A

2011 MS

Test Paper Code: MS

Time: 3 Hours **Maximum Marks: 300**

INSTRUCTIONS

1. This question-cum-answer booklet has **32** pages and has **25** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **- 2 (Negative two)** mark.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

2011 MS

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER						

Name:

Test Centre:

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

……………………………………………...
Signature of the Candidate

I have verified the information filled by the Candidate above.

……………………………………………...
Signature of the Invigilator

Special Instructions/ Useful Data

1. $\mathbb{R}$: Set of all real numbers.
2. $\mathbb{Q}$: Set of all rational numbers.
3. $\underset{\sim}{x}^T$: Transpose of a column vector $\underset{\sim}{x}$.
4. i.i.d.: independent and identically distributed.
5. $N(\mu, \sigma^2)$: Normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.
6. For a fixed $\lambda > 0$, X is $Exp(\lambda)$ random variable means that the probability density function of X is

$$f(x \mid \lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

7. $U(a,b)$: Continuous Uniform distribution on (a,b), $-\infty < a < b < \infty$.
8. $B(n, p)$: Binomial distribution with parameters $n \in \{1, 2, \ldots\}$ and $p \in (0,1)$.
9. $E(X)$: Expectation of a random variable X.
10. Based on the observations $(x_1, \ldots, x_n)$,

$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ is the sample mean

and $s_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$ is the sample variance.

11. t_n : Central Student's t random variable with n degrees of freedom.
12. $P(t_5 \le 2.57) = 0.975$, $P(t_5 \le 2.01) = 0.95$, $P(t_4 \le 2.78) = 0.975$, $P(t_4 \le 2.13) = 0.95$.
13. χ_n^2 : Central Chi-square random variable with n degrees of freedom.
14. $P(\chi_{20}^2 > 10.85) = 0.95$, $P(\chi_{10}^2 > 3.94) = 0.95$, $P(\chi_{20}^2 > 21.7) = 0.36$, $P(\chi_{10}^2 > 7.88) = 0.69$.

IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-25 (subjective questions) carry *twenty one* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 Let the function $f:[0,\infty)\to\mathbb{R}$ be given by $f(x)=x^2 e^{-x}$. Then the maximum value of f is

(A) e^{-1} (B) $4e^{-2}$ (C) $9e^{-3}$ (D) $16e^{-4}$

Q.2 An eigen-vector of the matrix $\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$ is

(A) $(1,2)^T$ (B) $(5,0)^T$ (C) $(0,2)^T$ (D) $(1,1)^T$

Q.3 Define $f,\, g:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases} x, & \text{if } x\in\mathbb{Q}, \\ \sin(x), & \text{if } x\in\mathbb{R}-\mathbb{Q} \end{cases} \quad \text{and} \quad g(x)=\begin{cases} x\ \sin(x)\ \sin(1/x), & \text{if } x\neq 0, \\ 0, & \text{if } x=0. \end{cases}$$

At $x=0$,

(A) both f and g are differentiable.
(B) f is differentiable but g is **NOT** differentiable.
(C) g is differentiable but f is **NOT** differentiable.
(D) neither f nor g is differentiable.

Q.4 Consider the series S_1 and S_2 given by:

$$S_1:\sum_{n=1}^{\infty}\frac{n^2+n+1}{n(n+1)} \quad \text{and} \quad S_2:\sum_{n=1}^{\infty}\frac{n^2+1}{n^2(n+1)}.$$

Then

(A) both S_1 and S_2 converge.
(B) S_1 converges and S_2 diverges.
(C) S_2 converges and S_1 diverges.
(D) both S_1 and S_2 diverge.

Q.5 The equation $x^{13}-e^{-x}+x-\sin(x)=0$ has

(A) no real root.
(B) more than two real roots.
(C) exactly two real roots.
(D) exactly one real root.

Q.6 Let D be the triangle bounded by the $y-$axis, the line $2y=\pi$ and the line $y=x$. Then the value of the integral $\iint_D \frac{\cos(y)}{y}\,dx\,dy$ is

(A) $\frac{1}{2}$ (B) 1

(C) $\frac{3}{2}$ (D) 2

Q.7 Let X be a random sample of size one from $U(\theta,\theta+1)$ distribution, $\theta\in\mathbb{R}$. For testing $H_0:\theta=1$ against $H_1:\theta=2$, the critical region $\{x : x>1\}$ has

(A) power = 1 and size = 1.
(B) power = 0 and size = 1.
(C) power = $1/2$ and size = 1.
(D) power = 1 and size = 0.

Q.8 Let $X_1,...,X_n$ be i.i.d. $B(1,\theta)$ random variables, $0<\theta<1$. Then, as an estimator of θ,

$$T(X_1,\ldots,X_n)=\frac{\sum_{i=1}^{n}X_i+\frac{\sqrt{n}}{2}}{n+\sqrt{n}}$$

is

(A) both consistent and unbiased.
(B) consistent but **NOT** unbiased.
(C) unbiased but **NOT** consistent.
(D) neither unbiased nor consistent.

Q.9 Let $X_1,\ X_2,\ X_3$ be i.i.d. $N(0,\ \theta^2)$ random variables, $\theta>0$. Then the value of k for which the estimator $\left(k\sum_{i=1}^{3}|X_i|\right)$ is an unbiased estimator of θ is

(A) $\frac{1}{3\pi}$ (B) $\sqrt{\frac{2}{9\pi}}$ (C) $\sqrt{\frac{\pi}{18}}$ (D) $\frac{2}{3\pi}$

Q.10 Let the random variables X and Y have the joint probability mass function

$$P(X=x, Y=y) = e^{-2}\binom{x}{y}\left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{x-y}\frac{2^{x}}{x!}; \qquad y=0,1,\ldots,x; \quad x=0,1,2,\ldots.$$

Then $E(Y)=$

(A) $\frac{1}{2}$ (B) 1

(C) $\frac{3}{2}$ (D) 2

Q.11 Let X_1 and X_2 be i.i.d. Poisson random variables with mean 1. Then $P(\max(X_1, X_2) > 1) =$

(A) $1-e^{-2}$ (B) $1-2e^{-2}$ (C) $1-3e^{-2}$ (D) $1-4e^{-2}$

Q.12 A fair die is rolled 3 times. The conditional probability of 6 appearing exactly once, given that it appeared at least once, equals

(A) $\dfrac{3\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)^{3}}$ (B) $\dfrac{\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{2}}{1-\left(\frac{5}{6}\right)^{3}}$

(C) $\dfrac{3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{2}}{1-\left(\frac{5}{6}\right)^{3}}$ (D) $\dfrac{\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)^{3}}$

Q.13 Let $X \sim B(2, 1/2)$. Then $E\left(\frac{2}{1+X}\right) =$

(A) $\frac{7}{6}$ (B) 1 (C) $\frac{6}{7}$ (D) $\frac{2}{3}$

Q.14 The moment generating function of an integer valued random variable X is given by

$$M_X(t) = \frac{1}{10}\left(2+e^{t}+4e^{2t}+3e^{3t}\right)e^{-t}.$$

Then $P(2X+5<7) =$

(A) $\frac{3}{10}$ (B) $\frac{7}{10}$ (C) 1 (D) $\frac{4}{10}$

Q.15 Let X_1 and X_2 be i.i.d. $Exp(3)$ random variables. Then $P(X_1 + X_2 > 1) =$

(A) $2\,e^{-3}$ (B) $3\,e^{-3}$ (C) $4\,e^{-3}$ (D) $5e^{-3}$

Space for rough work

Space for rough work

Space for rough work

Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question Number**. Do not write anything else on this page.**

Question Number	**Answer**	**Do not write in this column**
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

Number of Correct Answers		Marks	(+)
Number of Incorrect Answers		Marks	(–)
Total Marks in Questions 1-15			( )

Q.16 (a) Student population of a university has 30% Asian, 40% American, 20% European and 10% African students. It is known that 40% of all Asian students, 50% of all American students, 60% of all European students and 20% of all African students are girls. Find the probability that a girl chosen at random from the university is an Asian. (12)

(b) Let A_1, A_2 and A_3 be pairwise independent events with $P\left(A_i\right) = \frac{1}{2}$, $i = 1, 2, 3.$ Suppose that A_3 and $A_1 \cup A_2$ are independent. Find the value of $P\left(A_1 \cap A_2 \cap A_3\right)$. (9)

Q.17 Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2 + \frac{1}{4}, & \text{if } 0 \le x < \frac{1}{2}, \\ x + \frac{1}{8}, & \text{if } \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{x+1}{2}, & \text{if } \frac{3}{4} \le x < 1, \\ 1, & \text{if } x \ge 1. \end{cases}$$

Find the values of $P\left(0 \le X < \frac{1}{4}\right)$, $P\left(X \ge \frac{3}{4}\right)$ and $P\left(X = \frac{1}{2}\right)$. (21)

Q.18 (a) Let X be a continuous random variable with probability density function

$$f(x) = \frac{1}{2} e^{-|x-1|}; \qquad -\infty < x < \infty.$$

Find the value of $P(1 < |X| < 2)$. (12)

(b) Let X and Y be i.i.d. $U(0,1)$ random variables. Find the value of

$$P\left(\frac{1}{4} \leq X^2 + Y^2 \leq 1\right).$$ (9)

Q.19 (a) Let the random variables X_1 and X_2 have joint probability density function

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 \, e^{-x_1 x_2}}{2}, & \text{if } 1 < x_1 < 3, \, x_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the covariance between X_1 and X_2. (12)

(b) Let $X_1, \ldots, X_{100}$ be i.i.d. $U(-0.5, 0.5)$ random variables and let $T = X_1 + \cdots + X_{100}$. Using Chebyshev's inequality show that $P\left(T^2 \geq 25\right) \leq \dfrac{1}{3}$. (9)

Q.20 (a) Let $X_1, \ldots, X_n$ be a random sample from a population having a probability density function

$$f(x|\theta) = \begin{cases} \dfrac{4}{\theta} x^3 e^{-\frac{x^4}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Find the uniformly minimum variance unbiased estimator of θ. (9)

(b) Let $\overline{x} = 9$ and $s_x^2 = 6$ be the sample mean and sample variance, respectively, based on a random sample of size 3 from $N(\mu_1, \sigma^2)$. Also let $\overline{y} = 7$ and $s_y^2 = 4$ be the sample mean and sample variance, respectively, based on a random sample of size 3 from $N(\mu_2, 2\sigma^2)$, where $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma^2 > 0$ are unknown. Find a 95% confidence interval for $\mu_1 - \mu_2$. (12)

Q.21 Let $X_1, \ldots, X_{10}$ be a random sample of size 10 from a population having a probability density function

$$f(x \mid \theta) = \begin{cases} \dfrac{\theta}{x^{\theta+1}}, & \text{if } x > 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$. For testing $H_0 : \theta = 2$ against $H_1 : \theta = 4$ at the level of significance $\alpha = 0.05$, find the most powerful test. Also find the power of this test. (21)

Q.22 Let $f : [-1,1] \to \mathbb{R}$ be a continuous function.

(a) Show that $\sum_{n=1}^{\infty}\left(f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right)$ is a convergent series. (9)

(b) Further, if f is differentiable on $(0,1)$ and $|f'(x)| < 1$ for all $x \in (0,1)$, then show that

$$\sum_{n=1}^{\infty}\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right|$$

is a convergent series. (12)

Q.23 (a) Let $f:[0,1]\rightarrow\mathbb{R}$ be a continuous function such that $\int_0^1 f(t)\,dt=1$. Then show that there exists a point $c\in(0,1)$ such that $f(c)=3\,c^2$. (12)

(b) Find the general solution of

$$\frac{d^2y}{dx^2}-3\,\frac{dy}{dx}+2\,y=-\,e^x,$$

where it is given that $y=x\,e^x$ is a particular solution. (9)

Q.24 (a) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x, y) = x^2 + xy + y^2 - x - 100$. Find the points of local maximum and local minimum, if any, of f. (12)

(b) Find $\lim_{n \to \infty} \frac{4^{3n} \sin(n)}{3^{4n}}$. (9)

Q.25 (a) Consider the following matrix

$$A=\begin{pmatrix} \frac{\alpha}{\sqrt{11}} & \frac{1}{\sqrt{66}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & \frac{-4}{\sqrt{66}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{11}} & \frac{7}{\sqrt{66}} & \frac{\beta}{\sqrt{6}} \end{pmatrix}.$$

Find α and β so that A becomes an orthogonal matrix. Using these values of α and β, solve the system of equations

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$ (12)

(b) Let $A=\begin{pmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 3 & 5 \\ 1 & 6 & 10 & 14 \\ 1 & 4 & 4 & \gamma \end{pmatrix}$. Find γ so that the rank of A is two. (9)

Space for rough work

Space for rough work

Space for rough work

Space for rough work

Space for rough work

2011 MS Objective Part (Question Number 1 – 15)	
Total Marks	Signature

Subjective Part					
Question Number	Marks		Question Number	Marks	
16			21		
17			22		
18			23		
19			24		
20			25		
Total Marks in Subjective Part					

Total (Objective Part) :	
Total (Subjective Part) :	
Grand Total :	
Total Marks (in words) :	
Signature of Examiner(s) :	
Signature of Head Examiner(s) :	
Signature of Scrutinizer :	
Signature of Chief Scrutinizer :	
Signature of Coordinating Head Examiner :	