =====(i)

Let $Q(x_2, y_2)$ be any point on these tangents. Now locus of Q will be the equation of the pair of tangents drawn from P.

The line segment PQ is divided by the line AB in the ratio $-S_{11}:S_{22}$

$$\Rightarrow \frac{PB}{QB} = \left|\frac{S_{11}}{S_{22}}\right| \qquad \text{---- (ii)}$$

BUT $PB = \sqrt{S_{11}}, QB = \sqrt{S_{22}} \Rightarrow \frac{PB}{QB} = \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}}$ ---- (iii)

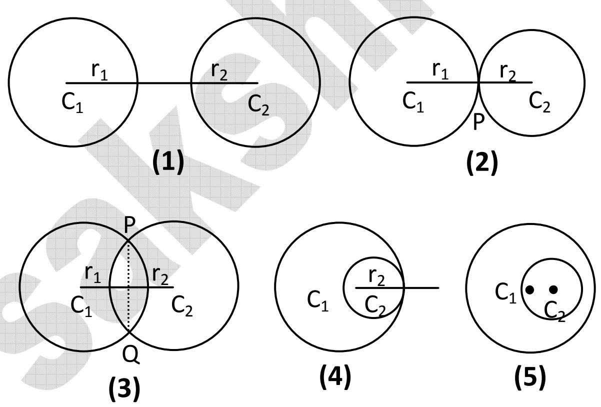
From ii) and iii) $\Rightarrow \frac{s^2{}_{11}}{s^2{}_{22}} = \frac{S_{11}}{S_{22}}$

$$\Rightarrow S_{11} S_{22} = S^2{}_{12}$$

Hence locus of $Q(x_2, y_2)$ is $S_{11} S = S^2{}_1$

Touching Circles: Two circles S = 0 and S′ = 0 are said to touch each other if they have a unique point P in common. The common point P is called point of contact of the circles S = 0 and S′ = 0.

Circle – Circle Properties: Let S = 0, S′ = 0 be two circle with centres C_1, C_2 and radii r_1, r_2 respectively.



1) If $C_1C_2 > r_1 + r_2$ then each circle lies completely outside the other circle.

2) If $C_1C_2 = r_1 + r_2$ then the two circles touch each other externally. The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ internally.

3) If $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ then the two circles intersect at two points P and Q. The chord $\overleftrightarrow{PQ}$ is called common chord of the circles.

4) If $C_1C_2 = |r_1 - r_2|$ then the two circles touch each other internally. The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ externally.

5) If $C_1C_2 < |r_1 - r_2|$ then one circle lies completely inside the other circle.

Common Tangents:

A line L = 0 is said to be a common tangent to the circle S = 0, S′ = 0 if L = 0 touches both the circles.

Definition:

A common tangent L = 0 of the circles S = 0, S′= 0 is said to be a direct common tangent of the circles if the two circles S = 0, S′ = 0 lie on the same side of L = 0.

Definition:

A common tangent L = 0 of the circles S = 0, S′= 0 is said to be a transverse common tangent of the circles if the two circles S = 0, S′ = 0 lie on the opposite (either) sides of L = 0.

Centres of Similitude:

Let S = 0, S′ = 0 be two circles. (i) The point of intersection of direct common tangents of S = 0, S′ = 0 is called external centre of similitude. (ii) The point of intersection of transverse common tangents of S = 0, S′ = 0 is called internal centre of similitude.

Theorem:

Let S = 0, S′ = 0 be two circles with centres C_1, C_2 and radii r_1, r_2 respectively. If A_1 and A_2 are respectively the internal and external centres of similitude circles S = 0, S′ = 0 then

(i) A_1 divides C_1C_2 in the ratio $r_1 : r_2$ internally.

(ii) A_2 divides C_1C_2 in the ratio $r_1 : r_2$ internally.

Very Short Answer Questions

1) Find the condition that the tangents Drawn from (0,0) to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be perpendicular to each other.

Sol. Let θ be the angle between the pair of

Tangents then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$

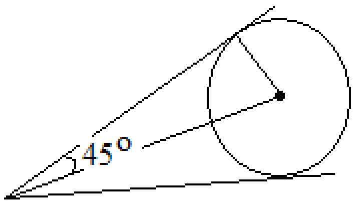
given $\theta = \frac{\pi}{2}$, radius r= $\sqrt{g^2 + f^2 - c}$

$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 + c = c$

$$\tan 45^0 = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}}$$

$\Rightarrow 1 = \frac{\sqrt{g^2+f^2-c}}{\sqrt{0+0+0+0+c}}$

$\Rightarrow g^2 + f^2 - c = c$



$\Rightarrow g^2 + f^2 = 2c$

This is the required condition

2) Find the chord of contact of (0,5) with respect to the circle

Sol . Equation of the circle is

$S = x^2 + y^2 - 5x + 4y - 2 = 0$

Equation of the chord of contact is $S_1 = 0$

$\Rightarrow x.0 + y.5 - \frac{5}{2}(x + 0) + 2(y+5) - 2 = 0$

$\Rightarrow 10y - 5x + 4y + 20 - 4 = 0$

$\Rightarrow -5x + 14y + 16 = 0$

$\Rightarrow 5x - 14y - 16 = 0$

3) **Find the polar of (1,2) with respect to $x^2 + y^2 = 7$.**

Sol. point (1,2) and circle is S= $x^2 + y^2 = 7$

Polar of P(x_1, y_1) with respect to s = 0 is S_1=0

$\Rightarrow$ x + 2y – 7 = 0 is the polar equation.

4) Find the polar of (3, -1) with respect to $2x^2 + 2y^2=11$.

Sol. Equation of circle is $2x^2 + 2y^2=11$

$\Rightarrow x^2 + y^2= \frac{11}{2}$. Point is (3,-1)

Equation of polar is S_1=0

$\Rightarrow xx_1 + yy_1 = a^2$

$\Rightarrow x(3) + (-1)\, y = \frac{11}{2}$

$\Rightarrow$ 6x – 2y – 11 = 0.

5) Find the polar of (1, -2) with respect to $x^2 + y^2 – 10x – 10y + 25 = 0$

Sol . Equation of the circle is

$x^2 + y^2 – 10x – 10y +25 = 0$

Polar of P (1, -2) is S_1 = 0

$\Rightarrow$ x.1 + y(-2) -5(x+1) -5(y-2) + 25 = 0

$\Rightarrow$ x – 2y – 5x – 5 – 5y + 10 + 25 = 0

$\Rightarrow$ -4x – 7y + 30 = 0

$\Rightarrow$ 4x + 7y – 30 = 0

6) Find the pole of ax + by + c = 0 (c $\neq$ 0) With respect to $x^2 + y^2 = r^2$.

Sol. Let (x_1, y_1) be pole. Then the polar equation is S_1=0.

$\Rightarrow xx_1 + yy_1 – r^2 = 0$ ______ (i)

But polar is ax + by + c = 0 ______(ii)

(i) and (ii) both are same lines

$$\Rightarrow \frac{x_1}{a} = \frac{y_1}{b} = \frac{-r^2}{c} \Rightarrow x_1 = -\frac{-a}{c}r^2, y_1 = \frac{-br^2}{c}$$

$$\therefore pole\left(\frac{-ar^2}{c}, \frac{-br^2}{c}\right)$$

7) Find the pole of 3x + 4y – 45 = 0 with Respect to $x^2 + y^2 -6x -8y+ 5 =0$.

Sol. Let (x_1, y_1) be pole.

Equation of polar is $S_1=0$

$xx_1 + yy_1 – 3(x+x_1) – 4 (y + y_1) + 5 = 0$

$x (x_1-3) + y(y_1-4) - 3x_1 – 4y_1+ 5 = 0$ – (i)

But equation of the polar is 3x + 4y – 45 = 0 – (ii)

Comparing (i) and (ii) we get

$$\frac{x_1-3}{3}=\frac{y_1-4}{4}=\frac{-3x_1-4y_1+5}{-45}=k$$

$$\frac{x_1-3}{3}=\frac{y_1-4}{4}$$

$4x_1 – 72 = 3y – 12$

$Y_1 = \frac{4}{3} x_1$

$$\frac{x_1-3}{3}=\frac{-3x_1-\frac{16}{3}x_1+5}{-45}=\frac{-9x_1-16x_1+15}{3(-45)}$$

$$x_1-3=\frac{-25x_1+15}{-45}$$ $20x_1 = 120 \Rightarrow x_1=6$

$Y= \frac{4}{3} x_1 = \frac{4}{3}.6 =8$

Pole is (6, 8).

8**) Find the pole of x-2y + 22 = 0 with respect to $x^2 + y^2 -5x + 8y + 6 = 0$**

∴ Pole is (2, -3)

9) Show that the points (-6, 1), (2, 3) are Conjugate points with respect to the circle $x^2 + y^2 – 2x + 2y + 1 = 0$

Sol. Polar of (2,3) w.r.t S= $\mathbf{x^2 + y^2 – 2x + 2y + 1 = 0}$ is $S_1 =0$

$\Rightarrow 2x+3y-1(x+2) + 1 (y+3)+1=0$

$\Rightarrow x + 4y + 2 = 0$……………(1)

Substituting (-6,1) in (i), then

(-6) + 4(1)+2=0

The point (-6,1) is a point on the polar of 2,3).

∴ (-6, 1) and (2,3) are conjugate with respect to circle.

II Method:

S= $x^2 + y^2 - 2x + 2y + 1 = 0$

Points are **(-6, 1), (2, 3)**

Now S_{12} = -6.2+1.3-(-6+2)+(1+3)+1

= -12+3+4+4+1 = 0.

Therefore given points are conjugate points.

10. Find the value of k if kx + 3y – 1 = 0, 2x +y + 5 = 0 are conjugate lines with respect to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$

Sol.

Given S = $x^2 + y^2 - 2x - 4y - 4 = 0$

Lines are kx + 3y – 1 = 0 and 2x +y + 5 = 0

Let (x_1, y_1) be the pole

Then polar is $S_1=0$.

$\Rightarrow$ x2 + $y^2 - 2x - 4y - 4 = 0$be

$\Rightarrow xx_1 + yy_1 - 1(x+x_1) - 2(y+y_1) - 4 = 0$

$\Rightarrow x(x_1-1) + y(y_1 - 2) - x_1 - 2y_1 - 4 = 0$ –(i)

Comparing (i) with 2x + y + 5 = 0

$$\frac{x_1-1}{2} = \frac{y_1-2}{1} = -\frac{x_1-2y_1-4}{5}$$

$$\frac{x_1-1}{2} = \frac{2(y-2)}{2} = \frac{-x-2y_1-4}{5} = \frac{x_1-1+2y_1-4-x_1-2y_1-4}{2+2+5}$$

$= \frac{-9}{9} = -1$

$x_1 = -1, y_1 = 1 \Rightarrow$ Pole (-1, 1)

kx + By – 1 = 0 is polar so it should satisfy

(-1, 1)

K(-1) + 3(1)-1 = 0

- k + 2 = 0

K = 2

11) Find the value of k if $x + y - 5 = 0$,

$2x + ky - 8 = 0$ are conjugate with respect To the circle $x^2 + y^2 - 2x - 2y - 1 = 0$

$k = 2$

12) Find the value of k if the points (4, 2) and (k, - 3) are conjugate points with respect to the circle $x^2 + y^2 - 5x + 8y + 6 = 0$

Sol. Equation of the circle is $x^2 + y^2 - 5x + 8y + 6 = 0$

Let P(4, 2) and Q(k, - 3)

Polar of P (4, 2) is $S_1 = 0$

$\Rightarrow x.4 + y.2 \frac{5}{2}(x + 4) + 4(y + 2) + 6 = 0$

$\Rightarrow 8x + 4y - 5x - 20 + 8y + 16 + 12 = 0$

$\Rightarrow 3x + 12y + 8 = 0$

P (4, 2), Q(k, -3) are conjugate point

Polar of P Passes through Q

$\therefore 3k - 36 + 8 = 0$

$3k = 28 \Rightarrow k = \frac{28}{3}$

13. **Discuss the relative position of the following pair of circles.**

i) $x^2 + y^2 - 4x - 6y - 12 = 0$

$x^2 + y^2 + 6x + 18y + 26 = 0.$

Sol. Centers of the circles are A (2,3), B(-3, -9)

Radii are $r_1 = \sqrt{4 + 9 + 12} = 5$

$r_2 = \sqrt{9 + 81 - 26} = 8$

$AB = \sqrt{(2 + 3)^2 + (3 + 9)^2}$

$= \sqrt{25 + 144} = 13 = r_1 + r_2$

$\therefore$ The circle touches externally.

ii) $x^2 + y^2 + 6x + 6y + 14 = 0$,

$x^2 + y^2 - 2x - 4y - 4 = 0$.

Sol. Centres are A(-3, -3), B(1, 2)

$r_1 = \sqrt{9+9-14} = 2$,

$r_2 = \sqrt{1+4+4} = 3$

$AB = \sqrt{(-3-1)^2 + (-3-2)^2}$

$= \sqrt{16+25} = \sqrt{41} > r_1 + r_2$

∴ Each circle lies on exterior of the other circle.

iii) $(x-2)^2 + (y+1)^2 = 9, (x+1)^2 + (y-3)^2 = 4$

Sol. Centers are A(2 -1). B(-1, 3)

$r_1 = \sqrt{4+1+4} = 3$, $r_2 = \sqrt{1+9-6} = 2$

$AB = \sqrt{(2+1)^2 + (-1-3)^2} = \sqrt{9+16}$

$= 5 = r_1 + r_2$

∴ The circles touch each other externally.

iv) $\mathbf{x^2+y^2 - 2x + 4y - 4 = 0}$,

$\mathbf{x^2+y^2 + 4x - 6y - 3 = 0}$

Sol. Center are A(1,-2), B(-2, 3)

$r_1 = \sqrt{1+4+4} = 3$, $r_2 = \sqrt{4+9+3} = 4$

$AB = \sqrt{(1+2)^2 + (-2-3)^2}$

$= \sqrt{9+25} = \sqrt{34} < r_1 + r_2$

$r_1 - r_2 < AB < r_2 + r_1$

∴ The circles intersect each other.

2) Find the number of possible common Tangents that exist for the following pairs of circles.

i) $S = x^2+y^2 + 6x + 6y + 14 = 0$, $S' = x^2+y^2 - 2x - 4y - 4 = 0$

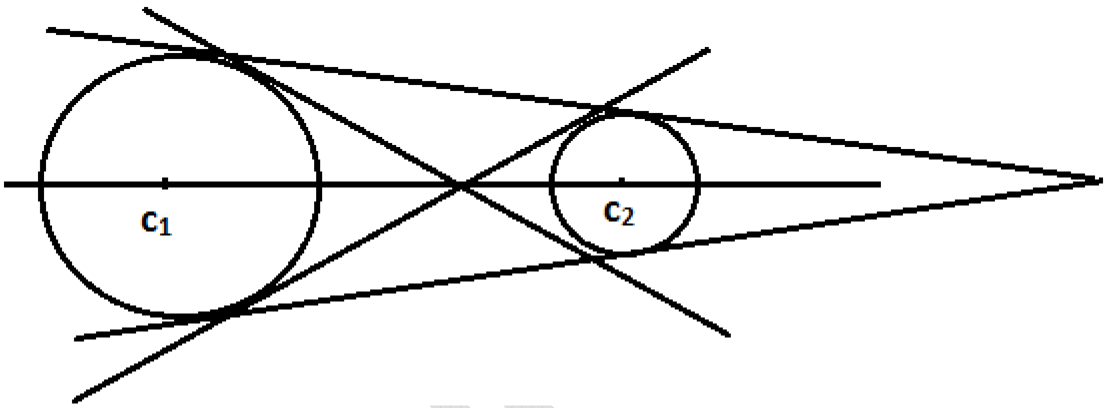
Sol.

$S = x^2+y^2 + 6x + 6y + 14 = 0$

C_1 (-3, -3) $r_1 = \sqrt{9+9-14} = 2$,

$S' = x^2+y^2 - 2x - 4y - 4 = 0$

$C_2 = (1, 2)$, $r_2 = \sqrt{1+4+4} = 0$



$C_1 . C_2 = \sqrt{(-3-1)^2 + (-3-2)^2}$

$= \sqrt{16+25} = \sqrt{41} > r_1 + r_2$

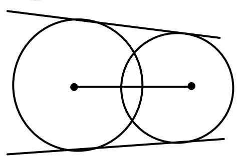
Each circle lies exterior of the other. Therefore No. of common tangents = 4

ii) $x^2+y^2 - 4x - 2y + 1 = 0; x^2+y^2 - 6x - 4y + 4 = 0.$

Sol .C_1 (2, 1) $C_2 = (3, 2)$

$r_1 = \sqrt{4+1-1}$ $r_2 = \sqrt{9+4-4}$

$= 2$ $\quad = 3$



C_1 $\quad$ C_2

$C_1\ C_2 = \sqrt{(2-3)^2 + (1-2)^2} = \sqrt{2}$

$C_1\ C_2 < r_1 + r_2$ intersect each other

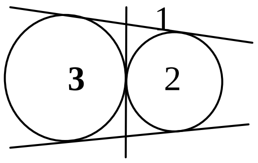
2 tangents (direct)

iii) $x^2+y^2 - 4x + 2y - 4 = 0$; $x^2+y^2 + 2x - 6y + 6 = 0$.

Sol. $C_1\ (2, -1)$ $\quad$ $C_2 = (-1, 3)$

$r_1 = \sqrt{4+1+4}\, r_1$ $\quad$ $r_2 = \sqrt{1+9-6} = 2$

$= 3$



$C_1 + C_2 = \sqrt{(2+1)^2 + (-1-3)^2}$

$= \sqrt{9+16} = 5$

$C_1\ C_2 = r_1 + r_2$ touch each other externally;

No. of common tangent s = 3.

iv) $x^2+y^2 = 4$; $x^2+y^2 - 6x + 8y + 16 = 0$

Sol. $C_1(0, 0)$ $\quad$ $C_2 = (3, 4)$

$r_1 = 2$ $\quad$ $r_2 = \sqrt{9+16-16} = 3$

$C_1 + C_2 = \sqrt{(0-3)^2 + (0-4)^2} = 5$

$r_1 + r_2 = C_1 + C_2$

$C_1\ C_2 = r_1 + r_2$ touch each other externally.

No. of common tangent s = 3.

V) $x^2+y^2 + 4x - 6y - 3 = 0$

$x^2+y^2 + 4x - 2y + 4 = 0$

Sol. $C_1\ (-2, 3)$ $\quad$ $C_2 = (-2, 1)$

$r_1 = \sqrt{4+9+3} = 4$

$r_2 \sqrt{4+1-4} = 1$

$C_1 . C_2 = \sqrt{(-2+2)^2 + (3-1)^2} = 2.$

$\Rightarrow C_1 + C_2 < r_1 - r_2$

One circle is inside the other.

$\therefore$ No common tangent =0.

14) **Find the internal centre of similitude for the circle $x^2+y^2 - 6x - 2y + 7 = 0$ and $x^2+y^2 - 2x - 6y + 9 = 0$**

Sol. $S = x^2+y^2 - 6x - 2y + 7 = 0$

$C_1 = (-3, 1)$, $r_1 = \sqrt{9+1-1} = 3$

$S' = x^2+y^2 - 2x - 6y + 9 = 0$

$C_2 = (1, 3)$, $r_2 = \sqrt{1+9-9} = 1$

The internal centre of similitude I divides the line of centres C_1C_2 internally in the ratio $r_1 : r_2 = 3 : 1$

Co-ordinates of I are $\left(\frac{1(-3)+3.1}{3+1}, \frac{1.1+3.3}{3+1}\right)$

$$=\left(\frac{-3+3}{4}, \frac{1+9}{4}\right) = \left(0, \frac{5}{2}\right)$$

15) Find the external centre of similitude of the circles $x^2+y^2 - 2x - 6y + 9 = 0$ and $x^2+y^2 = 4$

Sol. $S = x^2+y^2 - 2x - 6y + 9 = 0$ Centre C_1 (1,3) , $r_1 = \sqrt{1+9-9} = 1$ and

$S' = x^2+y^2 = 4$ centre C_2 (0,0), $r_2 = 2$

The external centre of 14similitude divides the line of centres C_1C_2 externally in the ratio $r_1 : r_2 = 1 : 2$

Co-ordinates of E are $\left(\frac{2.1-10}{2-1}, \frac{2.3-10}{2-1}\right)$

$= \left(\frac{2}{1}, \frac{6}{1}\right) = (2,6)$

Short Answer Questions

1) Find the angle between the tangents drawn from (3, 2) to the circle $x^2 + y^2 – 6x + 4y – 2 = 0$.

Sol. Equation of the circle is

$S \equiv x^2 + y^2 – 6x + 4y – 2 = 0$

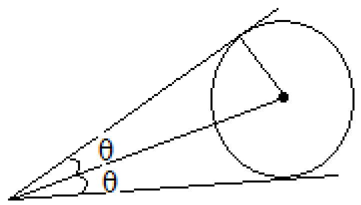
$r = \sqrt{9+4+2} = \sqrt{15}$

Point P(3,2)

$\Rightarrow S_{11} = 9 + 4 – 18 + 8 - 2 = 1$

Let 2θ be the angle between the tangents. Then

$\tan\theta = \frac{r}{\sqrt{S_{11}}} = \frac{\sqrt{15}}{1} = \sqrt{15}$



$\Rightarrow \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-15}{1+16} = -\frac{7}{8}$

Angle between the tangent at P = $\cos^{-1}\left(\frac{7}{8}\right)$

2. Find the locus of P where the tangents drawn from to $x^2 + y^2=a^2$ include an angle α

Sol. Equation of the circle is $S = x^2 + y^2 = a^2$

Radius = a

Let (x_1, y_1) be any point $.\Rightarrow S_{11} = x_1^2 + y_1^2 - a^2$

Let $2\theta(=\alpha)$ be the angle between the tangents. Then

$$\Rightarrow \tan\theta = \frac{r}{\sqrt{s_{11}}} = \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}$$

$$\Rightarrow \cos 2\theta = \frac{1 - \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}}{1 + \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}}$$

$$\Rightarrow \cos\alpha = \frac{x_1^2 + y_1^2 - 2a^2}{x_1^2 + y_1^2}$$

$\Rightarrow (x_1^2 + y_1^2)\cos\alpha = x_1^2 + y_1^2 - 2a^2$ **Locus of** (x_1, y_1) is

$(x^2 + y^2)$**Cos**$\alpha = x^2 + y^2 - 2a^2$

$2a^2 = (x^2 + y^2)(1 - \cos\alpha)$

$2a^2 = (x^2 + y^2)(2\sin^2\alpha/2)$

$x^2 + y^2 = \frac{a^2}{\sin^2\frac{\alpha}{2}} = a^2 \operatorname{cosec}^2\frac{\alpha}{2}$

3. Find the locus of P where the tangents drawn from P to $x^2 + y^2 = a^2$.

Sol.

$S = x^2 + y^2 = a^2$.

Radius = a

Let $(x_1\ y_1)$ be any point on the locus

$\Rightarrow S_{11} = x_1^2 + y_1^2 - a^2$

Let 2θ be the angle between the tangents. Then

$$\Rightarrow \tan\theta = \frac{r}{\sqrt{s_{11}}} = \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}$$

Given $2\theta = \frac{\pi}{2} \Rightarrow \tan\theta = \tan\frac{\pi}{4} = 1$

$$\therefore \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}} = 1$$

Squaring and cross – multiplying

$a^2 = x_1^2 + y_1^2 - a^2$

$\Rightarrow x_1^2 + y_1^2 - a^2$

Locus of P (x_1, y_1) is $x^2 + y^2 = 2a^2$

4. Find the slope of the polar of (1,3) with respect to the circle $x^2 + y^2 – 4x – 4y – 4 = 0$. Also find the distance from the centre to it.

Sol. Equation of the circle is

$S = x^2 + y^2 – 4x – 4y – 4 = 0$, center C =(2,2).

Polar of P (1,3) is $s_1=0$

$\Rightarrow x.1 + y.\ 3-2\ (x+1) – 2\ (y+3) – 4 = 0$

$\Rightarrow x + 3y – 2x – 2 – 2y – 6 – 4 = 0$

$\Rightarrow -x + y – 12 = 0$

Distance from the centre

$C(2,2) = \left|\frac{-2+2-12}{\sqrt{1+1}}\right|$

$= \frac{12}{\sqrt{2}} = 6 = \sqrt{2}$

5.If ax + by + c = 0 is the polar of (1,1) with respect to the circle $x^2 + y^2 – 2x + 2y + 1 = 0$ and H.C.F. of a,b,c is equal to one then find $a^2 + b^2 + c^2$.

Sol. Equation of the circle is

$S = x^2 + y^2 – 2x + 2y + 1 = 0$

Polar of (1, 1) w.r.to the circle is $S_1 = 0$.

$\Rightarrow x.1+y.1 – |(x + 1) +| (y + 1) + 1 = 0$

$\Rightarrow x + y - x - 1 + y + 1 + 1 = 0$

$\Rightarrow 2y + 1 = 0$

Given equation of the line ax + by + c = 0

Comparing (1) and (2)

$\frac{a}{0} = \frac{b}{2} = \frac{c}{1} = k$, say

a = 0, b = 2k, c= k

$a^2 + b^2 + c^2 = 0 + 4k^2 + k^2 = 5k^2$

H.C.F of (a,b,c) = 1 $\Rightarrow$ k = 1

$a^2 + b^2 + c^2 = 5\ (1)^2 = 5$

6.i) **Show that the circles $x^2+y^2 - 6x - 2y + 1 = 0$; $x^2+y^2 + 2x - 8y + 13 = 0$ Touch each other. Find the point of contact and the equation of the common tangent at their point of contact.**

Sol. Equations of the circles are

$S \equiv x^2+y^2 - 6x - 2y + 1 = 0$

Centers A (3, 1), radius $r_1 = \sqrt{9+1-1} = 3$

$S' \equiv x^2+y^2 + 2x - 8y + 13 = 0$

Centers B(-1,4) , radius $r_2 = \sqrt{1+16-13} = 2$

$AB = \sqrt{(3+1)^2 + (1-4)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$AB = 5 = 3+2 = r_1 + r_2$

$\therefore$ The circles touch each other externally. The point of contact P divides AB internally in the ratio $r_1 : r_2 = 3:2$

Co – ordinates of P are

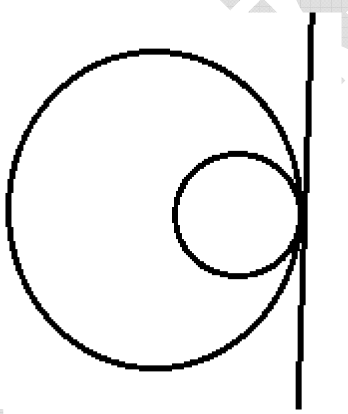
$\left(\frac{3(-1)+2.3}{5}, \frac{3.4+2.1}{5}\right)$ i.e.,p$\left(\frac{3}{5}, \frac{14}{5}\right)$

Equation of the common tangent is

$S_1 = 0$

$\Rightarrow -8x + 6y - 12 = 0 \Rightarrow 4x - 3y + 6 = 0$

ii) **Show that $x^2+y^2 - 6x - 9y + 13 = 0$, $x^2+y^2 - 2x - 16y = 0$ touch each other. Find the point of contact and the equation of common tangent at their point of contact.**



Sol.

Equations of the circles are

$S_1 \equiv x^2+y^2 - 6x - 9y + 13 = 0$

$S_2 \equiv x^2+y^2 - 2x - 16y = 0$

Centres are $A\left(3, \frac{9}{2}\right)$, B (1,8)

$r_1 = \sqrt{9\ \frac{81}{4} - 13} = \frac{\sqrt{65}}{2}$, $r_2 = \sqrt{1+64} = \sqrt{65}$

AB = $(3-1)^2 + \left(\frac{9}{2} - 8\right)^2 = \sqrt{4 + \frac{49}{4}} = \frac{\sqrt{65}}{2}$

AB = $|r_1 - r_2|$

∴ The circles touch each other internally. The point of contact 'P' divides AB

Externally in the ratio $r_1 : r_2 = \frac{\sqrt{65}}{2} : \sqrt{65} = 1:2$

Co-ordinates of p are

$$\left(\frac{1(1)-2(3)}{1-2}, \frac{1(8)-2\left(\frac{9}{2}\right)}{1-2}\right) = \left(\frac{-5}{-1}, \frac{-1}{-1}\right) = (5,1)$$

P = (5, 1)

∴ Equation of the common tangent is

$S_1 - S_2 = 0$

$-4x + 7y + 13 = 0$

$4x - 7y - 13 = 0$

7. Find the equation of the circle which touches the circle $x^2+y^2 - 2x - 4y - 20 = 0$ externally at (5, 5) with radius 5.

Sol. **S** = $x^2+y^2 - 2x - 4y - 20 = 0$

Centre C = (1, 2),

Radius r = $\sqrt{1+4+20} = 5$

Let (h, k) be the centre of the second circle.

Since circles are touching externally at (5,5) and they have equal radii, therefore

(5 , 5) = $\left(\frac{h+1}{2}, \frac{k+2}{2}\right)$ (midpoint)

$\frac{h+1}{2} = 5, \frac{k+2}{2} = 5$

h = 9 k = 8

Centre is (9, 8)

Equation of circle is

$(x-9)^2 + (y-8)^2 = 25$

$x^2+y^2 - 18x - 16y + 120 = 0$

8.. **Find the direct common tangents of the circles** $x^2+y^2 + 22x - 4y + 100 = 0$:

$x^2+y^2 - 22x + 4y + 100 = 0$.

Sol.

$x^2+y^2 + 22x - 4y + 100 = 0$

centreC_1 = (-11 ,2) ,

Radius $r_1 = \sqrt{121 + 4 + 100} = 15$

$x^2+y^2 - 22x + 4y + 100 = 0$

centre C_2 (11, -2)

Radius $r_2 = \sqrt{121 + 41 - 100} = 5$

External centre of similitude is $\left(\frac{33+11}{3-1}, \frac{-6-2}{3-1}\right) = (22, -4)$

Let m be the slope of the tangent.

Equation of the tangent is

$y+4 = m(x-22) \Rightarrow mx - y - (4+22) = 0$

This is a tangent to $x^2+y^2 - 22x + 4y + 100 = 0$

$\Rightarrow$ Radius = perpendicular from centre to this line.

$$\Rightarrow 5 = \left|\frac{11m+2-4-22m}{\sqrt{1+m^2}}\right|$$

$$\Rightarrow 25(1+m^2) = 121m^2 + 4 + 44m$$

$$\Rightarrow 96m^2 + 44m - 21 = 0$$

$$\Rightarrow 96m^2 + 72m - 28m - 21 = 0$$

$$\Rightarrow (4m+3)(24m-7) = 0$$

$$\Rightarrow m = \frac{-3}{4}, \frac{7}{24}$$

Equations of the tangents are

$y+4 = \frac{-3}{4}(x-22)$ and $y+4 = \frac{7}{24}(x-22)$

$\Rightarrow 3x + 4y - 50 = 0$ and 7x-24y-250=0

9. Find the transverse common tangents of $x^2+y^2-4x-10y+28=0$; $x^2+y^2+4x-6y+4=0$.

Sol. $\mathbf{x^2+y^2-4x-10y+28=0}$

$C_1 = (2,5)$, $r_1 = \sqrt{4+25-28} = 1$

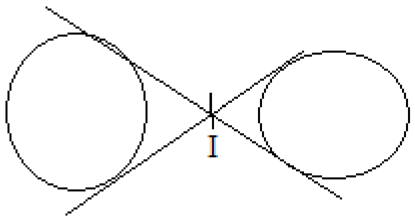
$\mathbf{x^2+y^2+4x-6y+4=0}$

$C_2 = (-2, 3)$, $r_2 = \sqrt{4+9-4} = 3$

$C_1C_2 = \sqrt{(2+2)^2+(5-3)^2}$

$= \sqrt{16+4} = \sqrt{20}$

$C_1C_2 > r_1 + r_2$



'I' divides C_1C_2 in the ratio 1:3

$I = \left[\frac{1.(-2)+3.2}{1+3}, \frac{1.3+3.5}{1+3}\right]$

$= \left[\frac{4}{4}, \frac{9}{2}\right] = \left[1, \frac{9}{2}\right]$

Let m be the slope of the tangent.

Equation of the tangent is

$y - \frac{9}{2} = m(x-1)$

$\Rightarrow 2mx - 2y + 9 - 2m = 0$

This line is tangent to $\mathbf{x^2+y^2-4x-10y+28=0}$

Radius = perpendicular distance.

$\Rightarrow 1 = \left|\frac{4m-10+9-2m}{\sqrt{4m^2+4}}\right|$

$\Rightarrow 1 = \frac{2m-1}{\sqrt{4m^2+4}}$

$\Rightarrow 4m = -3$

Since m^2 term is eliminated, the slope of the other line is not defined.(i.e., ∞)

Equation of the tangent with slope -3/4 is

$y-\frac{9}{2}=\frac{-3}{4}(x-1)$
$\Rightarrow 3x4y-21=0.$

Equation of the tangent having slope ∞ and passing through $\left[1,\frac{9}{2}\right]$ is

$x=1.$

10. Find the pair of tangents drawn from (4,10) to the circle $x^2+y^2=25$

Sol. Equation of the pair of tangents from (4, 10) to S=0 is $S_1^2 = S.\ S_{11}$

$\Rightarrow \left(\mathbf{x^2+y^2-25}\right)(16+100-25)=(4x+10y-25)$

$\Rightarrow 75x^2-9y^2-80xy250y+200x-2900=0$

Long Answer Questions

1) **Find the coordinates of the point of intersection of tangents at the points where $x+4y-14=0$ meets the circle $x^2+y^2-2x+3y-5=0$.**

Sol. Equation of the given circle is

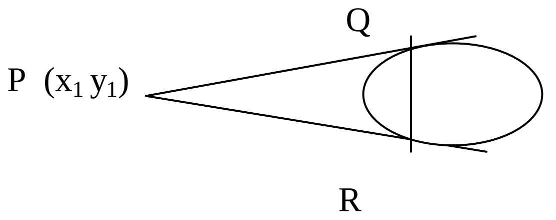
$S = x^2 + y^2 - 2x + 3y - 5 = 0.$

Equation of the line is $\mathbf{x + 4y - 14 = 0}$**---- (i)**

Let $P(x_1, y_1)$ be the point of intersection the tangents.

Equation (1) is chord contact of P with respect to S=0.

Equation of chord of contact is $S_1=0$



R

$\Rightarrow x x_1 + yy_1 - 1(x + x_1) + \frac{3}{2}(y + y_1) - 5 = 0$

$\Rightarrow 2x x_1 + 2yy_1 - 2x - x_1 + \frac{3}{2}(y + y_1) - 5 = 0$

$\Rightarrow 2(x_1 - 1)x + (2x_1 + 3)y$

$\Rightarrow 2x_1 - 3y_1 + 10 = 0$ –(2)

Comparing (1) and (2)

$\frac{2(x_1-1)}{1} = \frac{2y_1+3}{4} = \frac{2x_1-3y_1+10}{14}$

$2(x_1-1) = \frac{2y_1+3}{4}$

$\frac{2(x_1-1)}{1} = \frac{2y_1+3}{4} = \frac{2x_1-3y_1+10}{14}$

$2(x_1-1) = \frac{2y_1+3}{4}$

$8x_1 - 8 = 2y_1 + 3$

$8x_1 - 2y_1 = 11$ - (1)

$2(x_1-1) = \frac{2x_1-3y_1+10}{15}$

$28x_1 - 28 = 2x_1 - 3y_1 + 10$

$26x_1 + 3y_1 = 38$

$24x_1 - 6y_1 = 33$ (1) x3

$\underline{52x_1 + 6y_1 = 76}$ (2) x2

Adding $76x_1 = 109$

$x_1 = \frac{109}{76}$

From (3) $2y_1 = 8x_1 - 11 = 8 \times \frac{109}{76} - 11$

$= -\frac{218 - 209}{19} = \frac{9}{19}$

$y_1 = \frac{9}{38}$

$\therefore$ Co – ordinates of p are $\left(\frac{109}{76}, \frac{9}{38}\right)$

2) If the polar of the points on the circle $x^2 + y^2 = a^2$ with respect to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$ .Then prove that a,b,c are in Geometrical Progression.

Sol. let P (x_1, y_1) be a point on the circle $x^2 + y^2 = a^2$

$\Rightarrow x_1^2 + y_1^2 = a^2$ - (1)

Polar of P w.r.to the circle $x^2 + y^2 = b^2$ is

$xx_1 + yy_1 = b^2$

This is a tangent to the circle $\mathbf{x^2 + y^2 = c^2}$

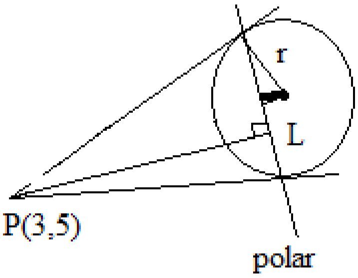
$\Rightarrow \frac{|0+0-b^2|}{\sqrt{x_1^2 + y_1^2}} = c \Rightarrow \frac{b^2}{a} = c$

$\Rightarrow b^2 = ac$

$\therefore$a, b, c are in Geometric Progression

3) **Tangents are drawn to the circle $x^2 + y^2 = 16$ from the point P (3,5). Find the area of the triangle formed by these tangents and the chord of contact of** P.

Sol. Equation of the circle is $S = x^2 + y^2 = 16$



Polar of (3 ,5) is 3x + 5y = 16

PL = length of the perpendicular from P to its polar

$= \frac{|9+25-16|}{\sqrt{9+25}} = \frac{16}{\sqrt{34}}$

Centre of the circle = C (0, 0)

d = Length of the perpendicular from c to polar

$= \frac{|0+0-16|}{\sqrt{34}} = \frac{16}{\sqrt{34}}$

Length of the chord = $2\sqrt{r^2 - d^2}$

$= 2\sqrt{16 - \frac{256}{34}} = 2\sqrt{\frac{544-256}{34}}$

$= 2\sqrt{\frac{288}{34}} = 24\sqrt{\frac{2}{34}}$

Area of Δ PQR = $\frac{1}{2}$ base. height

$= \frac{1}{2} . 24\sqrt{\frac{2}{34}} . \frac{18}{\sqrt{34}} = \frac{216\sqrt{2}}{34}$

$= \frac{108\sqrt{2}}{17}$ Sq. units.

4) Find the locus of the point whose polars with respect to the circles $x^2 + y^2 - 4x - 4y - 8 = 0$ and $x^2 + y^2 - 2x + 6y - 2 = 0$ are mutually perpendicular.

Sol.

Equation of the circles is

$S = x^2 + y^2 - 4x - 4y - 8 = 0$ - (1)

$S' = x^2 + y^2 - 2x + 6y - 2 = 0$ - (2)

Let P (x, y) be any position in the locus.

Equation of the polar of p w.r.to circle (1) is

$xx_1 \ yy_1 - 2(x + x_1) - 2(y + y_1) - 8 = 0$

$x(x_1 - 2) + y(y_1 - 2) - (2x_1 + 2y_1 + 8) = 0$ (3)

Polar of P w.r. to circle (2) is

$xx_1 + yy_1 - 1(x + x_1) - 3(y + Y_1) - 2 = 0$

$x_1 + yy_1 - x - x_1 + 3y + 3y_1 - 2 = 0$

$x(x_1 - 1) + y(y_1 + 3) - (x_1 + 3y_1 + 2) = 0$

(3) and (4) are perpendicular

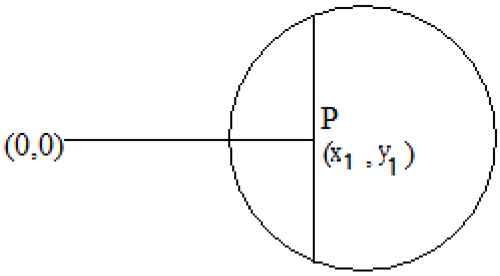
$\Rightarrow a_1 a_2 + b_1 b_2 = 0$

$(x_1 - 2)(x_1 - 1) + (y_1 - 2)(y_1 + 3) = 0$

$\Rightarrow x_1^2 + y_1^2 - 3x_1 + y_1 - 6 = 0$

Locus of $p(x_1, y_1)$ is $x^2 + y^2 - 3x + y - 4 = 0$

5) Find the locus of the foot of the perpendicular drawn from the origin to any chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ which subtends a right angle at the origin.



Sol.

Let $P(x_1, y_1)$ be the foot of the perpendicular from the origin on the chord.

Slop a OP = $\frac{y_1}{x_1}$

$\Rightarrow$ Slop of chord = $-\frac{x_1}{y_1}$

$\Rightarrow$ Equation of the chord is $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

$\Rightarrow yy_1 - y_1^2 = -xx_1 + x_1^2$

$\Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$

$\Rightarrow \frac{xx_1 + yy_1}{x_1^2 + y_1^2} = 1$-------- (1)

Equation of the circle is $x^2 + y^2 + 2fy + c = 0$ - (2) Hamogenising (2) with the help of (1). Then

$x^2 + y^2 + (2gx + 2fy)\frac{xx_1+yy_1}{x_1^2+y_1^2} + \frac{(xx_1+yy_1)^2}{(x_1^2+y_1^2)^2} = 0$

$x^2\left[1 + \frac{2gx_1}{x_1^2+y_1^2} + \frac{cx_1^2}{(x_1^2+y_1^2)}\right] + y^2\left[1 + \frac{2fy_1}{x_1^2+y_1^2} + \frac{cy_1^2}{(x_1^2+y_1^2)}\right] + (\ldots\ldots\ldots\ldots\ldots\ldots) xy = 0$

But above equation is representing a pair of perpendicular lines ,

Co – eff. of x^2 + co-eff of y^2 = 0

$1 + \frac{2gx_1}{x_1^2+y_1^2} + \frac{cx_1^2}{(x_1^2+y_1^2)^2} + 1 + \frac{2fy_1}{x_1^2+Y_1^2} + \frac{cy_1^2}{(x_1^2+y_1^2)^2} = 0$

$2 + \frac{2gx_1+2fy_1}{x_1^2+y_1^2} + \frac{(x_1^2+y_1^2)}{(x_1^2+y_1^2)} = 0$

$2 + \frac{2gx_1+2fy_1}{x_1^2+y_1^2} + \frac{c}{x_1^2+y_1^2} = 0$

$2\,(x_1^2 + y_1^2) + 2gx_1 + 2fy_1 + c = 0$

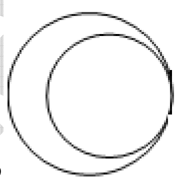
Locus of L (x_1, y_1) is

$2(x^2 + y^2) + 2g\,x + 2fy + c = 0$

6) Find the equation of the circle which touches $x^2+y^2 - 4x + 6y - 12 = 0$ (-1, -1) internally with a radius of 2.

Sol. $x^2+y^2 - 4x + 6y - 12 = 0$

$C_{1} = (2, -3)$, $r_1 = \sqrt{4+9+12} = 5$



(-1, 1)

Radius of required circle is $r_2 = 2$

Let centre of the second circle be

$C_2 = (h, k)$

Point of contact (-1, 1)

Since the two circles touch internally, point of contact divides line of centres externally in the ratio 5:2

$-1 = \frac{5h-4}{3}$ $\quad 1 = \frac{5k+6}{3}$

$h = \frac{1}{5}$, $\quad k = \frac{3}{5}$

Centre = (1/5, 3/5)

Equation of a circle with centre $\left(\frac{1}{5},\frac{-3}{5}\right)$ and radius 2 is given by

$\left(x-\frac{1}{5}\right)^2+\left(y+\frac{3}{5}\right)^2=4$

$Sx^2 + Sy^2 - 2x + 6y - 18 = 0$

7) Find the pair of tangents drawn from (1,3) to the circle $x^2+y^2 - 2x + 4y - 11 = 0$

Sol. $S= x^2+y^2 - 2x + 4y - 11 = 0$

Equation of pair of tangents from (3,2) to S=0 is $S.S_{11} = S_1^2$

$(x^2+y^2 - 2x + 4y - 11) (1+9-2+12-11)$

$= [x+ 3y - 1 (x+1) + 2 (y+3) - 11]^2$

$(x^2+y^2 - 2x + 4y - 11) 9 = (5y - 6)^2$

$9x^2+9y^2 - 18x + 36y - 99)$

$= 25y^2+36 - 60y$

$9x^2-16y^2 - 18x + 96y - 135 = 0$

Let θ be the angle between the pair of tangents. Then

$\text{Cos } \theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}} = \frac{|9-16|}{\sqrt{(25)^2}}$

$= \frac{|-7|}{25} = \frac{7}{25}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{7}{25}\right)$

8) Find the pair of tangents from the origin to the circle $x^2+y^2 +2gx + 2fy + c = 0$ and hence deduce a condition for these tangents to be perpendicular.

Sol. $S = x^2+y^{2+} 2gx + 2fy + c = 0$

Equation of pair of tangents from (0, 0) to S=0 is $S.S_{11} = S_1^2$

$(x^2+y^2 + 2gx + 2fy + c) (c) = [gx+fy+c]^2$

$\Rightarrow (x^2+y^2 + 2gx + 2fy + c)(c) = g^2x^2 + f^2y^2 + 2gfxy + 2gcx + 2fyc + c^2$

$\Rightarrow (gx + fy)^2 = c\,((x^2+y^2)$

Above tangents are perpendicular, then coefficient of x^2 + coefficient of y^2 =0

$\Rightarrow \mathbf{g}^2 - c + f^2 - c = 0$

$\Rightarrow \mathbf{g}^2 + f^2 = 2c$

6) From a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, two tangents are drawn to circles: $x^2 + y^2 + 2gx + 2fy + c \sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ $(0 < \alpha < \pi/2$ Prove that the angle between them is 2 α.

Sol. let (x_1, y_1) be a point on the circle

$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0.} \Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

Equation of the second circle is

S= $\mathbf{x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0}$

Equation of pair of tangents from $\mathbf{(x_1, y_1)}$ to S=0 is $S.S_{11} = S_1^2$

$$\left(x^2+y^2+2gx+2fy+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha\right)\left(x_1^2+y_1^2+2gx_1+2fy_1+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha\right)=$$
$$\left(xx_1+yy_1+g\left(x+x_1\right)+f\left(y+y_1\right)+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha\right)^2$$

$$\Rightarrow\left(x^2+y^2+2gx+2fy+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha\right)\left(-c+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha\right)=$$
$$\left(xx_1+yy_1+g\left(x+x_1\right)+f\left(y+y_1\right)+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha\right)^2$$

$[(-c + c\sin^2\alpha)+(g^2+f^2)\cos^2\alpha]S= (x(x_1+g) + y(y_1+f)+gx_1+fy_1+ c\sin^2\alpha + (g^2+f^2)\cos^2\alpha^2$

$[\cos^2\alpha\,(g^2+f^2-c)]S= [x(x_1+g) + y(y_1+f)+gx_1+fy_1+ c\sin^2\alpha + (g^2+f^2)\cos^2\alpha^2$

Let $g^2+f^2-c = r^2$

$[(\cos^2\alpha)r^2]S = [(x(x_1+g) + y(y_1+f)+gx_1+fy_1+c+(\cos^2\alpha).r^2)^2$

Coefficient of x^2 is $r^2\cos^2\alpha - (x_1+g)^2$

Coefficient of y^2 is $r^2\cos^2\alpha - (y_1+f)^2$

Coefficient of xy is

$h= \cos^2\alpha\, r^2 - 2(x_1+g)(y_1+f)^2$

Let θ be the angle between the tangents, then $\cos\theta = \frac{a+b}{\sqrt{(a-b)^2+4h^2}}$

$$= \frac{2r^2\cos^2\alpha - (x_1+g)^2 - (y_1+f)^2}{\sqrt{[(y_1+f)^2 - (x_1+g)^2]^2 + [4\cos^2\alpha . r^2 - (x_1+g)(y_1+f)}}$$

$= 2r^2 \cos^2\alpha - \frac{r^2}{r^2} = \frac{r^2 \sin 2\alpha}{r^2}$

Cos $\theta = \cos 2\alpha$

$\theta = 2\alpha$

Hence proved.