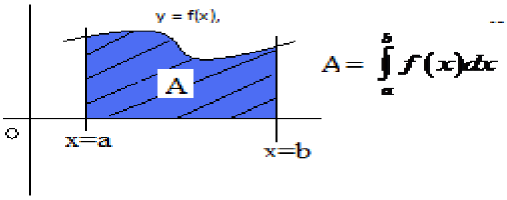
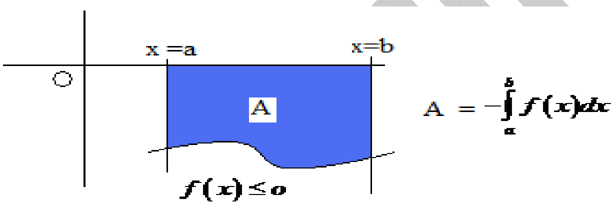
AREAS UNDER CURVES

1. **Let f be a continuous curve over [a, b]. If $f(x) \geq o$ in [a, b], then the area of the region bounded by y = f(x), x-axis and the lines x=a and x=b is given by** $\int_a^b f(x)dx$.

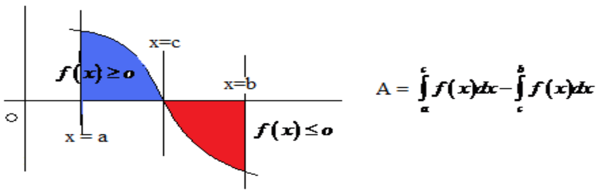


2. **Let f be a continuous curve over [a,b]. If $f(x) \leq o$ in [a,b], then the area of the region bounded by y = f(x), x-axis and the lines x=a and x=b is given by**

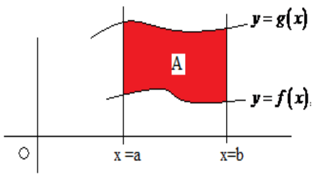
$-\int_a^b f(x)dx$



3. **Let f be a continuous curve over [a,b]. If $f(x) \geq o$ in [a, c] and $f(x) \leq o$ in [c, b] where a<c<b. Then the area of the region bounded by the curve y = f(x), the x-axis and the lines x=a and x=b is given by** $\int_a^c f(x)dx - \int_c^b f(x)dx$.

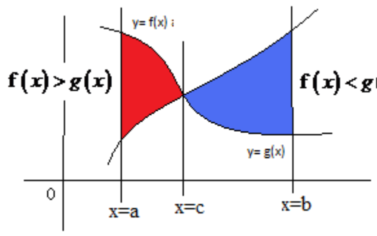


4. **Let f(x) and g(x) be two continuous functions over [a, b]. Then the area of the region bounded by the curves** $y = f(x), y = g(x)$ **and the lines x = a, x=b is given by** $\left| \int_a^b f(x)dx - \int_a^b g(x)dx \right|$ **.**



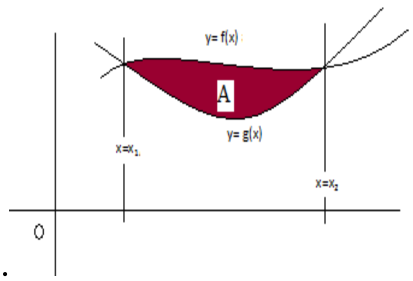
$A = \left| \int_a^b f(x)dx - \int_a^b g(x)dx \right|$

5. **Let f(x) and g(x) be two continuous functions over [a, b] and** $c \in (a, b)$**. If** $f(x) > g(x)$ **in (a, c) and** $f(x) < g(x)$ **in (c, b) then the area of the region bounded by the curves y= f(x) and y= g(x) and the lines x=a, x=b is given by** $\left| \int_a^c (f(x) - g(x))dx \right| + \left| \int_c^b (g(x) - f(x))dx \right|$



$\text{area} = \left| \int_a^c (f(x) - g(x))dx \right| + \left| \int_c^b (g(x) - f(x))dx \right|$

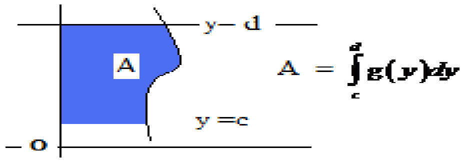
6. **let f(x) and g(x) be two continuous functions over [a, b] and these two curves are intersecting at X =x_1 and x= x_2 where** $x_1, x_2 \in (a, b)$ **then the area of the region bounded by the curves y= f(x) and y= g(x) and the lines x=x_1, x=x_2 is given by**



$A = \left| \int_{x_1}^{x_2} (f(x) - g(x))dx \right|$

$\left| \int_{x_1}^{x_2} (f(x) - g(x))dx \right|$.

Note: The area of the region bounded by x =g(y) where g is non negative continuous function in [c,d], the y axis and the lines y =c and y=d is given by $\int_c^d g(y)dy$.



Very Short Answer Questions

1. Find the area of the region enclosed by the given area

i) y = cos x , y = $1-\frac{2x}{\pi}$.

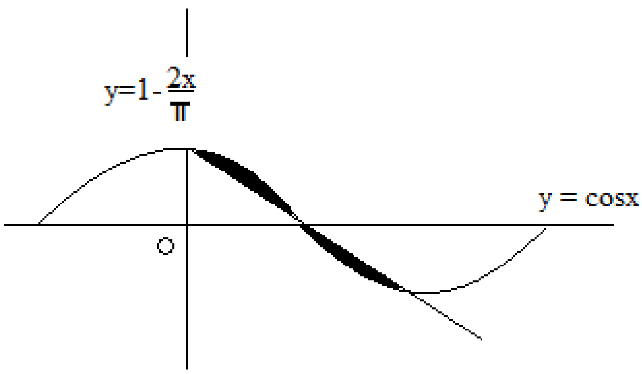
Sol: Equations of the given curves are

y = cosx ……..(1)

$y=1-\frac{2x}{\pi}$ ------ (2)

Solving (1) and (2)

$\cos x = 1-\frac{2x}{\pi}$



$\Rightarrow x = 0, \frac{\pi}{2}, \pi$

The curves are intersecting at $\Rightarrow x = 0, \frac{\pi}{2}, \pi$

in $\left(0,\frac{\pi}{2}\right)$, (1) > (2) and in $\left(\frac{\pi}{2},\pi\right)$, (2) > (1)

Therefore required area = $\int_0^{\frac{\pi}{2}} (y \text{ of (1)-y of (2)})dx + \int_{\frac{\pi}{2}}^{\pi} (y \text{ of (2)-y of (1)})dx =$

$$\int_0^{\frac{\pi}{2}} \left(\cos x - 1 + \frac{2x}{\pi}\right)dx + \int_{\frac{\pi}{2}}^{\pi} \left(1 - \frac{2x}{\pi} - \cos x\right)dx$$

$$= \left(\sin x - x + \frac{x^2}{\pi}\right)_0^{\frac{\pi}{2}} + \left(x - \frac{x^2}{\pi} - \sin x\right)_{\frac{\pi}{2}}^{\pi}$$

$$= 2 - \frac{\pi}{2}$$

2. **y = cos x , y = sin2x , x = 0,** $x = \frac{\pi}{2}$

Sol: Given curves y = cos x ---- (1)

y = sin 2x ---- (2)

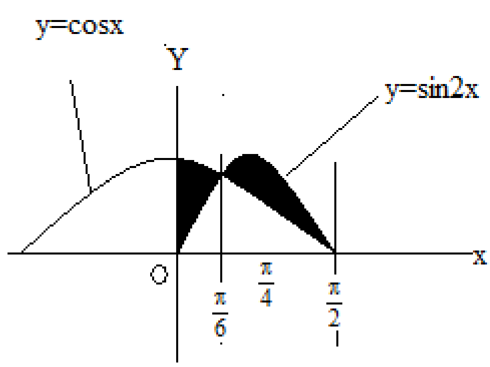
Solving (1) and (2) , cosx = sin2x

cosx-2sinx cosx=0 (where *sin(2x)* = 2 sin x cos x.)

cosx=0 and 1-2sinx =0

$x = \frac{\pi}{2}, \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

Given curve re intersecting at x=$\frac{\pi}{2}, \frac{\pi}{6}$



Required area =

$$\int_0^{\frac{\pi}{6}}(\cos x-\sin 2x)\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\sin 2x-\cos x)dx$$

$$=\left(\sin x+\frac{\cos 2x}{2}\right)_0^{\frac{\pi}{6}}+\left(-\frac{\cos 2x}{2}-\sin x\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=\left[\left(\frac{1}{2}+\frac{1}{4}\right)-\left(0+\frac{1}{2}\right)\right]-\left[\left(\frac{1}{2}-1\right)-\left(-\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}+\frac{1}{4}-\frac{1}{2}-\frac{1}{2}+\frac{1}{4}+\frac{1}{2}\;=\;\frac{1}{2}\text{sq.units}$$

3. $\mathbf{y=x^3+3,\ y=0, x=-1, x=2}$

Sol: $\mathbf{y=x^3+3,\ y=0, x=-1, x=2}$

Given curve is continuous in [-1.2] and y>0.

Area bounded by $\mathbf{y=x^3+3,\ x-axis, x=-1, x=2}$ is $\int_{-1}^{2} y\,dx$

$$=\int_{-1}^{2}\left(x^3+3\right)dx\;=\left(\frac{x^4}{4}+3x\right)_{-1}^{2}$$

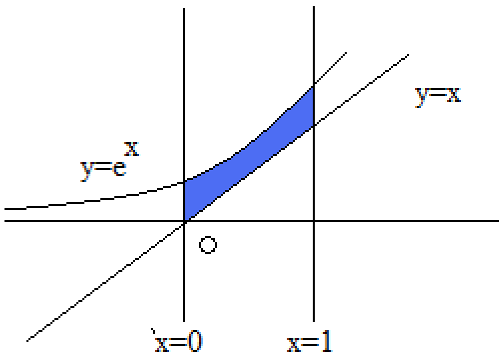
$$=\left(\frac{2^2}{4}+3.2\right)-\left(\frac{(-1)^4}{4}+3(-1)\right)$$

$$=12\frac{3}{4}\text{sq. units}$$

4. $\mathbf{y = e^x, y = x, x = 0, x = 1}$

Sol: Given curve is $\mathbf{y = e^x}$

Lines are y =x, x=0 and x=1.



Required area =

$$\int_0^1 \left(e^x - x\right)dx = \left(e^x - \frac{x^2}{2}\right)_0^1$$

$$= \left(e - \frac{1}{2}\right) - (1-0) = e - \frac{3}{2}$$

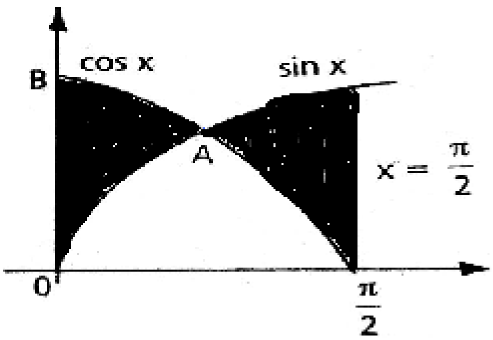
5. $y = \sin x,\ y = \cos x\ ;\ x = 0\ ,\ x = \frac{\pi}{2}$

Sol. Given curves y = sin x---- (1)

y = cos x ------- (2)

From (1) and (2), cosx = sinx

$\Rightarrow x = \frac{\pi}{4}$



Between 0 and, $\frac{\pi}{4}$ Cos x > sin x

Between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, Cos x < sin x

Required area

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)_0^{\pi/4} - (\sin x + \cos x)_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2(\sqrt{2} - 1) \text{ sq.units}.$$

6. $x = 4 - y^2, x = 0.$

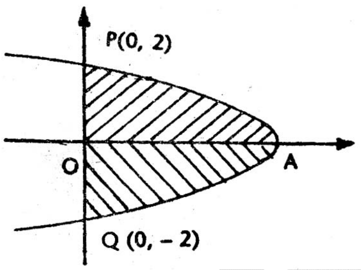
Sol: Given curve is $x = 4 - y^2$--- (1)

Put y=0 then x=4.

The parabola $x = 4 - y^2$ meets the x – axis at A(4,0).

Require area = region AQPA

Since the parabola is symmetrical about X – axis,



Required area = 2 Area of OAP

$$= 2\int_0^2 (4 - y^2)\,dy = 2\left(4y - \frac{y^3}{3}\right)_0^2$$

$$= 2\left(8 - \frac{8}{3}\right) = 2.\frac{16}{3} = \frac{32}{3} \text{ sq.units}$$

7. Find the area enclosed within the curve |x| + |y| = 1.

Sol: The given equation of the curve is |x|+|y|= 1 which represents ±x ±y = 1 representing four different lines forming a square.

Consider the line x + y = 1 ⇒ y = 1 – k

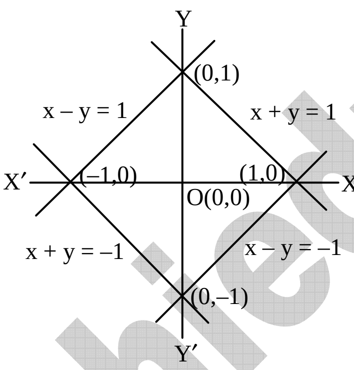
If the line touches the X-axis then y = 0 and one of the points of intersection with X-axis is (1, 0).

Since the curve is symmetric with respect to coordinate axes, area bounded by |x|+|y| = 1 is

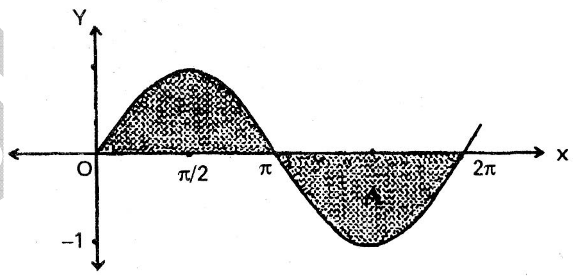
$$= 4\int_0^1 (1-x)dx$$

$$= 4\left(x - \frac{x^2}{2}\right)_0^1$$

$$= 4 - \frac{4}{2} = 2\,\text{sq.units.}$$



8. Find the area under the curve f(x) = sin x in (0, 2π)



Sol:

f(x) = sin x ,

We know that in $[0,\pi], \sin x \geq 0$ and $[\pi, 2\pi], \sin x \leq 0$

Required area $\int_{1}^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} (-\sin x)\, dx$

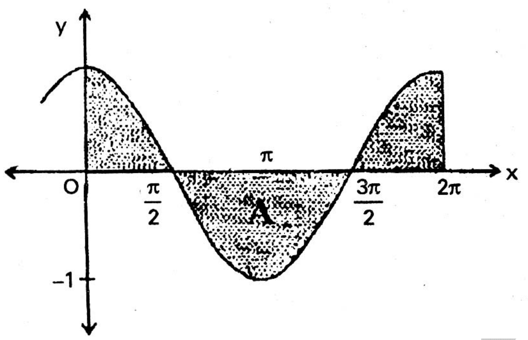
$(-\cos x)_0^{\pi} [\cos x]_{\pi}^{2\pi}$

$= \; -\cos\pi + \cos 0 + \cos 2\pi - \cos\pi$

$= -(-1)+1+1-(-1) = 1+1+1+1 = 4$

9. Find the area under the curve f(x) = cos x in [0, 2π].

Sol: We know that cos $x \geq 0$ in $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \pi\right)$ and ≤ 0 in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$



Required area

$$= \int_{0}^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos x)\, dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x dx$$

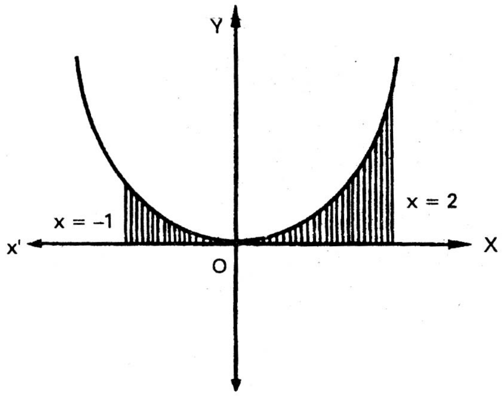
$$= (\sin x)_0^{\pi/2} + (-\sin x)_{\pi/2}^{3\pi/2} + (\sin x)_{3\pi/2}^{2\pi}$$

$$= \sin\frac{\pi}{2} - \sin 0 - \sin\frac{3\pi}{2} + \sin\frac{\pi}{2} + \sin 2\pi - \sin\frac{3\pi}{2}$$

$$= 1 - 0 - (-1) + 1 + 0 - (-1)$$

= 1 + 1 + 1 + 1 = 4.

10. Find the area bounded by the parabola $y = x^2$, the X–axis and the lines x =-1, x =2



Sol:

Required area = $\int_{-1}^{2} x^2 dx = \left(\frac{x^3}{3}\right)_{-1}^{2}$

$= \frac{1}{3}\left(2^3 - (-1)^3\right) = \frac{1}{3}(8+1) = \frac{9}{3} = 3$

11. Find the area cut off between the line y and the parabola $y = x^2 - 4x + 3$.

Sol:

Equation of the parabola is $y = x^2 - 4x + 3$

Equation of the line is y = 0

$x^2 - 4x + 3 = 0, (x-1)(x-3) = 0$, x = 1,3

The curve takes negative values for the values of x between 1 and 3.

Required area $= \int_{1}^{3} -(x^2 - 4x + 3)dx$

$= \int_{1}^{3} (-x^2 + 4x - 3)dx$

$= \left(-\frac{x^3}{3} + 2x^2 - 3x\right)_1^3$

$= (-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3\right)$

$= \frac{10}{2} - 2 = \frac{4}{3}$

Short Answer Questions

1. **$x = 2 - 5y - 3y^2$, $x = 0$.**

Sol:

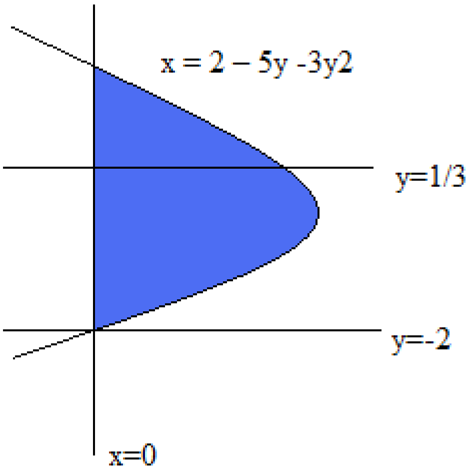
Given curve $x = 2 - 5y - 3y^2$ and $x = 0$

Solving the equations

$2 - 5y - 3y^2 = 0$,

$3y^2 + 5y - 2 = 0$

$\Rightarrow (y+2)(3y-1) = 0 \Rightarrow y = -2$ or $\frac{1}{3}$



Required area = $\int_{-2}^{\frac{1}{3}} \left(2 - 5y - 3y^2\right) dy$

$= \left(2y - \frac{5}{2}y^2 - y^3\right)_{-2}^{\frac{1}{3}}$

$= \left(\frac{2}{3} - \frac{5}{2}.\frac{1}{9} - \frac{1}{27}\right) - \left(-4 - \frac{5}{2}.4 + 8\right)$

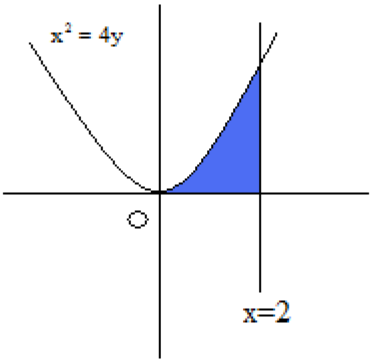
$= \left(\frac{2}{3} - \frac{5}{8} - \frac{1}{27}\right) + 6$

$= \frac{36 - 15 - 2 + 324}{54} = \frac{343}{54}$ sq. units

2. $x^2 = 4y,\ x = 2\ , y = 0$

Sol. Given curve $x^2 = 4y$,

X=2 and y=0 i.e., x- axis

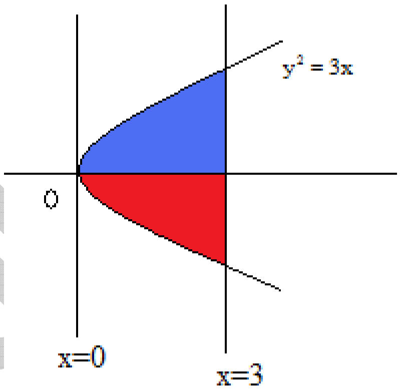


Required curve = $\int_0^2 y\,dx = \int_0^2 \frac{x^2}{4}dx =$

$$\left(\frac{x^3}{12}\right)_0^2 = \frac{8}{12} = \frac{2}{3} \text{ sq. units.}$$

3. $y^2 = 3x, x = 3$.

Given curve is $y^2 = 3x$ and the line is $x = 3$



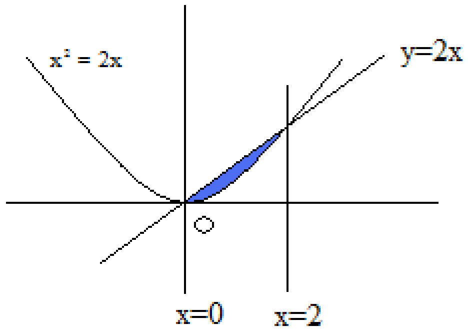
The parabola is symmetrical about X – axis Required area = 2 (area of the region bounded by the curve, x-axis, x=0 and x=3)

$$= 2\int_0^3 y\,dx = 2\int_0^3 \sqrt{3}.\ \sqrt{x}\,dx$$

$$= \left(2\sqrt{3}.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^3 = \frac{4\sqrt{3}}{3}.\left(3\sqrt{3}-0\right) = 12\,\text{sq.units}$$

4) $y = x^2, y = 2x$.

Sol:



Eliminating y, we get $x^2 = 2x$

$x^2 - 2x = 0$, x(x-2) =0

x = 0 or x = 2, y=0 or y = 4

Points of intersection are O(0,0), A(2,4)

Required area = $\int_0^2 \left(2x - x^2\right) dx$

$$= \left(x^2 - \frac{x^3}{3}\right)_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units}$$

5. y = sin 2x , y = $\sqrt{3}$ sin x, x =0, $x = \frac{\pi}{6}$.

Sol; y = sin 2x-------- (1)

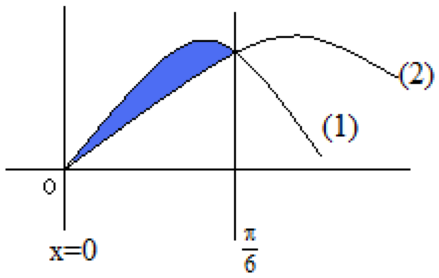
$y = \sqrt{3}\sin x$ _______(2)

Solving Sin2x = $\sqrt{3}\sin x$

$\Rightarrow$ 2sinx.cosx = $\sqrt{3}\sin x$

$\Rightarrow$ Sinx =0 **or** 2 cos $x = \frac{\sqrt{3}}{2}$

$$\Rightarrow x = 0\,,\ \cos x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{6}$$



Required area = $\int_0^{\frac{\pi}{6}} \left(\sin 2x - \sqrt{3}\sin x\right) dx$

$$= \left(-\frac{\cos 2x}{2} + \sqrt{3}\cos x\right)_0^{\frac{\pi}{6}}$$

$$= \left(-\frac{1}{4} + \sqrt{3}.\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} + \sqrt{3}\right)$$

$$= -\frac{1}{4} + \frac{3}{2} + \frac{1}{2} - \sqrt{3} = \frac{7}{4} - \sqrt{3}\,\text{sq. units}$$

6). $y = x^2, y = x^3$.

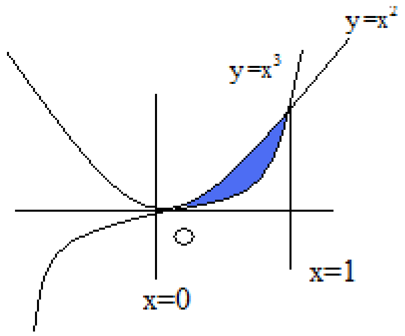
Sol: Given equations are $y = x^2$ ______(1)

$y = x^3$ _______(2)

From equation (1) and (2) $x^2 = x^3$

$x^3 - x^2 = 0,\ x^2(x-1) = 0$

$x = 0$ or 1



Required area = $\int_0^1 \left(x^2 - x^3\right)dx$

$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ sq. units

7). $y = 4x - x^2,\ y = 5 - 2x$.

Sol:

Given curves $y = 4x - x^2$ ________ (i)

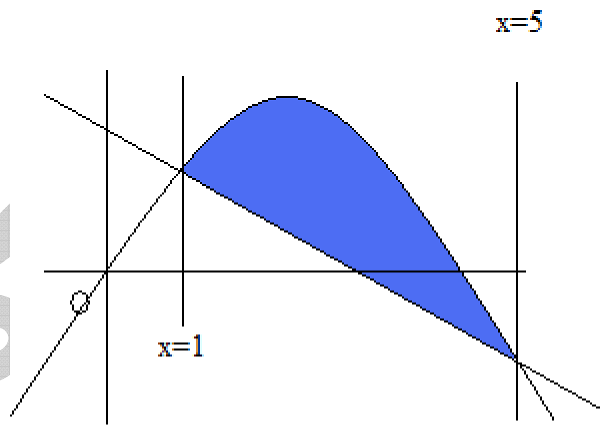
$y = 5 - 2x$ ________ (ii)

$y = -\left([x-2]^2\right) + 4,\ y-4=(x-2)^2$

Solving (i) and (ii) we get

$4x-x^2 = 5-2x,\ x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$, $X = 1, 5$



Required area $= \int_1^5 (yof(1) - yof(2))dx = \int_1^5 (4x - x^2 - 5 + 2x)dx$

$$= \int_1^5 (6x - x^2 - 5)dx = \int_1^5 \left(3x^2 - \frac{x^3}{3} - 5x\right)_1^5$$

$$= \left(75 - \frac{125}{3} - 25\right) - \left(3 - \frac{1}{3} - 5\right)$$

$$= 50 - \frac{125}{3} + 2 + \frac{1}{3}$$

$$= \frac{150 - 125 + 6 + 1}{3} = \frac{32}{3} \text{sq.units}$$

8. Find the area in sq.units bounded by the X-axis, part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates x = 2 and x = 4.

Sol: In [2, 4] we have the equation of the curve given by $y = 1 + \frac{8}{x^2}$.

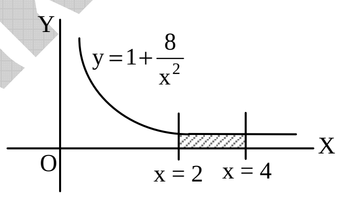
$\therefore$ Area bounded by the curve $y = 1 + \frac{8}{x^2}$.

X-axis and the ordinates x = 2 and x = 4 is

$$= \int_2^4 y\,dx = \int_2^4 \left(1 + \frac{8}{x^2}\right)dx$$

$$= \left[x - \frac{8}{x}\right]_2^4 = \left(4 - \frac{8}{4}\right) - \left(2 - \frac{8}{2}\right)$$

$$= 2 + 2 = 4 \text{sq.units.}$$



9. Find the area of the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Sol: Given equations of curves are

$y^2 = 4x$ … (1)

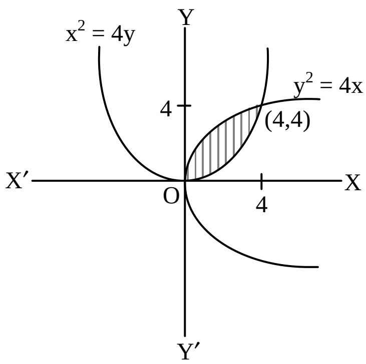
And $x^2 = 4y$ …(2)

Solving (1) and (2) the points of inter-section can be obtained.

$Y^2 = 4x \Rightarrow y^4 = 16x^2 \Rightarrow y^4 = 64y \Rightarrow y = 4$

$\therefore 4x = y^2 \Rightarrow 4x = 16 \Rightarrow x = 4$

Points of intersection are (0, 0) and (4, 4).



$\therefore$ Area bounded between the parabolas

$$= \int_0^4 \sqrt{4x}\,dx - \int_0^4 \frac{x^2}{4}dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{4}{3}(4^{3/2}) - \frac{1}{12}(64)$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{sq.units.}$$

10. Find the area bounded by the curve y = log x, the X-axis and the straight line x = e.

Sol: Area bounded by the curve $y = \log_c x$,

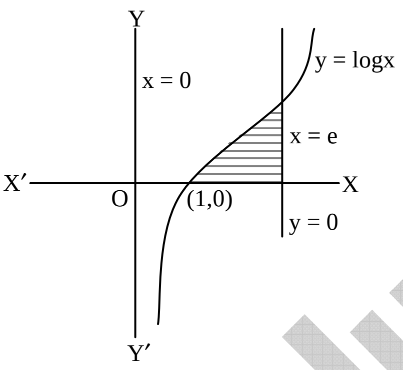
X-axis and the straight line x = e is

$$= \int_1^e \log_e x\,dx$$

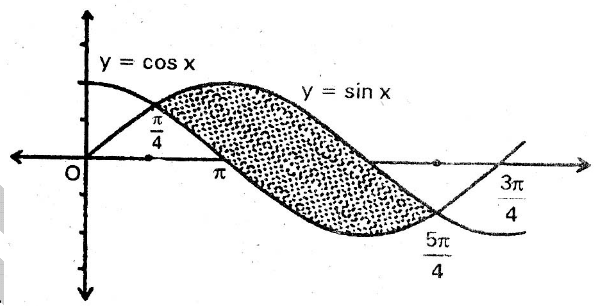
$$= [x\log x]_1^e - \int_1^e dx$$

($\because$ When x = e, $y = \log_e e = 1$)

= (e – 0) – (e – 1) = 1 sq.units.



11. Find the area bounded by y = sin x and y = cos x between any two consecutive points of intersection .



Sol:

Two consecutive points of intersection are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

$\sin x \geq \cos x$ for all $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Required area = $\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

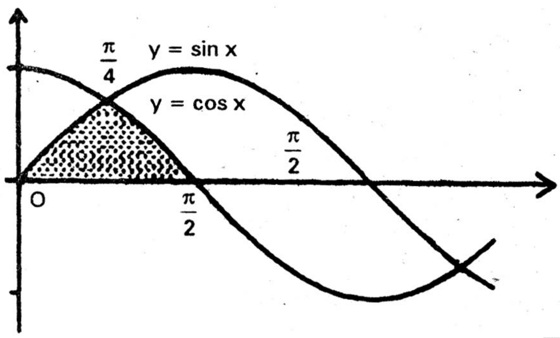
$$= \left(-\cos x - \sin x\right)_{\pi/4}^{5\pi/4}$$

$$= \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) + \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 4\frac{1}{\sqrt{2}} = 2\sqrt{2}$$

12. Find the area of one of the curvilinear triangles bounded by y = sin x, y = cos x and X – axis.

Sol:



In $\left(0, \frac{\pi}{4}\right) \cos x \ge \sin x$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right), \cos x \le \sin x$.

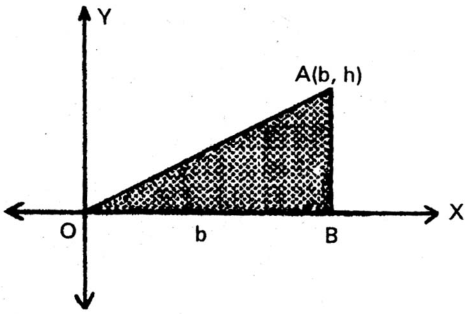
Required area = $\int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$

$$= \left(-\cos x\right)_0^{\pi/4} + \left(\sin x\right)_{\pi/4}^{\pi/2}$$

$$= -\cos\frac{\pi}{4} + \cos 0 + \sin\frac{\pi}{2} - \sin\frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}} = 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$$

13. Find the area of the right angled triangle with base b and altitude h, using the fundamental theorem of integral calculus.



Sol:

OAB is a right angled triangle and $\angle B = 90°$ take 'O' as the origin and OB as positive X – axis

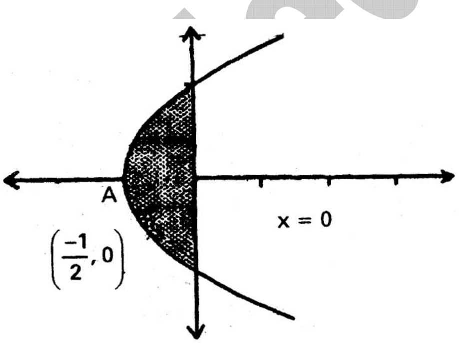
If OB = b and AB = h, then co – ordinates of A are (b, h)

Equation of OA is $y = \frac{h}{b}x$

Area of the triangle OAB = $\int_0^b \frac{h}{b} x dx$

$= \frac{h}{b}\left(\frac{x^2}{2}\right)_0^b = \frac{h}{b}\frac{b^2}{2} = \frac{1}{2}bh$.

14. Find the area bounded between the curves $y^2 - 1 = 2x$ and $x = 0$



Sol:

The parabola $y^2 - 1 = 2x$ meets

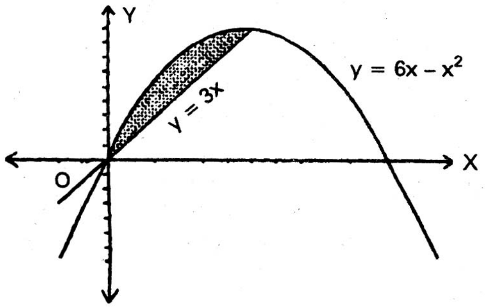
X – axis at A $\left(-\frac{1}{2}\ 0\right)$ and Y – axis at y = 1

y =-1. The curve is symmetrical about X – axis required area

$= \int_{-1}^{1}(-x)dy = \int_{-1}^{1} -\left(\frac{y^2-1}{2}\right)dy$

$$= \int_0^1 -(y^2 - 1)\,dy = \left(-\frac{y^3}{3} + y\right)_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

15. Find the area enclosed by the curves y = 3 and y = 6x- x².



Sol:

The straight line y = 3x meets the parabola

$y = 6x - x^2$. $3x = 6x - x^2, x^2 - 3x = 0$

$x(x - 3) = 0$, $x = 0$ or 3

Required area = $\int_0^3 (6x - x^2 - 3x)\,dx$

$$= \int_0^3 (3x - x^2)\,dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)_0^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}.$$

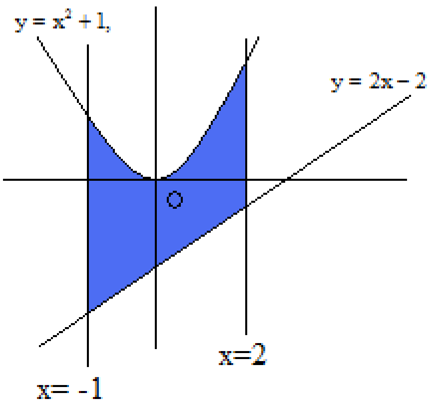
Long Answer Questions

1. $y = x^2 + 1, y = 2x - 2, x = -1, x = 2$.

Sol: Equation of the curves are

$y = x^2 + 1$ ________ (1)

$y = 2x - 2$ ________(2)



Area between the given curves

$$= \int_{-1}^{2} \left(f(x) - g(x)\right) dx$$

$$= \int_{-1}^{2} \left[\left(x^2 - 1\right) - (2x - 2)\right] dx$$

$$= \int_{-1}^{2} \left(x^2 - 2x + 3\right) dx$$

$$= \left(\frac{8}{3} - 4 + 6\right) - \left(-\frac{1}{3} - 1 - 3\right)$$

$$\frac{8}{3} + 2 + 4 + \frac{1}{3} = 3 + 6 = 9 \text{ sq.units}.$$

2. $y^2 = 4x,\ y^2 = 4(4-x)$

Sol: Equation of the curves are

$y^2 = 4x$ ____(1)

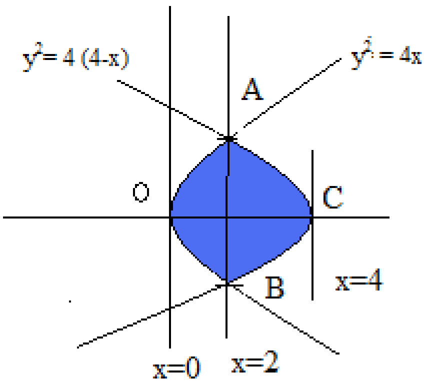
$y^2 = 4(4-x)$ _____(2)

Solving, we get

$4x = 4(4-x) \Rightarrow 2x = 4 \Rightarrow x = 2$

y=0 $\Rightarrow x=0$ and $x=4$

Given curves intersects at x=2 and those curves intersect the x axis at x=0 and x=4.



Required area is symmetrical about X – axis Area OACB

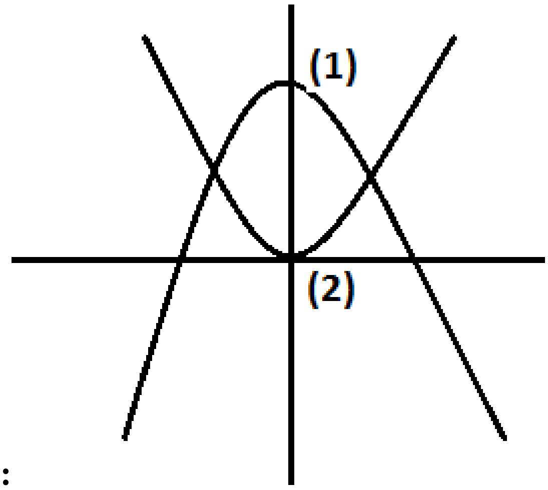
$$= 2\left[\int_0^2 2\sqrt{x}\,dx + \int_2^4 2\sqrt{4-x}\,dx\right]$$

$$= 2\left(2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^2 + 2\left\{\frac{(4-x)^{\frac{3}{2}}}{-\frac{3}{2}}\right\}_2^4$$

$$= 2\left[\frac{4}{3}\left(2\sqrt{2}\right) - \frac{4}{3}\left(-2\sqrt{2}\right)\right] = 2\left(\frac{8\sqrt{2}}{3} + \frac{8\sqrt{2}}{3}\right)$$

$$= 2\left(\frac{16\sqrt{2}}{3}\right) = \frac{32\sqrt{2}}{3} \text{ sq. units}$$

3. $y = 2 - x^2, y = x^2$



Sol:

$y = 2 - x^2$ ________(1)

$y = x^2$ ________(2)

FROM above equations,

$2 - x^2 = x^2, 2 = 2x^2 \text{ or } x^2 = 1$

$x = \pm 1$

Area bounded by two curves is

$$2 \times \int_{-1}^{1} (y \text{ of } (1) - y \text{ of } (2)) dx$$

$$= 2\int_{-1}^{1} (2 - x^2 - x^2) dx$$

$$= 2\int_{-1}^{1} (2 - 2x^2) dx = 2\left(2x - \frac{2x^3}{3}\right)_{-1}^{1}$$

$$= 2\left[2 - \frac{2}{3}\right] = \frac{8}{3} \text{ sq. units}.$$

4. Show that the area enclosed between the curve $y^2 = 12(x+3)$ and $y^2 = 20(5-x)$ is $64\sqrt{\frac{5}{3}}$.

Sol: Equation of the curves are

$y^2 = 12(x+3)$ ______(1)

$y^2 = 20(5-x)$ ________(2)

Eliminating y

12(x+3) = 20(5 – x)

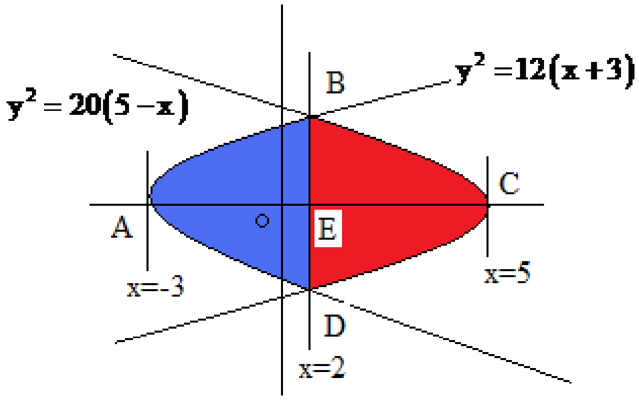
x + 9 = 25 – 5x, 8x = 16, x = 2

Given curves are intersecting on x=2.

The points of intersection of the curves and the x axis are x=5 and x=-3.

$y^2 = 12(2+3) = 60$

$y = \sqrt{60} = \pm\, 2\sqrt{15}$



The required area is symmetrical about X – axis

Required area =2x(AREA ABCOA)

= 2.(AREA ABEA + AREA BECB)

$$= 2\left[\int_{-3}^{2} 2\sqrt{3}\sqrt{x+3}\,dx + \int_{2}^{5} 2\sqrt{5}\sqrt{5-x}\,dx\right]$$

$$= 4\sqrt{3}\left(\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}\right)_{-3}^{2} + 4\sqrt{5}\left(\frac{(5-x)^{\frac{3}{2}}}{-\frac{3}{2}}\right)_{2}^{3}$$

$$= \frac{8\sqrt{3}}{3}\left(\frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}}\right)_{-3}^{2} + 4\sqrt{5}\left(\frac{(5-x)^{\frac{3}{2}}}{-\frac{3}{2}}\right)_{2}^{5}$$

$$= \frac{8\sqrt{3}}{3}\left(5^{\frac{3}{2}} - 0\right) - \frac{8\sqrt{5}}{3}\left[0 - 3^{\frac{3}{2}}\right]$$

$$= \frac{8\sqrt{3}}{3}.5\sqrt{5} + \frac{8\sqrt{5}}{3}\left[0 - 3^{\frac{3}{2}}\right]$$

$$= \frac{40.\sqrt{15}}{3} + \frac{24\sqrt{15}}{3} = \frac{64}{3}\sqrt{15} \text{ sq.units}$$

$$= 64\sqrt{\frac{15}{9}} \text{ sq. units} = 64\sqrt{\frac{5}{3}} \text{ sq. units.}$$

5. Find the area of the region $\{(x,y)/x^2 - x - 1 \le y \le -1\}$

Sol. Let the curves be $y = x^2 - x - 1$ ------(1)

And **$y = -1$** **--------- (2)**

$$y = x^2 - x - 1 = \left(x - \frac{1}{2}\right)^2 - \frac{5}{4}$$

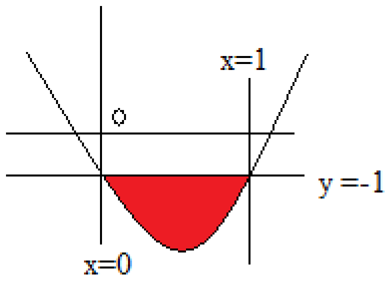
$y = \frac{5}{4} - \left(x - \frac{1}{2}\right)^2$ is a parabola with

Vertex $\left(\frac{1}{2}, -\frac{5}{4}\right)$

From (1) and (2),

$$x^2 - x - 1 = -1 \Rightarrow x^2 - x = 0 \Rightarrow x = 0, x = 1$$

Given curves are intersecting at x=0 and x=1.



Required area = $\int_0^1 \left(y \text{ of } (1) - y \text{ of } (2)\right)dx$

$$A = \left|\int_0^1 \left(x^2 - x - 1\right)dx - \int_0^1 (-1)\,dx\right|$$

$$= \left|\int_0^1 \left(\frac{x^3}{3} - \frac{x^2}{2} - x\right) - {}_0^1[-x]\right| = \frac{1}{6}\text{sq.units}$$

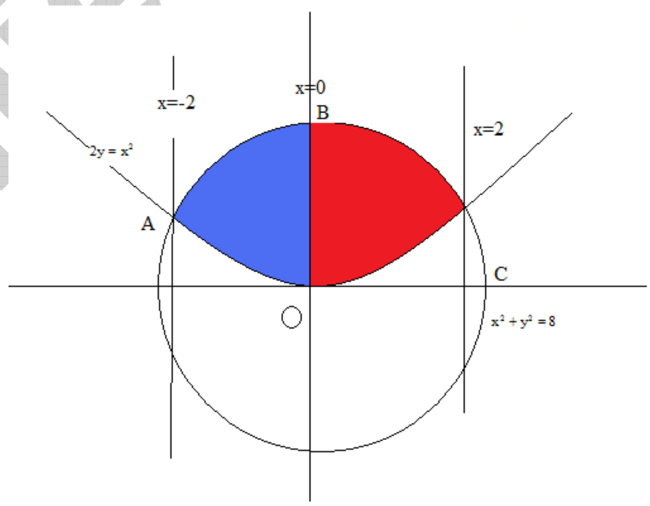
6. The circle $x^2 - y^2 = 8$ is divided into two parts by the parabola $2y = x^2$. Find the area of both the parts.

Sol:

$x^2 + y^2 = 8$ _______ (1)

$2y = x^2$ ________(2)

Eliminating Y between equations (1) and (2)



Let $x^2 = t$, $4t + t^2 = 32$, $t^2 + 4t - 32 = 0$

$(t + 8)(t - 4) = 0$

$t = -8$ (not possible) $x^2 = 4 \Rightarrow x = \pm 2$

Given curves are intersecting at x=2 and x=-2.

$$\text{AREA OBCO} = \int_0^2 \sqrt{8-x^2}\,dx - \int_0^2 \frac{x^2}{2}\,dx$$

$$= \left[\frac{1}{2}\times.\sqrt{8-x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right]_0^2 - \left[\frac{x^3}{6}\right]_0^2$$

$$= \frac{1}{2}.2.2 + 4.\frac{\pi}{4} - \frac{8}{6} = \frac{2}{3} + \pi$$

As curve is symmetric about Y – axis, total area ABCOA= 2. OBCO

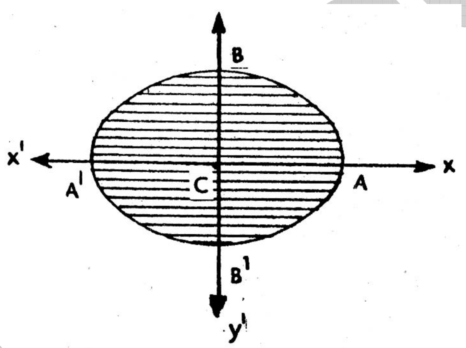
$$= 2\left(\frac{2}{3}+\pi\right) = \frac{4}{3} + 2\pi \text{sq. units}.$$

AREA of the circle = $\pi r^2 = 8\pi$

Remain part = $8\pi - \left(\frac{4}{3} + 2\pi\right)$

$$= \left(6\pi - \frac{4}{3}\right)\text{sq. units}.$$

7. Show that the area of the region bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$(ellipse) is π ab . Also deduce the area of the circle $x^2+y^2=a^2$.



Sol:

The ellipse is symmetrical about X and Y axis Area of the ellipse = 4 Area of CAB= $4.\frac{\pi}{4}$ ab

Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$CAB = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\ dn$$

$$= \frac{b}{a}\left(\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0$$

$$= \frac{b}{a}\left(0 + \frac{a^2}{2}.\frac{\pi}{2} - ab\right) = \frac{\pi a^2}{4}.\frac{b}{a} = \frac{\pi}{4}ab$$

(From prob. 8 in ex 10(a)) $= \pi ab$

Substituting b = a, we get the circle

$x^2 + y^2 = a^2$

Area of the circle = $\pi a(a) = \pi a^2$ sq. units.

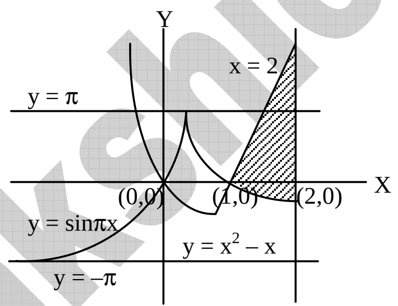
8. Find the area of the region enclosed by the curves $y = \sin \pi x$, $y = x^2 - x$, $x = 2$.

Sol: The graphs of the given equations

$y = \sin \pi x$ … (1)

and $y = x^2 - x$, $x = 2$ are shown below.

X	–2	–1	0	1	2	3
$y = x^2 - x$	6	+2	0	0	2	6



Required area bounded by

$y = \sin \pi$, $y = x^2 - x$, $x = 2$ is given by

$$= \left| \int_1^2 \sin \pi x \, dx - \int_1^2 (x^2 - x) dx \right|$$

$$= \left| -\left(\frac{\cos \pi x}{\pi}\right)_1^2 - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^2 \right|$$

$$= \left| -\left[\frac{\cos 2\pi}{\pi} - \frac{\cos \pi}{\pi}\right] - \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] \right|$$

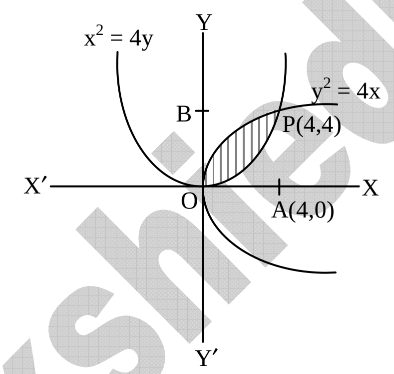
$$= \left| -\frac{1}{\pi}[1+1] - \left[\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right] \right|$$

$$= \left| -\frac{2}{\pi} - \left[\frac{2}{3} + \frac{1}{6}\right] \right|$$

$$= \left| -\frac{2}{\pi} - \frac{5}{6} \right| = \frac{2}{\pi} + \frac{5}{6} \text{ sq.units.}$$

9. **Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by the lines $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.**

Sol:



The given curves are $y^2 = 4x$ ...(1)

and $x^2 = 4y$ ...(2)

Solving $y^4 = 16x^2 = 64y$

$\Rightarrow y(y^3 - 64) = 0$

$\Rightarrow y = 0$ or $y = 4$

When $y = 4$ we have $4x = 16 \Rightarrow x = 4$.

$\therefore$ Points of intersection of parabola is P(4, 4).

$\therefore$ Area bounded by the parabolas

$$=\int_0^4 2\sqrt{x}\,dx-\int_0^4 \frac{x^2}{4}dx$$

$$=\int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$$

$$=2\left(\frac{2}{3}\right)(x^{3/2})_0^4-\frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$=\frac{4}{3}(8)-\frac{1}{4}\left(\frac{64}{3}\right)$$

$$=\frac{32}{3}-\frac{16}{3}=\frac{16}{3}\text{sq.units.}$$

Area of the square formed = $(OA)^2 = 4^2 = 16$

Since the area bounded by the parabolas

$x^2 = 4y$ and $y^2 = 4x$ is $\frac{16}{3}$ sq.units. which is one third of the area of square we conclude that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by x = 0,

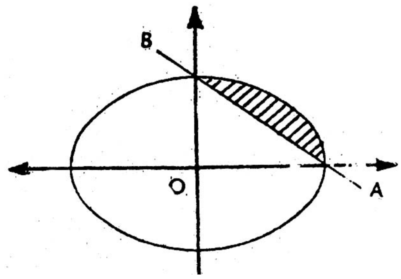
x = 4, y = 0, y = 4 into three equal parts.

10. **Let AOB be the positive quadrant of the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with OA = a, OB = b. Then show that the area bounded between the chord AB and the arc AB of the ellipse is $\frac{(\pi-2)ab}{4}$.**

Sol: Let OA = a, OB = b

Equation of AB is $\frac{x}{a}+\frac{y}{b}=1$

$\frac{y}{b}=1-\frac{x}{a},\ y=b\left(1-\frac{x}{a}\right)$



Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

Required area

$$=\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}\right)dx-\int_0^a b\left(1-\frac{x}{a}\right)dx$$

$$==\frac{b}{a}\left[x\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$-b\left(x-\frac{1}{a}.\frac{x^2}{2}\right)_0^a$$

$$=\frac{b}{a}\left[0+\frac{a^2}{2}.\sin^{-1}1-(0+0)\right]\ -b\left[a-\frac{a^2}{2a}-0\right]$$

$$=\frac{b}{a}.\frac{a^2}{2}.\frac{\pi}{2}-\frac{ab}{2}=\frac{ab}{4}(\pi-2)\text{sq.units}$$

11. Find the area enclosed between $y=x^2-5x$ **and y = 4-2x.**

Sol: Equations of the curves are

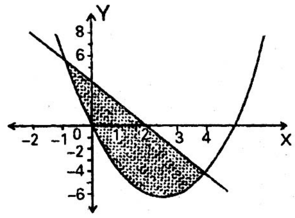
$y = x^2 -5x$ ……….(1)

$y = 4-2x$ ……….(2)

$x^2-5x=4-2x\,,\,x^2-5x=4-2x$

$x^2-3x-4=0$

$(x+1)(x-4)=0$ x =-1,4



Required area $\int_{-1}^{4}\left[(4-2x)-(x^2-5x)\right]dx$

$= \int_{-1}^{4}(4+3x-x^2)dx = \left(4x+\frac{3}{2}x^2-\frac{x^3}{3}\right)_{-1}^{4}$

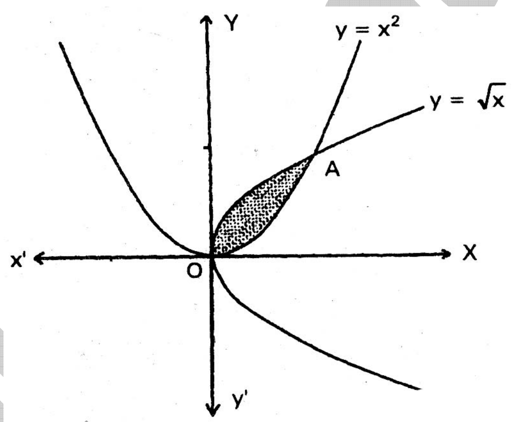
$= \left(16+\frac{3}{2}16-\frac{64}{3}\right)-\left(-4+\frac{3}{2}+\frac{1}{3}\right)$

$= 16+24-\frac{64}{3}+4-\frac{3}{2}-\frac{1}{3}$

$= 44-\frac{64}{3}-\frac{3}{2}-\frac{1}{3}$

$= \frac{264-128-9-2}{6}=\frac{125}{6}$

12. Find the area bounded between the curves $y=x^2,\ y=\sqrt{x}$ **.**



Sol:

Equations of the given curves are

$y=\sqrt{x}$ ………(1)

$y=x^2$ …………(2)

$\therefore\ \sqrt{x}=x^2 \Rightarrow x^4=x$

$x(x^3-1)=0$, x=0 or x =1

$\therefore$ The curves intersect at O(0,0) A(1,1)

Required area = $= \int_0^1 \left(\sqrt{x} - x^2\right) dx$

$= \left(\frac{2}{3} \times \sqrt{x} - \frac{x^3}{3}\right)_0^1 - \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

13. Find the area bounded between the curves $y^2 = 4ax,\ x^2 = 4by\,(a > 0, b > 0)$**.**

Sol: Equations of the given curves are

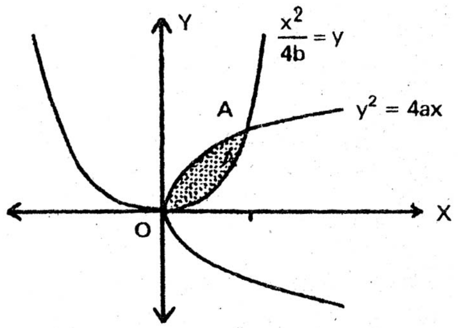
$y^2 = 4ax$ ……….(1)

$x^2 = 4by$ ………..(2)

From equation (2) y = $\frac{x^2}{4b}$

Substituting in (1) $\left(\frac{x^2}{4b}\right)^2 = 4ax$

$x^4 = (16b^2)\,|4ax|$



$x\left[x^3 - 64b^2a\right] = 0$

X = 0, x = 4 $(b^2a)^{1/3}$

Area bounded will be

$= \int_0^{4(b^2a)^{1/3}} \left[\sqrt{4ax} - \frac{x^2}{4b}\right] dx$

$= {}_0^{4(b^2a)^{1/3}} \left[(4a)^{1/2}\, x^{3/2}.\frac{2}{3} - \frac{x^3}{12b}\right]$

$$= \left[(4a)^{1/2}\, 8\left(b^2a\right)^{\frac{1}{3}\cdot\frac{3}{2}} \frac{2}{3} - \frac{4^3\left(b^2a\right)^{3\cdot\frac{1}{3}}}{12b} \right]$$

$$= \left[2ab\frac{16}{3} - \frac{64.b^2a}{12b} \right] = ab\left(\frac{32}{3} - \frac{16}{3}\right)$$

$$= \frac{16}{3}\ \text{ab sq.units}$$