CHAPTER 11 PARTIAL DIFFERENTIATION

TOPICS:

1. DEFINITION, FIRST AND SECOND ORDER PARTIAL DERIVATIVES
2.HOMOGENEOUS FUNCTIONS AND EULARS THEOREM.

PARTIAL DIFFERENTIATION

Let $u = f(x, y)$ be a function of two independent varaibles x and y.

(i) If $\underset{h\to 0}{Lt}\dfrac{f(x+h,\ y)-f(x,y)}{h}$ exists then the limit is called the partial derivatie of u with respect to x. It is denoted by $\dfrac{\partial u}{\partial x}$ or u_x or $\dfrac{\partial f}{\partial x}$ or f_x .

(ii) If $\underset{k\to 0}{Lt}\dfrac{f(x,y+k)-f(x,y)}{k}$ exists then the limit is called the partial derivative of u with respect to y. It is denoted by $\dfrac{\partial u}{\partial y}$ or u_y or $\dfrac{\partial f}{\partial y}$ or f_y.

$\therefore \dfrac{\partial u}{\partial x} = \underset{h\to 0}{Lt}\dfrac{f(x+h,\ y)-f(x,y)}{h}$ and $\dfrac{\partial u}{\partial y} = \underset{k\to 0}{Lt}\dfrac{f(x,y+k)-f(x,y)}{k}$

Note : (i)The partial derivative of u w.r.t. x is the ordinary derivative of u w.r.t. x treating the other variable y (and its functions) as constant

(ii) The partial derivative of u w.r.t. y is the ordinary derivative of u w.r.t. y treating the other variable x (and its functions) as constant.

DIFFERENTIATION OF COMPOSITE FUNCTIONS

1. If V = g(U) and U = f(x, y) then (i) $\dfrac{\partial V}{\partial x} = \dfrac{dV}{dU}.\dfrac{\partial U}{\partial x}$, (ii) $\dfrac{\partial V}{\partial y} = \dfrac{dV}{dU}.\dfrac{\partial U}{\partial y}$

2. If Z = f(x, y) and x = g(t); y = h(t) then $\dfrac{dZ}{dt} = \dfrac{\partial Z}{\partial x}.\dfrac{dx}{dt} + \dfrac{\partial Z}{\partial y}\dfrac{dy}{dt}$ is called the total differential coefficient of Z w.r.t. t.

3. If f(x, y) = c where c is constant, then $\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y}$.

PARTIAL DERIVATIVES OF SECOND ORDER

Definition : If U = f(x, y) then $\dfrac{\partial U}{\partial x}, \dfrac{\partial U}{\partial y}$ are called the partial derivatives of first order and they are functions of x, y. The partial derivatives of $\dfrac{\partial U}{\partial x}$ and $\dfrac{\partial U}{\partial y}$, if they exist, are called the second order partial derivatives . They are denoted by

$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)=\frac{\partial^2 U}{\partial x^2}\equiv U_{xx}\equiv f_{xx}$, $\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)=\frac{\partial^2 U}{\partial y\partial x}\equiv U_{yx}\equiv f_{yx}$, $\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)=\frac{\partial^2 U}{\partial x\partial y}\equiv U_{xy}\equiv f_{xy}$ and $\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)=\frac{\partial^2 U}{\partial y^2}\equiv U_{yy}\equiv f_{yy}$.

HOMOGENEOUS FUNCTION:

A function $u=f(x,y)$ is said to be a homogeneous function of degree n in the variables in x and y if $f(kx,ky)=k^n f(x,y)$ for all k or $f(x,y)=x^n f\left(\frac{y}{x}\right)$ or $f(x,y)=y^n f\left(\frac{x}{y}\right)$.

EULER'S THEOREM

If $u=f(x,y)$ is a homogeneous function of degree n in the variables x,y then $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=nU$

Proof:

Since $u=f(x,y)$ is a homogeneous function of degree n, we have $U=x^n g(y/x)$ where $g(y/x)$ is function of y/x.

$$\therefore \frac{\partial U}{\partial x}=x^n.g'(y/x)\left(\frac{-y}{x^2}\right)+nx^{n-1}.g(y/x) \quad \text{-- (1)}$$

and $$\frac{\partial U}{\partial y}=x^n.g'(y/x)\left(\frac{1}{x}\right)$$

$$\therefore x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=x\left(x^n g'(y/x)\left(-\frac{y}{x^2}\right)\right)+nx^n g(y/x)+yx^{n-1}.g'(y/x)$$

$=n.x^n.g(y/x)==$ nU.

Note 1: If $U=f(x,y,z)$ is a homogeneous function of degree n in x,y,z then $x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}+z\frac{\partial U}{\partial z}=nU$

Therorem

If $U=f(x,y)$ is a homogeneous function of degree n in x,y

then $x^2\frac{\partial^2 U}{\partial x^2}+2xy\frac{\partial^2 U}{\partial x\partial y}+y^2\frac{\partial^2 U}{\partial y^2}=n(n-1)U$ **.**

Proof:

Since $U=f(x,y)$ is a homogeneous function of degree n, by Euler's theorem, we have

$$x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=nU \text{ ------(1)}$$

Differentiating (1) partially w.r.t. x we get

$$x\frac{\partial^2 U}{\partial x^2}+\left(\frac{\partial U}{\partial x}\right).1+y\frac{\partial^2 U}{\partial x\partial y}=n\frac{\partial U}{\partial x}\Rightarrow x\frac{\partial^2 U}{\partial x^2}+y\frac{\partial^2 U}{\partial x\partial y}=(n-1)\frac{\partial U}{\partial x} \text{ -- (2)}$$

Differentiating (1) w.r.t. y partially we get,

$$x\frac{\partial^2 U}{\partial y\partial x}+y\frac{\partial^2 U}{\partial y^2}+\frac{\partial U}{\partial y}.1=n\frac{\partial U}{\partial y}\Rightarrow x\frac{\partial^2 U}{\partial y\partial x}+y\frac{\partial^2 U}{\partial y^2}=(n-1)\frac{\partial U}{\partial y} \text{ -------(3)}$$

(2) . x + (3) . y $\Rightarrow x^2\frac{\partial^2 U}{\partial x^2}+2xy\frac{\partial^2 U}{\partial x\partial y}+y^2\frac{\partial^2 U}{\partial y^2}=n(n-1)U$.

PARTIAL DIFFERENTIATION

EXERCISE – 11(a)

I.

1. Find $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$ for

i) $z=3xe^{y^2}+4y$

differentiate partially w.r.t x,

Sol: $\frac{\partial z}{\partial x}=3e^{y^2}\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial x}(4y)=3e^{y^2}$

differentiate partially w.r.t y,

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(3xe^{y^2}\right)+\frac{\partial}{\partial y}(4y)=6xye^{y^2}+4$$

ii) $z=\log\left(y+\frac{x}{y^2}\right)$

Sol: $z=\log\left(y+\frac{x}{y^2}\right)$

differentiate partially w.r.t x,

$$\frac{\partial z}{\partial x}=\frac{1}{y+\frac{x}{y^2}}\frac{\partial}{\partial x}\left(y+\frac{x}{y^2}\right)=\frac{y^2}{y^3+x}\left[\frac{1}{y^2}\right]=\frac{1}{y^3+x}$$

differentiate partially w.r.t y,

$$\frac{\partial z}{\partial y}=\frac{1}{\left(y+\frac{x}{y^3}\right)}\frac{\partial}{\partial y}\left(y+\frac{x}{y^2}\right)=\left(\frac{y^2}{y^3+x}\right)\left(1+x\left(\frac{-2}{y^3}\right)\right)$$

$$=\left(\frac{y^2}{y^3+x}\right)\left(\frac{y^3-2x}{y^3}\right)=\frac{\left(y^3-2x\right)}{y\left(y^3+x\right)}$$

iii) $\mathbf{z=Tan^{-1}\left(\frac{y^2}{x}\right)}$

Sol: differentiate partially w.r.t x,

$$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{y^2}{x}\right)^2}\frac{\partial}{\partial x}\left[\frac{y^2}{x}\right]=\frac{x^2}{x^2+y^4}\left(\frac{-y^2}{x^2}\right)=\frac{-y^2}{x^2+y^4}$$

differentiate partially w.r.t y,

$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{y^2}{x}\right)}.\frac{\partial}{\partial y}\left[\frac{y^2}{x}\right]$$

iv) $\mathbf{z=\frac{\cos x}{\sin y}}$ **ans:** $\frac{\partial z}{\partial x}=\frac{-\sin x}{\sin y}$ & $\frac{\partial z}{\partial y}=\frac{-\cos x}{\sin^2 y}(\cos y)$

$=-\cos x.\cot y.\mathrm{cosec}\, y$

v) $\mathbf{z=xe^y+ye^x}$. **ans.** $\frac{\partial z}{\partial x}=e^y+ye^x$, $\frac{\partial z}{\partial y}=xe^y+e^x$

vi) $\mathbf{z=\frac{1}{\sqrt{1+x+y^2}}}$

Sol: differentiate partially w.r.t x,

$$\frac{\partial z}{\partial x}=\frac{-1}{2}\left(1+x+y^2\right)^{-3/2}\frac{\partial}{\partial x}\left(1+x+y^2\right)=\frac{-1}{2}\left(1+x+y^2\right)^{-3/2}$$

differentiate partially w.r.t y,

$$\frac{\partial z}{\partial y}=-\frac{1}{2}\left(1+x+y^2\right)^{-3/2}\frac{\partial}{\partial y}\left(1+x+y^2\right)$$

$$=\frac{-1}{2}\left(1+x+y^2\right)^{-3/2}(2y)=\frac{-y}{\left(1+x+y^2\right)^{3/2}}$$

vii) $\mathbf{x = \sin\left(x^2 - y\right)}$ **[ans.** $= 2x\cos\left(x^2 - y\right), -\cos\left(x^2 - y\right)$

2. For the following functions f, show that $\mathbf{f_{xx} + f_{yy} = 0}$ **.**

i) $\mathbf{f\left(x\right) = x^2 - y^2}$

ii) $\mathbf{e^x \sin y}$

Sol: $f = e^x \sin y$

Differentiate f partially w.r.t x,

$f_x = e^x \sin y,$

differentiate f_x partially w.r.t x,

$f_{xx} = e^x \sin y$

$f = e^x \sin y$

differentiate f partially w.r.t y,

$f_y = e^x \cos y,$

differentiate f_x partially w.r.t y,

$\therefore f_{yy} = -e^x \sin y$

$\therefore f_{xx} + f_{yy} = 0$

iii) **f=** $\sin x . \cosh y$

3. If $\mathbf{v = \pi r^2 h}$**, show that** $\mathbf{r v_r + 2h v_h = 4v}$

4. If $\mathbf{z = \sin\left(x - y\right) + \log\left(x + y\right)}$ **show that** $\mathbf{z_{xx} = z_{yy}}$ **.**

Sol: $\mathbf{z = \sin\left(x - y\right) + \log\left(x + y\right)}$

Differentiate partially w.r.t x,

$z_x = \cos\left(x - y\right) + \frac{1}{x + y}$

Again differentiate partially w.r.t x,

$$z_{xx} = -\sin\left(x - y\right) - \frac{1}{\left(x + y\right)^2} \quad \text{---(1)}$$

Differentiate z partially w.r.t y,

$z_y = \cos\left(x - y\right) + \frac{1}{x + y}$

Aganin differentiate partially w.r.t y,

$$z_{yy} = -\sin(x-y) - \frac{1}{(x+y)^2} \quad \text{---(2)}$$

$\therefore$ From (1) and (2), we get $z_{xx} = z_{yy}$

5. If $u^3\left(1+a^3\right) = 8(x+ay+b)^3$ then show that $u_x^3 + u_y^3 = 8$

Sol: $u^3 = \dfrac{8(x+ay+b)^3}{1+a^3}$

$$u = \frac{2}{3\sqrt{1+a^3}}(x+ay+b)$$

differentiate partially w.r.t x $\Rightarrow$ $u_x = \dfrac{2}{3\sqrt{1+a^3}}$

differentiate partially w.r.t y $\Rightarrow u_y = \dfrac{2a}{3\sqrt{1+a^3}}$ $\therefore$ $u_x^3 + u_y^3 = \dfrac{8}{1+a^3} + \dfrac{8^3}{1+a^3} = \dfrac{8\left(1+a^3\right)}{1+a^3} = 8$

6. If $au+b = a^2x+y$, then show that $u_x u_y = 1$

7. If $z = Ae^{-p^2t}\cos px$, then prove that $z_{xx} = z_t$

Sol: $z = Ae^{-p^2t}\cos px$

differentiate partially w.r.t x,

$$z_x = A.e^{-p^2t}(-p\sin px)$$

$$= -Ap.e^{-p^2t}\sin Px$$

Again differentiate partially w.r.t x

$$z_{xx} = -Ap^2.e^{-p^2t}.\cos px \quad \text{---(1)}$$

differentiate z partially w.r.t t,

$$z_t = A\cos px.e^{-p^2t}\left(-p^2\right) = -Ap^2e^{-p^2t}\cos px$$

---(2)

From (1) and (2) we get $z_{xx} = z_t$

II.

1. Find all the first and second order partial derivatives of the following functions f.

i) sin (xy)

ii) $\tan\left(\tan^{-1}x+\tan^{-1}y\right)$

iii) $e^{x}\cos y$

iv) $e^{x^{y}}$

i) z = sin (xy)

differentiate partially w.r.t x,

$z_x = \cos(xy).y = xy\cos(x.y)$

Again differentiate partially w.r.t x,

$z_{xx} = -y\sin(xy).y = -y^2\sin(xy)$

Differentiate z partially w.r.t y,

$z_y = (\cos xy).x = x\cos xy$

Again differentiate partially w.r.t x,

$z_{xy} = (z_y)_x = x(-\sin xy.y)+\cos xy$

$= -xy\sin(xy)+\cos(xy)$

Differentiate z_y partially w.r.t y,

$z_{yy} = -x\sin(xy).x = -x^2\sin(xy)$

$(z_x)_y = (y\cos xy)_y$

$= y(\sin(xy))x+\cos xy$

$= \cos xy - x.y.\sin(xy)$

ii) z= $\tan\left(\tan^{-1}x+\tan^{-1}y\right)$

iii) $e^{x}\cos y$

Sol: z = $e^{x}\cos y$

differentiate partially w.r.t x,

$z_x = e^x\cos y,$ again diff. partially w.r.t x,

$z_{xx} = e^x\cos y$

differentiate z partially w.r.t y,

$z_y = -e^x \sin y,$

again diff. partially w.r.t y,

$z_{yy} = -e^x \cos y$

Differentiate z_y partially w.r.t x,

$z_{yx} = (z_y)_x = -e^x . \sin y$

$z_x = e^x \cos y$

$z_y = z_{xy} = z_{yx} = -e^x . \sin y$

iv) $\mathbf{z = e^{x^y}}$

2. For the following functions f, show that $\mathbf{f_{xx} + f_{yy} = 0}$.

i) $\mathbf{\dfrac{y}{x^2 + y^2}}$

Sol: $f = \dfrac{y}{x^2+y^2}$ differentiate partially w.r.t x, $f_x = \dfrac{-y(2x)}{(x^2+y^2)^2}$

Again differentiate partially w.r.t x,

$$f_{xx} = \frac{-2y(x^2+y^2)^2 + 2xy.2(x^2+y^2).2x}{\left[x^2+y^2\right]^4} = \frac{2y(x^2+y^2)\left[-x^2-y^2+4x^2\right]}{(x^2+y^2)^4}$$

$$= \frac{2(-x^2-y^2+4x^2)}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

Differentiate f partially w.r.t y,

$$f_y = \frac{1(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

differentiate partially w.r.t y,

$$f_{yy} = \frac{-2y(x^2+y^2)^2 - 2(x^2+y^2)(x^2-y^2)(2y)}{(x^2+y^2)^4} = \frac{2y(x^2+y^2)\left[-x^2-y^2-2x^2+2y^2\right]}{(x^2+y^2)^4}$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} \qquad \therefore f_{xx} + f_{yy} = 0$$

ii) $\mathbf{Tan^{-1}\left(\frac{y}{x}\right)}$

Sol: $f = \tan^{-1}\left(\frac{y}{x}\right)$

differentiate partially w.r.t x,

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{-y}{x}\right] = \frac{-x^2 y}{x^2+y^2}.\frac{1}{x^2} \qquad = \frac{-y}{x^2+y^2}$$

differentiate partially w.r.t x,

$$f_{xx} = \frac{+y(2x)}{\left(x^2+y^2\right)^2}$$

differentiate f partially w.r.t y,

$$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{1}{x} = \frac{x}{x^2+y^2}$$

differentiate partially w.r.t y, $f_{yy} = \frac{-2xy}{\left(x^2+y^2\right)^2}$ $\quad \therefore f_{xx}+f_{yy}=0$

iii) $\mathbf{f=\log\left(x^2+y^2\right)}$

iv) $\mathbf{e^{-x}\left(x\sin y - y\cos y\right)}$

Sol: $\mathbf{f = e^{-x}\left(x\sin y - y\cos y\right)}$

differentiate partially w.r.t x,

$$f(x) = e^{-x}\left(x\sin y - y\cos y\right)$$

$$f_x = e^{-x}\left(x\sin y - y\cos y\right) + e^{-x}\left(\sin y\right)$$

differentiate partially w.r.t x,

$$f_{xx} = e^{-x}\left(x\sin y - y\cos y\right) - e^{-x}.\sin y - e^{-x}.\sin y \ = e^{-x}\left(x\sin y - y\cos y - 2\sin y\right)$$

differentiate f partially w.r.t y,

$$f_y = e^{-x}\left(x\cos y - \cos y + y\sin y\right)$$

differentiate partially w.r.t y,

$$f_{yy} = e^{-x}\left(-x\sin y + \sin y + \sin y + y\cos y\right) = e^{-x}\left(-x\sin y + y\cos y + 2\sin\right)$$

$$\therefore f_{xx}+f_{yy}=0$$

v) $\mathbf{e^{x}\left(x\cos y - y\sin y\right)}$

vi) $\mathbf{e^{2xy}\cos\left(y^2-x^2\right)}$

vii) $e^{2x}\left(A\sin 2y + B\cos 2y\right)$

Sol: $f = e^{2x}\left(A\sin 2y + B\cos 2y\right)$

Diff. f partially w.r.t x, $f_x = 2e^{2x}\left(A\sin 2y + B\cos 2y\right)$

Again diff. partially w.r.t x, $f_{xx} = 4e^{2x}\left(A\sin 2y + B\cos 2y\right)$

Diff, f partially w.r.t. y, $f_y = e^{2x}\left(2A\cos 2y - 2B\sin 2y\right)$

Again diff. partially w.r.t. y, $f_{yy} = e^{2x}\left(-4A\sin 2y - 4B\cos 2y\right)$

$= -4e^{2x}\left(A\sin 2y + B\cos 2y\right) = -f_{xx}$

$\Rightarrow f_{xx} + f_{yy} = 0$

viii) $f = e^{-2xy}\sin\left(x^2 - y^2\right)$

3. If r, θ, x and y are connected by the equations $r = \left(x^2 + y^2\right)^{1/2}; \theta = Tan^{-1}\left(\frac{y}{x}\right)$**. find** r_x, r_y, θ_x **and** θ_y**. Also verify that** $r_y\theta_y + r_x\theta_x = 0$**.**

Sol: $r = \sqrt{x^2 + y^2}$

differentiate partially w.r.t x,

$$r_x = \frac{1}{2\sqrt{x^2 + y^2}}.2x = \frac{x}{\sqrt{x^2 + y^2}}$$

differentiate r partially w.r.t y,

$$r_y = \frac{1}{2\sqrt{x^2 + y^2}}.2y = \frac{y}{\sqrt{x^2 + y^2}}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

differentiate partially w.r.t x,

$$\theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2\frac{\left(x^2 + y^2\right)}{x^2}} = -\frac{y}{x^2 + y^2}$$

differentiate partially w.r.t y,

$$\theta_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2}.\frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$r_y.\theta_x + r_x.\theta_y = \frac{y}{\sqrt{x^2+y^2}}.\frac{x}{x^2+y^2} \qquad + \frac{x}{\sqrt{x^2+y^2}}\left(-\frac{y}{x^2+y^2}\right) = \frac{xy-xy}{\sqrt{x^2+y^2}} = 0$$

4. If $z = \tan(y+ax)+(y-ax)^{1/2}$, find $z_{xx} - a^2 z_{yy}$.

Sol: $z = \tan(y+ax)+(y-ax)^{1/2}$

Differentiate partially w.r.t x,

$$z_x = a\sec^2(y+ax)+\frac{1}{2}(-a)(y-ax)^{-1/2}$$

Differentiate partially w.r.t x,

$$z_{xx} = 2a^2\sec^2(y+ax).\tan(y+ax) - \frac{a^2}{4}(y-ax)^{-3/2}$$

Differentiate z partially w.r.t y,

$$z_y = \sec^2(y+ax)+\frac{1}{2}(y-ax)^{-1/2}$$

Differentiate partially w.r.t y,

$$z_{yy} = 2\sec^2(y+ax)\tan(y+ax)-\frac{1}{4}(y-ax)^{-3/2}$$

$$\therefore z_{xx} - a^2 z_{yy} = 2a^2\sec^2(y+ax)\tan(y+ax) \quad -\frac{a^2}{4}(y-ax)^{-3/2}$$

$$-2a^2\sec^2(y+ax)\tan(y+ax)+\frac{a^2}{4}(y-ax)^{-3/2} = 0$$

$$\therefore z_{xx} - a^2 z_{yy} = 0$$

5. If $\left(2z - ay^2 - 2b^2\right) = 16ax$, show that $z_y = xyz_x^2$

6. If $(z+a)e^{x+ay} = b$, then show that $z_x(z+z_x) = -z_y$

7. If $u^2 = \dfrac{1}{x^2+y^2+z^2}$, then show that $\sum \dfrac{\partial^2 u}{\partial x^2} = 0$.

Exercise – 11(b)

I.

1. Which of the following are homogeneous functions?

i) $\mathbf{f(x,y)=x^{1/3}.y^{3/4},\tan^{-1}\left(\frac{y}{x}\right)}$

Sol: $f(x,y)=x^{1/3}.y^{3/4},\tan^{-1}\left(\frac{y}{x}\right)$ $f(kx,ky)=(kx)^{1/3}(ky)^{3/4}.\tan^{-1}\left(\frac{ky}{kx}\right)$

$= k^{1/3}.x^{1/3}.k^{3/4}.y^{3/4}\tan^{-1}\left(\frac{y}{x}\right)$

$=k^{\frac{1}{3}+\frac{3}{4}}.x^{\frac{1}{3}}.y^{\frac{3}{4}}\tan^{-1}\left(\frac{y}{x}\right)$

$=k^{\frac{13}{12}}f(x,y)$

$f(x,y)$ is a homogeneous function.

ii) $\mathbf{f(x,y)=\frac{3x}{y}+\log\left(\frac{y}{x}\right)}$

Ans ; $f(x,y)$ is a homogeneous function.

iii) $\mathbf{f(x,y)=\log y+2\log x}$

Sol: $f(kx_1\,ky)=\log ky+2\log kx$

$=\log k+\log y+2(\log k+\log x)$

$=3\log k+(\log y+2\log x)$

$=3\log k+f(x,y)$

$f(x,y)$ is not a homogeneous function.

iv) $\mathbf{u(x,y)=xf\left(\frac{y}{x}\right)+y.g\left(\frac{x}{y}\right)}$

Ans : homogeneous function.

v) $\mathbf{f(x,y)=\left(x^3+y^3\right)^{3/2}}$

Ans. f is a homogeneous function.

vi) $\mathbf{f(x,y)=x^{1/3}y^{1/3}+x^{2/3}.y^{1/3}}$

Sol: $f(kx,ky)=(kx)^{1/3}(ky)^{1/3}+(kx)^{2/3}.(ky)^{1/3}$

$=k^{2/3}.x^{1/3}.y^{1/3}.+k.x^{2/3}.y^{1.3}$

$=k^{2/3}\left(x^{1/3}.y^{1/3}+k^{1/3}.x^{2/3}.y^{1/3}\right)$

$\neq k^n f(x,y)$

$f(x, y)$ is not a homogeneous function.

II.

1. Verify Euler's theorem for the following.

i) $\mathbf{f(x,y)=\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}}$

Sol. $\mathbf{f(x,y)=\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}}$

Differentiate partially w.r.t

$$x,\ \frac{\partial f}{\partial x}=\frac{\frac{1}{2}(x)^{-1/2}\left[\sqrt{x}+\sqrt{y}\right]-\frac{1}{2}(x)^{-1/2}(\sqrt{x}-\sqrt{y}}{\left[\sqrt{x}+\sqrt{y}\right]^2}=\frac{\frac{1}{2}(x)^{-1/2}\left[\sqrt{x}+\sqrt{y}-\sqrt{x}+\sqrt{y}\right]}{\left[\sqrt{x}+\sqrt{y}\right]^2}$$

$$\frac{\partial f}{\partial y}=\frac{(x)^{-1/2}(y)^{1/2}}{\left[\sqrt{x}+\sqrt{y}\right]^2}$$

$$x\frac{\partial f}{\partial y}=\frac{(xy)^{1/2}}{\left[\sqrt{x}+\sqrt{y}\right]^2}$$

Differentiate partially w.r.t y,

$$\frac{\partial f}{\partial y}=\frac{\frac{-1}{2}(y)^{-1/2}\left[\sqrt{x}+\sqrt{y}\right]-\frac{1}{2}(y)^{-1/2}\left\{\sqrt{x}-\sqrt{y}\right\}}{\left[\sqrt{x}+\sqrt{y}\right]^2}=\frac{-\frac{1}{2}(y)^{-1/2}\left[2\sqrt{x}\right]}{\left[\sqrt{x}+\sqrt{y}\right]^2}$$

$$y\frac{\partial f}{\partial y}=\frac{-(xy)^{1/2}}{\left[\sqrt{x}+\sqrt{y}\right]^2}$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=o$$

Degree of the given function is 0.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf=0.f=0$$

Hence $f(x, y)=\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ holds Euler's theorem.

ii) $f(x,y) = \tan^{-1}\frac{y}{x}$

iii) $\mathbf{f(x, y) = \frac{x^2y}{x^3+y^3}}$

Sol. $f(x,y) = \frac{x^2y}{x^3+y^3} \to f(kx,ky) = \frac{k^3}{k^3}\frac{x^2y}{x^3+y^3} = k^0\frac{x^2y}{x^3+y^3}$

$= k^0 f$

f is a homogeneous function of degree 0.

$\therefore x\frac{\partial f}{\partial x} + \frac{\partial f}{ay} = 0.f = 0$

differentiate f partially w.r.t x,

$$\frac{\partial f}{\partial x} = \frac{2xy(x^3+y^3) - 3x^2.x^2y}{(x^3+y^3)^2} \quad x\frac{\partial f}{\partial x} = \frac{2x^2y(x^3+y^3) - 3x^5.y}{(x^3+y^3)^2}$$

differentiate f partially w.r.t y,

$$\frac{\partial f}{\partial y} = \frac{x^2(x^3+y^3) - 3y2.x^2y}{(x^3+y^3)^2}$$

$$y\frac{\partial f}{\partial y} = \frac{yx^5 + y^4x^2 - 3y^4x^2}{(x^3+y^3)^2}$$

$\therefore x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0.f = 0$

Hence eulers theorem verified.

iv) $\mathbf{f(x, y) = x\tan^{-1}\left(\frac{y}{x}\right) + xe^{x/y}}$

2) **If** $\mathbf{u = x\phi\left(\frac{y}{x}\right) + y\psi\left(\frac{y}{x}\right)}$ **then show that** $\mathbf{x^2u_{xx} + 2xy\ u_{xy} + y^2u_{yy} = 0}$

3) **If** $\mathbf{u = \tan^{-1}\left(\frac{x^3+y^3}{x^3+y^3}\right)}$ **then show that** $\mathbf{xu_x + yu_y = 0}$ **(May.'06)**

Sol. $u = \tan^{-1}\left(\frac{x^3+y^3}{x^3+y^3}\right)$

$\tan u = \frac{x^3-y^3}{x^3+y^3}$

$$let\ \tan u = z(x,y) = \frac{x^3 - y^3}{x^3 + y^3} \to f(kx,ky) = \frac{k^3}{k^3}\frac{x^3 - y^3}{x^3 + y^3}$$

$$= k^0 \frac{x^3 - y^3}{x^3 + y^3} = k^0 z$$

tanu is a homogeneous function degree Zero in x and y

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 0.\tan u.$$

$$x.\sec^2 u.\frac{\partial u}{\partial x} + y.\sec^2 y.\frac{\partial u}{\partial y} = 0$$

$$\sec^2 u\left(x.\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = 0 \Rightarrow x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0$$

i.e., $x.u_u + y.u_y = 0$

PROBLEMS FOR PRACTICE

1. **If z = e^{ex} sin by, where a and b are real Constants, find $Z_{Z,}Z_y, Z_{xx}, Z_{xy}, Z_{yy}$ and Z_{yz}.**

2. **Find all the first and second order partial Derivatives for f(x, y) = e^{x-2y}**

3. **Find all the first and second order partial derivatives for f(x, y) = sin (ax + by) where a and b are real constants.**

4. **If u (x, y) = $\frac{x^2y}{x^3+y^3}$, show that $xu_x + yu_y = 0$**

5. **If z = log (tan x + tan y), show that $(\mathrm{Sin}2x)z_z + (\sin 2y)z_y = 2$**

6. **If $u = e^{xy}$, show that $u(u_{xx} + u_{yy}) = (u_x^2 + u_y^2)$**

7. **If $z = f(x^2 + y^2)$, show that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$.**

8. **If $u = 3xy - y^3 + (y^2 - 2x)^{3/2}$, then show tat $u_{xx}.u_{yy} - (u_{xy})^2 = 0$**

9. **If $f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$; $x \neq 0, y \neq 0$, show that $f_{xy} = \frac{x^2 - y^2}{x^2 + y^2}$.**

10. If $r^2=(x-a)^2+(y-b)^2$, find the value of $r_{xx}+r_{yy}$

Sol: Given $r^2=(x-a)^2+(y-b)^2$ ---(1)

Differentiating (1) partially with respect to x, we get $2r.r_x=2(x-a)\Rightarrow r.r_x=x-a$

Differentiating again partially with respect to x, we get $r.r_{xx}+r_x{}^2=1$ ---(2)

Similarly differentiating (1) partially with respect to x

$2r.r_y=2(y-h)\Rightarrow r.r_y=y-b$ ---(3)

Differentiating (3) partially with respect to y, we get $r.r_{yy}+r_y{}^2=1$ ---(4)

Adding equation (2) and (4), we obtain

$r(r_{xx}+r_{yy})=r_x{}^2+r_y{}^2=2$

$r(r_{xx}+r_{yy})=2-r_x{}^2-r_y{}^2 \quad =2-\frac{(x-a)^2}{r^2}-\frac{(y-b)^2}{r^2} \quad =2-\frac{(x-a)^2+(y-b)^2}{r^2}$

$2-\frac{r^2}{r^2}=2-1=1 \therefore r_{xx}+r_{yy}=\frac{1}{r}$

11. If $z=\frac{y}{x}f(x+y)$, then show that $xz_x+yz_y=\frac{y}{x}(x+y)f'(x+y)$ Where $f'(x+y)=\frac{df}{du}$ and $u=x+y$.

12. If $z=e^{x+y}+f(x)+g(y)$, show that $z_{xy}=e^{x+y}$.

13. If $z=xf(y)+yg(x)$, show that $z+xy\ z_{xy}=xz_x+yz_y$.

14. If $u(x,y)=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$, then show that $xu_x+yu_y=0$.

15. Verify Euler's theorem for the function $f(x,y)=\frac{x^2+y^2}{x+y}$.

16. Using Euler's theorem show that $xu_x+yu_y=\frac{1}{2}\tan u$ for the function $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$.

17. If $u=\sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then $x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=2\cot u$

18. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that $xu_x + yu_y = \sin 2u$.

Sol: Given that $\tan u = \frac{x^3+y^3}{x+y}$

Write $z = \frac{x^3+y^3}{x+y}$. Then z is a homogeneous function of degree 2 and z = tan u.

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$ ---(1)

But $\frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}$

and $\frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$ ---(2)

Form (1) and (2), $x\left(\sec^2 u\right)u_x + y\left(\sec^2 u\right)u_y = 2\tan u$

i.e., $xu_x + yu_y = 2\tan u.\cos^2 u$

$= \left(2\sin u\cos u\right) = \sin 2u$

19. If $u = \log v$ and $v(x,y)$ is a homogeneous function of degree n, then prove that $xu_x + yu_y = n$.

Sol: Given $u = \log v \Rightarrow v = e^u$

Then by Euler's theorem,

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$ ---(1)

Now from $u = \log v,\ v = v(x,y)$

We get

$\frac{\partial u}{\partial x} = \frac{du}{dv}.\frac{\partial v}{\partial x} = \frac{1}{v}.\frac{\partial v}{\partial x}$

i.e., $\frac{\partial v}{\partial x} = v\frac{\partial u}{\partial x}$ ---(2)

and $\frac{\partial u}{\partial y} = \frac{du}{dv}.\frac{\partial v}{\partial y} = \frac{1}{v}.\frac{\partial v}{\partial y}$

i.e., $\frac{\partial v}{\partial y} = v.\frac{\partial u}{\partial y}$ ---(3)

Substituting the value of

we get $x.v.\frac{\partial u}{\partial x} + y.v.\frac{\partial u}{\partial y} = nv$

i.e., $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$

(or) $xu_x + yu_y = n$.

20. If $x^x.y^y.Z^z = C$, then prove that $\frac{\partial Z}{\partial x} = -\frac{(1+\log x)}{(1+\log Z)}$

Sol: Given $x^x.y^y.Z^Z = c$

$\log\ (x^x.y^y.Z^Z) = \log c$

x log x + y log y + X log Z = log c

Differentiating partially w. r. t. x

$\left(x.\frac{1}{x} + \log x\right) + \left(Z\frac{1}{Z} + \log Z.1\right)\frac{\partial z}{\partial x} = 0$

$(1 + \log Z)\ \frac{\partial Z}{\partial x} = -(1+\log x)$

$\frac{\partial Z}{\partial y} = -\frac{(1+\log x)}{(1+\log Z)}$