B

2007 – MS

Test Paper Code : MS

Time : 3 Hours   Maximum Marks : 300

INSTRUCTIONS

1. The question-cum-answer book has **36** pages and has **25** questions. Please ensure that the copy of the question-cum-answer book you have received contains all the questions.
2. Write your **Roll Number, Name** and **the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each objective question, you will be awarded **6 (six)** marks if you have written only the correct answer.
   (b) In case you have not written any answer for a question, you will be awarded **0 (zero)** mark for that question.
   (c) In all other cases, you will be awarded **–2 (minus two)** marks for the question.
   (d) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing later only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Logarithmic Tables / Calculator of any kind / cellular phone / pager / electronic gadgets are not allowed.**
11. The question-cum-answer book must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this book.
12. Refer to special instructions/useful data on the reverse.

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READ THE INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER						

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer book.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

....

Signature of the Invigilator

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Special Instructions / Useful Data

1. For an event A, $P(A)$ denotes the probability of the event A.
2. For events A and B, $P(A \mid B)$ denotes the conditional probability of A given B.
3. Complement of event A is denoted by A^c.
4. For a random variable X, $E(X)$ denotes the expectation of X and $V(X)$ denotes the variance of X.
5. For random variables X and Y, $\mathrm{Cov}(X,Y)$ denotes the covariance between X and Y.
6. $\overline{X} = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$, $X_{(1)} = \min\{X_1,\ldots,X_n\}$, $X_{(n)} = \max\{X_1,\ldots,X_n\}$.
7. For the random variable Z having a normal distribution with mean 0 and variance 1, $P(Z \leq 1) = 0.841$.
8. $n!$ denotes the factorial of n.
9. The determinant of a square matrix A is denoted by $\det(A)$.
10. $\mathbb{R}$: The set of all real numbers.
11. $\mathbb{R}^n$: n-dimensional Euclidean space.
12. $f'(x)$ and $f''(x)$ denote the first and second derivatives, respectively, of the function $f(x)$ with respect to x.

DO NOT WRITE ON THIS PAGE

IMPORTANT NOTE FOR CANDIDATES

- **Attempt ALL the 25 questions.**
- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-25 (subjective questions) carry twenty one marks each.**
- **Write the answers to the objective questions in the Answer Table for Objective Questions provided on page 7 only.**

1. Let the random variable X have binomial distribution with parameters 3 and θ. A test of hypothesis $H_0 : \theta = 3/4$ against $H_1 : \theta = 1/4$ rejects H_0 if $X \le 1$. Then the test has

   (A) size = 5/32, power = 27/32 (B) size = 5/32, power = 18/32

   (C) size = 15/32, power = 27/32 (D) size = 1/32, power = 31/32

2. Let X be a random variable having probability density function

$$f(x;x_0,\alpha) = \begin{cases} \dfrac{\alpha x_0^\alpha}{x^{\alpha+1}}, & x > x_0 \\ 0, & x \le x_0 \end{cases}$$

   where $\alpha > 0$, $x_0 > 0$. If $Y = \ln\left(\dfrac{X}{x_0}\right)$, then $P(Y > 3)$ is

   (A) $e^{-3\alpha x_0}$ (B) $1 - e^{-3\alpha x_0}$ (C) $e^{-3\alpha}$ (D) $1 - e^{-3\alpha}$

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y, z) = (x + y, x - z)$. Then the dimension of the null space of T is

   (A) 0 (B) 1 (C) 2 (D) 3

Space for rough work

4. Let $X_1, X_2, \ldots, X_{2n}$ be random variables such that $V(X_i) = 4$, $i = 1, 2, \ldots, 2n$ and Cov $(X_i, X_j) = 3$, $1 \leq i \neq j \leq 2n$. Then $V(X_1 - X_2 + X_3 - X_4 + \cdots + X_{2n-1} - X_{2n})$ is

(A) n (B) $2n$ (C) $3n - 2$ (D) $n + 1$

5. Let X_1 and X_2 be independent random variables, each having exponential distribution with parameter λ. Then, the conditional distribution of X_1 given $X_1 + X_2 = 1$ is

(A) Exponential with mean 2

(B) Beta with parameters $\lambda / 2$ and $\lambda / 2$

(C) Uniform on the interval $(0,1)$

(D) Gamma with mean 2λ

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(0, \theta)$. Then the uniformly minimum variance unbiased estimator (UMVUE) of θ is

(A) $\left(\frac{n+1}{n}\right) X_{(n)}$ (B) $X_{(1)} + X_{(n)}$ (C) $2\overline{X}$ (D) $X_{(n)}$

Space for rough work

7. Let A be a 4×4 nonsingular matrix and B be the matrix obtained from A by adding to its third row twice the first row. Then $\det(2A^{-1}B)$ equals

(A) 2 (B) 4 (C) 8 (D) 16

8. Independent trials consisting of rolling a fair die are performed. The probability that 2 appears before 3 or 5 is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$

9. Let $X_1, X_2, \ldots, X_6$ be independent random variables such that

$$P(X_i = -1) = P(X_i = 1) = \frac{1}{2}, \; i = 1,2,3,\ldots,6.$$

Then $P\left[\sum_{i=1}^{6} X_i = 4\right]$ is

(A) $\frac{3}{32}$ (B) $\frac{3}{4}$ (C) $\frac{3}{64}$ (D) $\frac{3}{16}$

Space for rough work

10. Let 1, x and x^2 be the solutions of a second order linear non-homogeneous differential equation on $-1 < x < 1$. Then its general solution, involving arbitrary constants C_1 and C_2, can be written as

(A) $C_1(1-x)+C_2(x-x^2)+1$

(B) $C_1x+C_2x^2+1$

(C) $C_1(1+x)+C_2(1+x^2)+1$

(D) $C_1+C_2x+x^2$

11. Let

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then

(A) $f'(x)$ is continuous at $x = 0$

(B) $f''(x)$ is continuous at $x = 0$

(C) $f'(0)$ exists

(D) $f''(0)$ exists

12. Let E and F be two events such that $0 < P(E) < 1$ and $P(E \mid F) + P(E \mid F^c) = 1$. Then

(A) E and F are mutually exclusive

(B) $P(E^c \mid F) + P(E^c \mid F^c) = 1$

(C) E and F are independent

(D) $P(E \mid F) + P(E^c \mid F^c) = 1$

Space for rough work

13. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with mean $1/\lambda$. The maximum likelihood estimator of the median of the distribution is

(A) $\frac{\overline{X}}{(\ln 2)}$ (B) $\overline{X}(\ln 2)$ (C) $\frac{(\ln 2)}{\overline{X}}$ (D) $\ln(2\overline{X})$

14. $\lim_{n\to\infty} \frac{1-2+3-4+5-6+\ldots+(-2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}$ equals

(A) ∞ (B) $1/2$ (C) 0 (D) $-1/2$

15. By changing the order of integration, the integral

$$\int_0^1 \int_1^{e^x} f(x,y)\,dy\,dx$$

can be expressed as

(A) $\int_0^1 \int_1^{\ln y} f(x,y)\,dx\,dy$ (B) $\int_0^1 \int_0^{\ln y} f(x,y)\,dx\,dy$

(C) $\int_1^e \int_1^{e^y} f(x,y)\,dx\,dy$ (D) $\int_1^e \int_{\ln y}^1 f(x,y)\,dx\,dy$

Space for rough work

Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page.

Question No.	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

No. of Correct Answers		Marks	(+)
No. of Incorrect Answers		Marks	(−)
Total Marks in Question Nos. 1-15			( )

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16. (a) Let $f(x) = x^3 + 3x - 2, x \in \mathbb{R}$. Show that the equation $f(x) = 0$ has only one real root. Also, find x_0 in the interval $(0,1)$ such that the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is parallel to the line joining the points $(0, -2)$ and $(1, 2)$. (9)

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with

$$T(1,1) = (0,0,1) \text{ and } T(1,2) = (0,1,1).$$

Then find the linear transformation $T(x, y)$. Also, find the associated matrix referred to the standard bases. (12)

17. (a) Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 2 - x^2$ and the line $y = x$, while the top of the solid is bounded by the plane $z = x + 2$. (9)

(b) Find all the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{\left(n + \frac{1}{n}\right)}$$

converges. (12)

18. The cumulative distribution function of a random variable X is

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x+k}{5}, & k \le x < k+1, \ k = 0,1,2 \\ 1, & x \ge 3. \end{cases}$$

Find :

(a) $P(X = j)$ for all non-negative integers j

(b) $P(X > 2)$

(c) $P(-1 \le X < 1)$.

(21)

19. Let $X_1,\ldots,X_n$ be independent random variables with X_k having normal distribution with mean $k\theta$ and variance σ^2 for $k=1,2,\ldots,n$. Find the maximum likelihood estimator of θ based on $X_1,\ldots,X_n$. Show that it is an unbiased and consistent estimator of θ. (21)

20. Let the joint probability mass function of random variables X and Y be given by

$$P(X = m, Y = n) = \frac{e^{-1}}{(n-m)!\,m!\,2^n}, \quad m = 0,1,2,\ldots,n; \quad n = 0,1,2,\ldots$$

Find the marginal probability mass functions of X and Y. Also, find the conditional probability mass function of X given $Y = 5$, and that of Y given $X = 6$. (21)

21. Let $X_1,\ldots,X_n$ $(n \geq 2)$ be a random sample from a distribution having the probability mass function

$$P(X = x) = \theta(1-\theta)^x,\ x = 0,1,2,\ldots$$

where $0 < \theta < 1$. Show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic. Find the uniformly minimum variance unbiased estimator (UMVUE) of θ. (21)

22. Find the continuous solution of

$$\frac{dy}{dx} + y = g(x), \; 0 \le x < \infty; \; y(0) = 2,$$

where

$$g(x) = \begin{cases} 3, & 0 \le x < \pi/2 \\ \cos x, & x \ge \pi/2. \end{cases} \tag{21}$$

23. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables each with mean 4 and variance 4. Show that for large n,

$$0.5 \leq P\left[16n - 12\sqrt{n} \leq \sum_{i=1}^{n} X_{2i} X_{2i-1} \leq 16n + 12\sqrt{n}\right] \leq 0.9. \qquad (21)$$

24. An urn contains ten balls of which M (an unknown number) are white. To test the hypothesis $H_0 : M = 3$ against $H_1 : M = 7$, three balls are drawn at random from the urn without replacement. If X is the number of white balls drawn, show that the most powerful test rejects H_0 if $X \geq k$, where k is a constant. Find the power, if the size of this test is $11/60$. (21)

25. (a) Evaluate the integral $\iint_R e^{(x^2+y^2)/2}\,dx\,dy$, where R is the region bounded by the lines $y = 0$ and $y = x$, and the arcs of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. (9)

(b) Let

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Determine whether the function is continuous and differentiable at $(0,0)$. (12)

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2007 – MS Objective Part (Q. Nos. 1 – 15)	
Total Marks	**Signature**

Subjective Part					
Q. No	**Marks**	**Signature**	**Q. No.**	**Marks**	**Signature**
16			21		
17			22		
18			23		
19			24		
20			25		
Total Marks in Subjective Part					

Total (Objective Part) :	
Total (Subjective Part) :	
Grand Total :	
Total Marks (in words) :	
Signature of Examiner(s) :	
Signature of Head Examiner(s) :	
Signature of Scrutinizer :	
Signature of Chief Scrutinizer :	
Signature of Coordinating Head Examiner :	

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