MAXIMA AND MINIMA - 2

GREATEST AND LEAST VALUES

Definition: Let f be a function defined on a set A and $l \in f(A)$. Then l is said to be

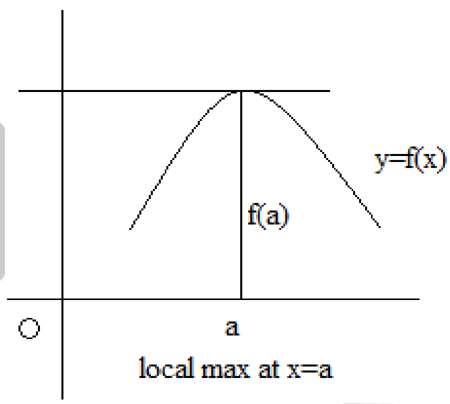
(i) the maximum value or the greatest value of f in A if $f(x) \leq l \ \forall x \in A$.

(ii) the minimum value or the least value of f in A if $f(x) \geq l \ \forall x \in A$

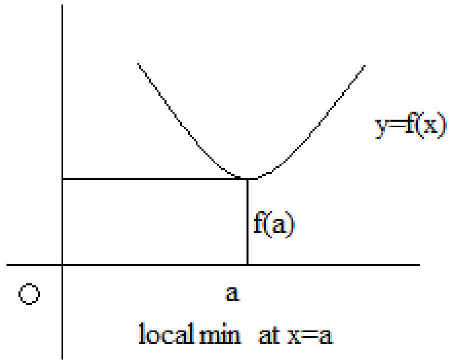
LOCAL MAXIMUM AND LOCAL MINIMUM VALUES

Let f be a function defined in a nbd of a point ‘a’ then f is said to have

(i)a local maximum (value) or a relative maximum at a if $\exists$ a $\delta > 0$ such that $f(x) < f(a) \ \forall x \in (a-\delta, a) \cup (a, a+\delta)$. In this case a is called a point of local maximum of f and $f(a)$ is its local maximum value.



(ii) a local minimum (value) or relative minimum at a if $\exists\, a\ \delta > 0$ such that $f(x) > f(a) \ \forall x \in (a-\delta, a) \cup (a, a+\delta)$. In this case a is called a point of local minimum and $f(a)$ is its local minimum value.



THEOREM:

Let f be a differentiable function in a nbd of a point a. The necessary condition for f to have local maximum or local minimum at a is $f'(a)=0$.

FIRST DERIVATIVE TEST

Let f be a differentiable function in a nbd of a point a and $f'(a)=0$. Then

(i) $f(x)$ has a relative maximum at x =a if $\exists$ a $\delta>0$ such that $x\in(a-\delta,a) \Rightarrow f'(x)>0$ and $x\in(a,a+\delta) \Rightarrow f'(x)<0$.

(ii) $f(x)$ has a relative minimum at x= a if $\exists$ a $\delta>0$ such that $x\in(a-\delta,a) \Rightarrow f'(x)<0$ and $x\in(a,a+\delta) \Rightarrow f'(x)>0$.

(iii) $f(x)$ has neither a relative maximum nor a minimum at x=a if $f'(x)$ has the same sign for all $x\in(a-\delta,a)\cup(a,a+\delta)$.

SECOND DERIVATIVE TEST

Let $f(x)$ be a differentiable function in a nbd of a point 'a' and let $f''(a)$ exist.

(i) If $f'(a)=0$ and $f''(a)<0$ then $f(x)$ has a relative maximum at $f(x)$ and the maximum value at a is $f(a)$.

(ii) If $f'(a)=0$ and $f''(a)>0$ then $f(x)$ has a relative minimum at $f(x)$ and the minimum value at a is $f(a)$.

ABSOLUTE MAXIMA AND ABSOLUTE MINIMA

Let f be a function defined on [a, b]. Then

(i) Absolute maximum of f on [a, b] = Max.{ $f(a), f(b)$ and all relative maximum values of f in (a,b) }.

(ii) Absolute minimum of f on [a, b] = Min.{ $f(a), f(b)$ and all relative minimum values of f in (a,b) }.

Note : 1) maximum and minimum values are called extrimities.

2) If $f^1(a) = 0$, then f is said to be stationary at a and f(a) is called the stationary value of f. and (a, f(a)) is called a stationary point of f.

Exercise

1. Find the point at which the local maxima or local minima (if any) are attained for the following function. Also find the local maximum or local minimum values as the case man be.

i) $f(x) = x^3 - 3x$

Sol. $f(x) = x^3 - 3x \Rightarrow f^1(x) = 3x^2 - 3$ and $f^{11}(x) = 6x$

$\therefore$ For maximum or minimum $f^1(x) = 0$

$\Rightarrow 3x^2 - 3 = 0 \quad \Rightarrow x^2 - 1 = 0 \quad \Rightarrow x = \pm 1$

$f''(1) = 6(1) = 6 > 0$

$\therefore f(x)$ has minimum at x = 1and that minimum value is $f(1) = 1^3 - 3(1) = -2$

$f''(-1) = 6(-1) = -6 < 0$

$\therefore$ f(x) has maximum value at x = -1

And that maximum value is $f(-1) = (-1)^3 - 3(-1) \quad = -1 + 3 = 2$

ii) $f(x) = \mathbf{x^3 - 6x^2 + 9x + 15}$

Sol. $f(x) = \mathbf{x^3 - 6x^2 + 9x + 15} \Rightarrow f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$

$\therefore$ For maximum or minimum $f'(x) = 0$

$$\Rightarrow 3x^2 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1 \text{ or } 3$$

Now $f''(1) = 6(1) - 12 = -6 < 0$

$\therefore f(x)$ has maximum value at x = 1

Max. value is $f(1) = 1^3 - 6(1)^2 + 9(1) + 15 \; = 1 - 6 + 9 + 15 = 19$

$f''(3) = 6(3) - 12 = 18 - 12 = 6 > 0$

$\therefore f(x)$ has minimum value at x = 3

Min. value is $f(3) = 3^3 - 6.3^2 + 9.3 + 15 \; = 27 - 54 + 27 + 15 = 15$

iii) $\mathbf{f(x) = (x-1)(x+2)^2}$

ans. f has min. value at x = 0 and that min = -4

F has max. value at $x = -2$ and that is 0

iv) $\mathbf{f(x) = \frac{x}{2} + \frac{2}{x} (x > 0)}$

Sol: $\mathbf{f(x) = \frac{x}{2} + \frac{2}{x} (x > 0)}$

$$\Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2} \text{ and } f''(x) = \frac{4}{x^3}$$

$\therefore$ For max. or min. $f'(x) = 0$

$$\Rightarrow \frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$

$f''(2)=\frac{4}{2^3}=\frac{1}{2}>0\,(\text{Since } x>0)$

$\therefore f(x)$ has min. value at x = 2

Min. value is $f(2)=\frac{2}{2}+\frac{2}{2}=1+1=2$

v) $\mathbf{f(x)=\frac{1}{x^2+2}}$

ans: max f(0) = 1/2

vi) $\mathbf{f(x)=x\sqrt{1-x}\,(0<x<1).}$

vii) $\mathbf{f(x)=-(x-1)^3(x+1)^2}$

critical values at x = 0, min value at x =-1 and that min = 0, max at $x=-\frac{1}{5}$

and max = 3456/3125

2. Prove that the following functions do not have maxima or minima.

i) $\mathbf{f(x)=e^x}$

Sol: $f'(x)=e^x$ and $f''(x)=e^x$

$\therefore$ For maxima or minima $f'(x)=0\Rightarrow e^x=0$ which is not true i.e., there is no value of x satisfying $f^1(x)$ =0.

Hence it has no maxima or minima.

ii) $\mathbf{f(x)=\log x}$

Sol: $f'(x)=\frac{1}{x}$ and $f'(x)=-\frac{1}{x^2}\neq$o for all x.

$\Rightarrow f'(x)\neq 0$

$\Rightarrow f(x)$ has no maxima or minima.

iii) $\mathbf{f(x) = x^3 + x^2 + x + 1}$

Sol: $f'(x) = 3x^2 + 2x + 1 \neq o$ for all x.

$\Rightarrow f'(x) \neq 0 \;\Rightarrow$ It has no maximum or minimum values.

3. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals.?

i) $\mathbf{f(x) = x^3 \text{ in } [-2, 2]}$

Sol: $f'(x) = 3x^2$ and $f''(x) = 6x$

And $3x^2>0$ for all vaules of x. hence f is an increasing function.

Therefore it greatest value is f(2) and least value is f(-2)

$\Rightarrow f(-2) = (-2)^3 = -8$ And $f(2) = 2^3 = 8$

ii) $\mathbf{f(x) = (x-1)^2 + 3 \text{ in } [-3, 1]}$

Sol: $\mathbf{f(x) = (x-1)^2 + 3 \text{ in } [-3, 1]} \Rightarrow f'(x) = 2(x-1)$

$f'(x) = 0 \Rightarrow x = 1.$

Now $f(-3) = (-3-1)^2 + 3 = 16 + 3 = 19$

$f(1) = (1-1)^2 + 3 = 0 + 3 = 3$

Max. value = 19

Min.value =3

iii) $\mathbf{f(x) = \sin x + \cos x \text{ in } [0, \pi]}$

Sol: $\mathbf{f(x) = \sin x + \cos x \text{ in } [0, \pi]} \Rightarrow f^1(x) = \cos x - \sin x$

Now $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \;\Rightarrow x = \dfrac{\pi}{4}$

Now $f(0) = \sin o + \cos 0 = 0 + 1 = 1$

$$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}==\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

$$f(\pi)=\sin\pi+\cos\pi=0-1=-1$$

Abslolute max $=$ max of $\{ f(0),f(\frac{\pi}{4}),f(\pi)\}$ $=$ max of $\{1,\sqrt{2},-1\}=\sqrt{2}$

Absolute min = min of $\{ f(0),f(\frac{\pi}{4}),f(\pi)\}$ = -1.

iv) $\mathbf{f(x)=4x^3-8x^2+1 \text{ in} [-1,1]}$

ANS: Minimum value is –11

Maximum value is 1.

v) $\mathbf{f(x)=x+\sin 2x \text{ in } [0,2\pi]}$

ANS: Minimum value = 0

Maximum value is = 2π

II

1. Find two positive numbers whose sum is 12 and the sum of the squares is minimum.

Sol: let x and y be the given numbers $\Rightarrow$ x + y = 12

$\Rightarrow y = 12 - x$

Let $f(x)=x^2+y^2 \Rightarrow f(x)=x^2+(12-x)^2$

$=x^2+144+x^2-24x$

$=2x^2-24x+144$

$\Rightarrow f'(x)=4x-24$ and $f''(x)=4$

For max or min, $f'(x)=0 \Rightarrow 4x-24=0$

$\Rightarrow 4x-24 \Rightarrow x=\frac{24}{4}=6$

$f''(x) = 4 > 0$

$f(x)$ has minimum when x = 6

$y = 12 - x = 12 - 6 = 6$

$\therefore$ The numbers are 6, 6.

2. Find two positive numbers x and y such that x + y = 60 and xy^2 is maximum.

Sol: let x and y be the given numbers $\Rightarrow$ $x + y = 60 \Rightarrow y = 60 - x$ ---(1)

Let $f(x) = xy^3 = x(60-x)^3$

$f' = x\,3(60-x)^2(-1) + (60-x)^3$

$= -3x(60-x)^2 + (60-x)^3$

$= (60-x)^2[-3x + 60 - x]$

$= (60-x)^2(60-4x) = 4(60-x)^2(15-x)$

$f'' = 4\left[(60-x)^2(-1) + (15-x)2(60-x)(-1)\right]$

$= 4(60-x)[-60 + x - 30 + 2x]$

$= 4(60-x)(3x-90)$

$= 12(60-x)(x-30)$

For maximum or minimum $f' = 0$

$\Rightarrow 4(60-x)^2(15-x) = 0$

$\Rightarrow x = 60$ or $x = 15$; x cannot be 60

$\therefore x = 15 \Rightarrow y = 60 - 15 = 45$

$f''(15) = 12(60-15)(15-3x) < 0$

$\Rightarrow$ f has maximum when x =15

$\Rightarrow y = 45.$

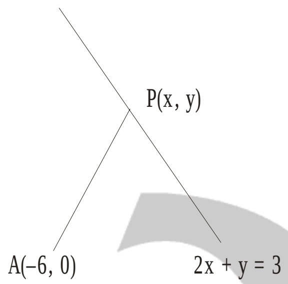
$\therefore$ Required numbers are 15, 45.

3. Find the shortest distance from $(-6,0)$ to $2x + y = 3$.

Sol: Equation of the line is $2x + y = 3$

$y = 3 - 2x$. let $A(-6,0)$

let $P(x, y)$ be any point on the line



$AP^2 = (x+6)^2 + y^2$

$= (x+6)^2 + (3-2x)^2$

$= x^2 + 12x + 36 + 9 + 4x^2 - 12x = 5x^2 + 45$

Let $f(x) = 5x^2 + 45$

$\Rightarrow f'(x) = 10x$ and f" = 10

For max or min $f'(x) = o \Rightarrow 10x = 0 \Rightarrow x = 0$

$f''(x) = 10 > 0$

f(x) is minimum when x = 0

$AP^2 = 45 \Rightarrow AP = \sqrt{45} = 3\sqrt{5}$

Shortest distance $= 3\sqrt{5}$ units.

4. Find the shortest distance from $(-6,0)$ to $x^2-y^2+16=0$.

Sol: let $P(x,y)$ be any point on the line

$\Rightarrow x^2-y^2+16=0$

$\Rightarrow \therefore y^2=x^2+16$

$A(-6,0)$ is the given point.

$AP^2=(x+6)^2+y^2=(x+6)^2+x^2+16=x^2+12x+36+x^2+16=2x^2+12x+52$

Let $f(x)=2x^2+12x+52$

$\Rightarrow f'(x)=4x+12$ and f"=4

For max or min , $f'(x)=0 \Rightarrow 4x+12=0$

$\Rightarrow 4x=-12$

$\Rightarrow x=\frac{-12}{4}=-3$

$f''(x)=4>0$

$f(x)$ is minimum when $x=-3$

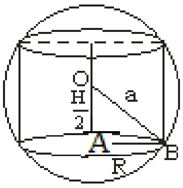
X= -3 $\Rightarrow y^2=x^2+16=9+16=25;\ y=5$

$\therefore$ Shortest distance $AP=\sqrt{(-3+6)^2+25}$

$=\sqrt{9+25}=\sqrt{34}$

5. Find the dimensions of the right circular cylinder of greatest volume, that can be inscribed in a sphere of radius a.

Sol: Suppose R is the radius and H is the height of the cylinder. Then OA = H/2.



From ΔOAB, $OB^2 = OA^2 + AB^2$

$$\Rightarrow a^2 = R^2 + \frac{H^2}{4}$$

$$\Rightarrow R^2 = a^2 - \frac{H^2}{4}$$

Volume of the cylinder V = $\pi R^2 H$

$$\Rightarrow V = \pi H\left(a^2 - \frac{H^2}{4}\right)$$

Let $f(H) = \pi\left(a^2 H - \frac{H^2}{4}\right)$

$\Rightarrow f'(H) = \pi\left(a^2 - \frac{3H^2}{4}\right)$, here H is the variable.

And $f''(H) = \pi\left(-\frac{6H}{4}\right)$

For max or min $f^1 = 0$ $\Rightarrow \pi\left(a^2 - \frac{3H^2}{4}\right) = 0$

$$\therefore a^2 - \frac{3H^2}{4} = 0 \Rightarrow \frac{3H^2}{4} = ah$$

$$\Rightarrow H^2 = \frac{4a^2}{3} \Rightarrow H = \frac{2a}{\sqrt{3}}$$

$$f''(H) = \pi\left(\frac{-6H}{4}\right) < 0$$

$f(H)$ is maximum when $H = \frac{2a}{\sqrt{3}}$

$$\Rightarrow R^2 = a^2 - \frac{H^2}{4} = a^2 - \frac{a^2}{3} = \frac{2a^2}{3}$$

$$\Rightarrow R = \frac{\sqrt{2}a}{\sqrt{3}}$$

Dimensions of greatest cylinder are

Base of the cylinder = $\frac{\sqrt{2}a}{\sqrt{3}}$

Height of the cylinder = $\frac{2a}{\sqrt{3}}$

6. **For what values of x > 0, the ratio of ln x to x is greatest**

Sol: Let $f(x) = \frac{l n x}{x}$

$$\Rightarrow f'(x) = \frac{x\frac{1}{x} - l n x.1}{x^2} = \frac{1 - l n x}{x^2}$$

For max or min $f'(x) = 0 \Rightarrow \frac{1 - l n x}{x^2} = 0$

$\Rightarrow$ 1- log x = 0

$\therefore x = e$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - l n x)2x}{x^4} = -\frac{(3 - 2\ln x)}{x^3}$$

$$f''(e) = -\left(\frac{3-2}{e^3}\right) = \frac{1}{e^3} < 0$$

$f(x)$ is greatest when x = e

i.e., $\frac{l\,n\,x}{x}$ is greatest when x = e.

III.

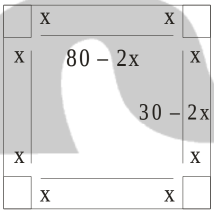
1. From a rectangular sheet of dimensions 30cm×80cm four equal squares of side x cm are removed at the corners and the sides are then tuned up so as to form an open rectangular box. What is the value o x, so that the volumes of the box is the greatest ?

Sol: length of the sheet = 80 , breadth = 30.

Let the Side of the square $= x$

Length of the box $= 80 - 2x = l$

Breadth of the box $= 30 - 2x = b$



Height of the box = x = h

Volume = $lbh = (80-2x)(30-2x).x$

$= x\left(2400 - 200x + 4x^2\right)$

$f(x) = 4x^3 - 220x^2 + 2400x$

$\Rightarrow f'(x) = 12x^2 - 440x + 2400$

$= 4\left[3x^2 - 110x + 600\right]$ and $f'' = 4(6x - 110)$

For max or min $f'(x) = 0 \Rightarrow 3x^2 - 110x + 600 = 0$

$$x = \frac{110 \pm \sqrt{12100 - 7200}}{6}$$

$$= \frac{110 \pm 70}{6} = \frac{180}{6} \text{ or } \frac{40}{6} = 30 \text{ or } \frac{20}{3}$$

If $x = 30,\ b = 30 - 2x = 30 - 2(30) = -30 < 0$

$$\Rightarrow x \neq 30 \qquad \therefore x = \frac{20}{3}$$

$$f''(x) = 24x - 440$$

When $x = \frac{20}{3}, f''(x) = 24.\frac{20}{3} - 440$

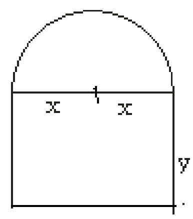
$$= 160 - 440 \ = -280 < 0$$

$\Rightarrow$ f(x) is maximum when $x = \frac{20}{3}$

Volume of the box is maximum when $x = \frac{20}{3}$ cm.

2. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be 20 feet, find the maximum area.

Sol: Let the length of the rectangle be 2x and breadth be y so that radius of the semi-circle is x.



Perimeter $= 2x + 2y + \pi.x = 20$

$$\Rightarrow 2y = 20 - 2x - \pi x$$

$$\Rightarrow y = 10 - x - \frac{\pi}{2}.x$$

$$\text{Area} = 2xy + \frac{\pi}{2}.x^2 \;= 2x\left(10 - x - \frac{\pi x}{2}\right) + \frac{\pi}{2}x^2$$

$$= 20x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$$

Let $f(x) = 20x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$

$\Rightarrow f' = 20 - 4x - 2\pi x + \pi x$ and $f'' = -4 - 2\pi + \pi = -4 - \pi$ for max or min

for max or min $f'(x) = 0 \Rightarrow 20 - 4x - \pi x = 0$

$$\Rightarrow (\pi + 4)x = 20$$

$$\Rightarrow x = \frac{20}{\pi + 4}$$

$$f''(x) = -4 - \pi < 0$$

$\Rightarrow f(x)$ has a maximum when $x = \frac{20}{\pi + 4}$

$$y = 10 - x - \frac{\pi}{2}x = 10 - \frac{20}{\pi + 4} - \frac{\pi}{2}\frac{20}{\pi + 4}$$

$$= \frac{10\pi + 40 - 20 - 10\pi}{\pi + 4} \quad = \frac{20}{\pi + 4}$$

Maximum area $= 2xy + \frac{\pi}{2}.x^2$

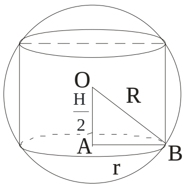
$$= \frac{40}{\pi + 4}.\frac{20}{\pi + 4} + \frac{\pi}{2}\frac{400}{(\pi + 4)^2}$$

$$= \frac{800 + 200\pi}{(\pi + 4)^2} = \frac{200(\pi + 4)}{(\pi + 4)^2}$$

$$= \frac{200}{\pi + 4}\text{sq.feet.}$$

3. Show that when curved surface area of a cylinder inscribed in a sphere of radius R is a maximum, then the height of the cylinder is $\sqrt{2}R$.

Sol: let r be the radius and h be the height of the cylinder.



From ΔOAB, $OA^2 + AB^2 = OB^2$

$$\Rightarrow r^2 + \frac{h^2}{4} = R^2;\ r^2 = R^2 - \frac{h^2}{4}$$

Curved surface area $= 2\pi rh \qquad = 2\pi\sqrt{R^2 - \frac{h^2}{4}}.h$

$$= \pi h\sqrt{4R^2 - h^2}$$

Let $f(h) = \pi h\sqrt{4R^2 - h^2}$

$$\Rightarrow \quad f'(h) = \pi\left[h.\frac{1}{2\sqrt{4R^2 - h^2}}(-2h) + \sqrt{4R^2 - h2}.1\right]$$

$$= \pi.\frac{-h^2 + 4R^2 - h^2}{\sqrt{4R^2 - h^2}} = \frac{2\pi\left(2R^2 - h^2\right)}{\sqrt{4R^2 - h^2}}$$

For max or min $f'(h) = 0$

$$\therefore 2R^2 - h^2 = 0$$

$$\Rightarrow \frac{2\pi\left(2R^2 - h^2\right)}{\sqrt{4R^2 - h^2}} = 0$$

$$\Rightarrow \ h^2 = 2R^2 \Rightarrow h = \sqrt{2}R$$

$$\Rightarrow \sqrt{4R^2-h^2}(-2h)+(2R^2-h^2)$$

And $f''(h)=2\pi\dfrac{\frac{d}{dh}\sqrt{4R^2-h^2}}{4R^2-h^2}=-\dfrac{4\pi h+0}{\sqrt{4R^2-^2}}<0\left[\text{when } h=\sqrt{2}R\right]$

$f(h)$ is greatest when $h=\sqrt{2}R$

i.e., Height of the cylinder $=\sqrt{2}R$

4. A wire of length *l* is cut into two parts which are bent respectively in the form of a square and a circle. What are the lengths of the pieces of wire so that the sum of the areas is least

Sol: Suppose x is the side of the square and r is the radius of the circle.

Given $4x+2\pi r=l \qquad \Rightarrow 4x=l-2\pi r$

$$\Rightarrow x=\frac{l-2\pi r}{4}$$

Sum of the areas $=x^2+\pi r^2$

Let $f(r)=\dfrac{(l-2\pi r)^2}{16}+\pi r^2$

Then $f'(r)=\dfrac{2(l-2\pi r)}{16}(-2\pi)+2\pi r$

and $f''(r)=\dfrac{\pi^2}{2}+2\pi$

For max/min f '(r) = 0

$$\Rightarrow f'(r)=\frac{2(l-2\pi r)}{16}(-2\pi)+2\pi r=0$$

$$\Rightarrow f'(r)=0\Rightarrow\frac{-\pi}{4}(l-2\pi)+2\pi r=0$$

$$\Rightarrow \frac{\pi}{4}(l-2\pi r)=2\pi r$$

$$\Rightarrow r = \frac{l}{2(\pi+4)}$$

$$\Rightarrow x = \frac{l - 2\pi r}{4} = \frac{1}{4}\left(l \frac{-\pi l}{\pi+4}\right) = \frac{\pi l + r l - \pi l}{4\pi + 4} = \frac{4l}{4(\pi+4)} \Rightarrow \qquad = \frac{l}{\pi+4}$$

$$\Rightarrow 4x = \frac{4l}{\pi+4}$$

And f "(x) >0

$\therefore f''(r)$ is least when $r = \frac{l}{2(\pi+4)}$

Sum of the area is least when the wire is cut into pieces of length.

$\frac{\pi l}{\pi+4}$ and $\frac{4l}{\pi+4}$

Equation 1

1. Find the intervals in which the following function are strictly increasing or strictly decreasing. 6 – 9x – x^2

Sol. Let f(x) = 6 – 9x – x^2

$f'(x) = -9 - 2x$

f(x) is increasing if $f'(x) > 0$

$\Rightarrow -9 - 2x > 0 \Rightarrow 2x + 9 < 0 \Rightarrow x < \frac{-9}{2}$

f(x) is increasing if $x \in \left(-\infty, \frac{-9}{2}\right)$

f(x) is decreasing if $f'(x) < 0$

$\Rightarrow 2x + 9 > 0 \Rightarrow x > \frac{-9}{2}$

f(x) is decreasing of $x \in \left(\frac{-9}{2}, \infty\right)$.

2. Find the intervals in which the function f(x) = $\sin^4 x + \cos^4 x$ $\forall$ $x \in \left(0, \frac{\pi}{2}\right)$ is increasing and decreasing.

Sol. $f(x) = \sin^4 x + \cos^4 x$

$f(x) = (\sin^2 x)^2 + (\cos^2 x)^2$

$= (\sin^2 x + \cos^2 x) - 2\sin^2 x \cos^2 x$

$= 1 - \frac{1}{2}\sin^2 2x$

$f'(x) = \frac{-1}{2} 2\sin 2x \cdot \cos 2x(2)$

$= -2\sin 2x \cdot \cos 2x$

$= -4\sin 4x$

Let $0 < x < \frac{\pi}{4}$

$\therefore$ f(x) is decreasing if $f'(x) < 0$

$-\sin x < 0$

$\sin x > 0$

$\therefore x \in \left(0, \frac{\pi}{4}\right)$

f(x) is increasing if $f'(x) > 0$

$-\sin x > 0$

$\sin x < 0$

$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Find the points of local extrema (if any) and local extrema of the following functions each of whose domain is shown against the function.

3. $f(x) = \sin x, [0, 4\pi)$.

Sol. Given $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\Rightarrow f''(x) = -\sin x$

For max or min,

$f'(x) = 0$

$\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

i) $f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1 < 0$

$f(x) = \sin\frac{\pi}{2} = 1$

$\therefore$ Point of local maximum $x = \frac{\pi}{2}$

Local maximum = 1

ii) $f''\left(\frac{3\pi}{2}\right) = -\sin\frac{3\pi}{2} = -1 > 0$

$f(x) = \sin\frac{3\pi}{2} = -1$

$\therefore$ Point of local minimum x = $\frac{3\pi}{2}$

Local minimum x = –1

iii) $f''\left(\frac{5\pi}{2}\right) = -\sin\frac{5\pi}{2} = -1 < 0$

$f(x) = \sin\frac{5\pi}{2} = 1$

$\therefore$ Point of local maximum x = $\frac{5\pi}{2}$

Local maximum x = 1

iv) $f''\left(\frac{7\pi}{2}\right) = -\sin\frac{7\pi}{2} = 1 > 0$

$f(x) = \sin\frac{7\pi}{2} = -1$

$\therefore$ Point of local minimum x = $\frac{7\pi}{2}$

Local minimum x = –1

4. $\mathbf{f(x) = x\sqrt{(1-x)}\ \forall x \in (0,1)}$

Sol. $f'(x) = x\frac{1}{2\sqrt{1-x}}(-1) + \sqrt{1-x}\cdot 1$

$= \frac{-x}{2\sqrt{1-x}} + \sqrt{1-x}$

$= \frac{-x+2-2x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$

For max. or min. f′(x) = 0

$$\Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow 2-3x = 0 \Rightarrow x = \frac{2}{3}$$

$$f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3}\frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$$

5. Use the first derivative test to find local extrema of f(x) = x^2 – 6x + 8 on R.

Sol. $f(x) = x^2 - 6x + 8$

$f'(x) = 2x - 6 \Rightarrow f''(x) = 2$

for maximum or minimum $f'(x) = 0$

$2x - 6 = 0 \Rightarrow x = 3$

$f''(3) = 2 > 0$

$\therefore$ Point of local minimum x = 3.

Local minimum = –1.

6. Find local maximum or local minimum of f(x) = –sin 2x – x defined on [–π/2, π/2].

Sol. $f(x) = -\sin 2x - x$

$f'(x) = -2\cos 2x - 1$

$f''(x) = 4\sin 2x$

Thus the starting point are $x = \frac{\pi}{3}, \frac{\pi}{3}$ at

$$x = \frac{\pi}{3}, f''\left(\frac{\pi}{3}\right) = 4\sin\frac{2\pi}{3} = \frac{4\sqrt{3}}{2} > 0$$

$$f\left(\frac{\pi}{3}\right) = -\sin\frac{2\pi}{3} - \frac{\pi}{3}$$

$$\text{at } x = \frac{\pi}{3} f''\left(\frac{\pi}{3}\right) = -4\sin\frac{2\pi}{3} = -\frac{4\sqrt{3}}{2} < 0$$

$$f\left(-\frac{\pi}{3}\right) = +\sin\frac{2\pi}{3} + \frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

Local minimum = $-\frac{\sqrt{3}}{2}-\frac{\pi}{3}$

Local maximum = $\frac{\sqrt{3}}{2}+\frac{\pi}{3}$

7. Find the maximum profit that a company can make, if the profit function is given by P(x) = –41 + 72x – 18x^2.

Sol. P(x) = –41 + 72x – 18x^2

$\frac{dp(x)}{dx}=72-36x$

For maxima or minima, $\frac{dp}{dx}=0$

$72-36x=0\Rightarrow x=2$

$\frac{d^2p}{dx^2}=-36<0$

∴ The profit f(x) is maximum for x = 2

The maximum profit will be

P(2) = – 41 + 72(2) – 18(4) = 31

8. The profit function P(x) of a company selling x items per day is given by P(x) = (150 – x) x – 1000. Find the number of items that the company should manufacture to get maximum profit. Also find the maximum profit.

Sol. Given that the profit function

P(x) = (150 – x) x – 1000

For maximum or minimum $\frac{dp}{dx}=0$

$(150-x(1)-x(-1)=0$

$150-2x=0\Rightarrow x=75$

Now $\frac{d^2p}{dx^2}=-2<0$

∴ The profit P(x) is maximum for x = 75

The company should sell 75 terms a day

The maxima profit will be P(75) = 4625.

9. Find the absolute maximum and absolute minimum of f(x) = $8x^3 + 81x^2 - 42x - 8$ on [–8, 2].

Sol. $f(x) = 8x^3 + 81x^2 - 42x - 8$

$f'(x) = 24x^2 + 162x - 42$

For maximum or minimum, $f'(x) = 0$

$24x^2 + 162x - 42 = 0$

$4x^2 + 27x - 7 = 0$

$4x(x + 7) - 1(x + 7) = 0$

$(x + y)(4x - 1) = 0$

$x = -7 \text{ or } \frac{1}{4}$

$$f(-8) = 8(-8)^3 + 81(-8)^2 - 42(-8) - 8$$
$$= -8(512) + 81(64) + 336 - 8$$
$$= -4096 + 5184 + 336 - 8$$
$$= 5520 - 4104 = 1416$$

$$f(2) = 8(2)^3 + 81(2)^2 - 42(2) - 8$$
$$= 64 + 324 - 84 - 8 = 296$$

$$f\left(\frac{1}{4}\right) = 8\left(\frac{1}{4}\right)^3 + 8\left(\frac{1}{4}\right)^2 - 42\left(\frac{1}{4}\right)$$
$$= \frac{8}{64} + \frac{81}{16} - \frac{42}{4}$$
$$= \frac{8 + 324 - 672}{164} = -\frac{852}{64} = -\frac{213}{16}$$

$f(-7) = 1246$

Absolute maximum = 1416

Absolute minimum = $\frac{-213}{16}$

10. Find the maximum area of the rectangle that can be formed with fixed perimeter 20.

Sol. Let x and y denote the length and the breadth of a rectangle respectively. Given that the perimeter of the rectangle is 20.

i.e., $2(x + y) = 20$

i.e., $x + y = 10$ …(1)

Let A denote the area of rectangle. Then

$A = xy$ ...(2)

Which is to be minimized, equation (1) can be expressed as

$y = 10 - x$ ...(3)

from (3) and (2), we have

$A = x(10 - x)$

$A = 10x - x^2$ ...(4)

Differentiating (4) w.r.t. x we get

$$\frac{dA}{dx} = 10 - 2x \quad \text{...(5)}$$

The stationary point is a root of $10 - 2x = 0$

$\therefore$ $x = 5$ is the stationary point.

Differentiating (5) w.r.t. x, we get

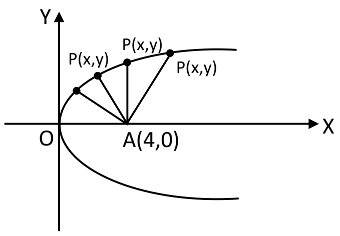
$$\frac{d^2A}{dx^2} = -2$$

which is negative. Therefore by second derivative test the area A is maximized at

$x = 5$ and hence $y = 10 - 5 = 5$, and the maximum area is $A = 5(5) = 25$.

11. Find the point on the graph $y^2 = x$ which is the nearest to the point (4, 0).

Sol.



Let P(x, y) be any point on $y^2 = x$ and

A(4, 0). We have to find P such that PA is minimum. Suppose PA = D. The quantity to be minimized is D.

$$D = \sqrt{(x-4)^2 + (y-0)^2} \quad \text{...(1)}$$

P(x, y) lies on the curve, therefore

$y^2 = x$ ...(2)

from (1) and (2), we have

$$D = \sqrt{(x-4)^2 + x}$$

$$D = \sqrt{(x^2 - 7x + 16)} \qquad \ldots(3)$$

Differentiating (3) w.r.t. x, we get

$$\frac{dD}{dx} = \frac{2x-7}{2} \cdot \frac{1}{\sqrt{x^2 - 7x + 16}}$$

Now $\frac{dD}{dx} = 0$

Gives x = 7/2. Thus 7/2 is a stationary point of the function D. We apply the first derivative test to verify whether D is minimum at x = 7/2

$$\left(\frac{dD}{dx}\right)_{x=3} = -\frac{1}{2} \cdot \frac{1}{\sqrt{9-12+16}}$$

and it is negative

$$\left(\frac{dD}{dx}\right)_{x=4} = \frac{1}{2} \cdot \frac{1}{\sqrt{16-28+16}}$$

and it is positive

$\frac{dD}{dx}$ changes sign from negative to positive. Therefore, D is minimum at x = 7/2. Substituting x = 7/2 in (2) we have $y^2 = 7/2$.

$$\therefore y = \pm\sqrt{\frac{7}{2}}$$

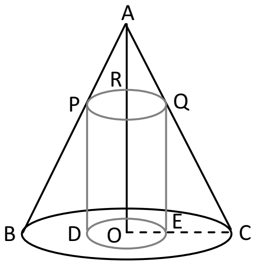
Thus the points $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$ and $\left(\frac{7}{2}, -\sqrt{\frac{7}{2}}\right)$ are nearest to A(4, 0).

12. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Sol. Let O be the center of the circular base of the cone and its height be h. Let r be the radius of the circular base of the cone.

Then AO = h, OC = r

Let a cylinder with radius x(OE) be inscribed in the given cone. Let its height be u.



i.e. RO = QE = PD = u

Now the triangles AOC and QEC are similar.

Therefore, $\frac{QE}{OA} = \frac{EC}{OC}$

i.e., $\frac{u}{h} = \frac{r-x}{r}$

$\therefore u = \frac{h(r-x)}{r}$

Let S denote the curved surface area of the chosen cylinder. Then

$S = 2\pi\, xu$

As the cone is fixed one, the values of r and h are constants. Thus S is function of x only.

Now, $\frac{dS}{dx} = 2\pi h(r-2x)/r$ and $\frac{d^2S}{dx^2} = -\frac{4\pi h}{r}$

The stationary point of S is a root of

$$\frac{dS}{dx} = 0$$

i.e., $\pi(r-2x)/r = 0$

i.e., $x = \frac{r}{2}$

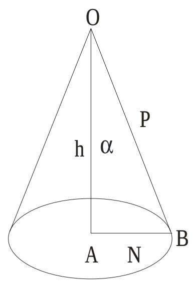
$\frac{d^2S}{dx^2} < 0$ for all x, therefore $\left(\frac{d^2S}{dx^2}\right)_{x=r/2} < 0$

Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is r/2.

Additional Problems for Practice

1. Show that the semi-vertical angle of the right circular cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Sol: let r be the base, h be the height, l be the slant height and α be the semi-vertical angle of the cone.



From ΔOAB, $\sin\alpha = \frac{r}{l}$, $\cos\alpha = \frac{h}{l}$

Volume of the cone is $v = \frac{1}{3}Ar^2h$

$= \frac{\pi}{3}\left(l^2\sin^2\alpha\right)\left(l\cos\alpha\right) \quad = \frac{\pi l^3}{3}\sin^2\alpha.\cos\alpha$

Let $f(\alpha) = \frac{\pi l^3}{3}.\sin^2\alpha.\cos\alpha$

$\Rightarrow \ f'(\alpha) = \frac{\pi l^3}{3}(\sin^2\alpha(-\sin\alpha) + \cos\alpha$

$2\sin\alpha\cos\alpha$

$= \frac{\pi l^3}{3}\sin\alpha\left(2\cos^2\alpha - \sin^2\alpha\right)$

And $f''(\alpha) = \frac{\pi l^3}{3}\{\sin\alpha(-4\cos\alpha\sin\alpha - 2\sin\alpha\cos\alpha) + \left(2\cos^2\alpha - \sin^2\alpha\cos\alpha\right)$

For max or min $f'(\alpha) = 0$

$\Rightarrow \frac{\pi l^3}{3}\sin\alpha\left(2\cos^2\alpha - \sin^2\alpha\right) = 0 \ \ \sin\alpha = 0$ or $2\cos^2\alpha - \sin^2\alpha = 0$

$\sin^2 \alpha = 2\cos^2 \alpha \Rightarrow \tan^2 \alpha = 2$

$\tan \alpha = \sqrt{2} \Rightarrow \left(\alpha = \tan^{-1}\left(\sqrt{2}\right)\right)$

When $\alpha = \tan^{-1}\sqrt{2}, 2\cos^2 \alpha = \sin^2 \alpha = 0$

$f''(\alpha) = \frac{\pi l^3}{3} - 6\sin^2 \alpha \cos \alpha / < 0$

$f(\alpha)$ is maximum, when $\alpha = \tan^{-1}\sqrt{2}$

The volume of the cure is maximum when semi-vertical angle is $\tan^{-1}\sqrt{2}$.

6. **Assume that the petrol burnt (per hour) in driving a motor boat varies as cube of its velocity. Show that the most economic speed. When going against a current of 6k per hour is 9 km. per hour.**

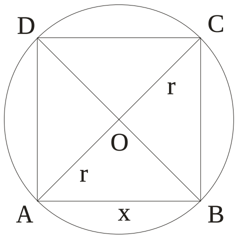
Ans: most economical speed is 9 kmph.

7. **Let $A(0,a)$, $B(0,b)$ be two fixed points $P(x,0)$ a variable point. Show that when acute angle $\angle APB$ is maximum, $x^2 = ab$.**

8. **Show that in the area of a rectangle inscribed a circle is maximum when it is a square. ?**

Sol: Let r be the radius of the circle x be the length and y be the breadth of the rectangle.

Then diagonal of rectangle = diameter of the circle =2r.



From ΔABC, $x^2 + y^2 = 4r^2 \quad \Rightarrow y = \sqrt{4r^2 - x^2}$

Area $A = xy = x\sqrt{4r^2 - x^2}$

Let $f(x) = x\sqrt{4r^2 - x^2}$

$$\Rightarrow f'(x) = x.\frac{1}{2\sqrt{4r^2 - x^2}}(-2x) + \sqrt{4r^2 - x^2 - 1}$$

$$= \frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}} = \frac{2\left(2r^2 - x^2\right)}{\sqrt{4r^2 - x^2}}$$

For max or min $f'(x) = 0$

$$\Rightarrow \frac{2\left(2r^2 - x^2\right)}{\sqrt{4r^2 - x^2}} = 0$$

$$\Rightarrow 2.r^2 - x^2 = 0 \Rightarrow x^2 = 2r^2$$

$$\Rightarrow x = \sqrt{2}r$$

$$f''(x) = \frac{2\left(\sqrt{4r^2 - x^2}.(-2x) - \left(2r^2 - x^2\right)\frac{(-2x)}{2\sqrt{4r^2 - x^2}}\right)}{\left(4r^2 - x^2\right)}$$

When $x = \sqrt{2}r$, we have $2r^2 - x^2 = 0$

$$f''(x) = \frac{-4x\sqrt{4r^2 - x^2}}{4r^2 - x^2} = \frac{-4x}{\sqrt{4r^2 - x^2}} < 0$$

$\Rightarrow$ f is max when $x = \sqrt{2}r$

$$\Rightarrow y^2 = 4r^2 - x^2 = 4r^2 - 2r^2 = 2r^2$$

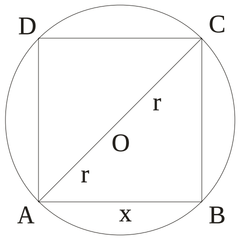
$$\Rightarrow y = \sqrt{2}r$$

Therefore $x = y = \sqrt{2}r$

Hence the rectangle is a square.

9. Find the rectangle of maximum perimeter that can be inscribed in a circle.

Sol: let x be the length, y be the breadth of the rectangle, r be the radius of the circle. Then diagonal = 2r.



From ΔABC, $x^2 + y^2 = 4r^2; y = \sqrt{4r^2 - x^2}$

Perimeter = $2(x+y) = 2\left(x + \sqrt{4r^2 - x^2}\right)$

Let $f(x) = 2\left(x + \sqrt{4r^2 - x^2}\right)$

$$\Rightarrow f'(x) = 2\left(1 - \frac{x}{\sqrt{4r^2 - x^2}}\right)$$

For max or min $f'(x) = 0$

$$\Rightarrow 1 - \frac{x}{\sqrt{4r^2 - x^2}} = 0$$

$$\Rightarrow \frac{x}{\sqrt{4r^2 - x^2}} = 1$$

$$\Rightarrow x^2 = 4r^2 - x^2$$

$$\Rightarrow 2x^2 = 4r^2 \Rightarrow x = \sqrt{2}r$$

$$f''(x) = \frac{-2\left[\sqrt{4r^2 - x^2}.1 - x\frac{(-x)}{2\sqrt{4r^2 - x^2}}(-2x)\right]}{4r^2 - x^2}$$

$$= \frac{-2\left(4r^2 - x^2 + x^2\right)}{2\left(4r^2 - x^2\right)^{3/2}} \qquad = \frac{-4r^2}{2\left(4r^2 - x^2\right)^{3/2}}$$

$$f''\left(\sqrt{2}r\right)=\frac{-4r^2}{2\left(4r^2-2r^2\right)^{\frac{3}{2}}}<0$$

Therefore f is max when x = $\sqrt{2}r$.

$\Rightarrow y^2=4r^2-x^2=4r^2-2r^2=2r$

$y=\sqrt{2}r$ i.e., x = y

f(x) is maximum when x = y

i.e., when the rectangle becomes a square.

10. Show that $f(x)=\sin(x)(1+\cos x)$ has a maximum value at $x=\pi/3$

Sol: Given $f(x)=\sin x(1+\cos x)$

$f'(x)=\sin x(-\sin x)+(1+\cos x).\cos x$

$=\cos x+\cos^2 x-\sin^2 x$

$=\cos x+\cos^2 x-1+\cos^2 x$

$=2\cos^2 x+\cos x-1$

For max or min $f'(x)=0$

$\Rightarrow 2\cos^2 x+\cos x-1=0$

$\Rightarrow (\cos x+1)(2\cos x-1)=0$

$\Rightarrow \cos x=-1 \text{ or } \cos x=\frac{1}{2}$

Now $f''(x)=-4\cos x.\sin x-\sin x$

$=-\sin x(4\cos x+1)$

when $\cos x=\frac{1}{2}, \sin x=\frac{\sqrt{3}}{2}$

$f''(x) = -\cos\frac{\sqrt{3}}{2}(2+1) < 0$

$f(x)$ is maximum when $\cos x = \frac{1}{2}$

i.e., $x = \frac{\pi}{3}$

11. Show that $f(x) = sim^m x.\cos^n x$ has maximum value of $x = \tan^{-1}\sqrt{\frac{m}{n}}, (mn > 0)$.

Sol: Given $f(x) = sim^m x.\cos^n x$

$f'(x) = \sin^m x.\cos^{n-1} x(-\sin x) + \cos^n x.m\sin^{m-1} x.\cos x$

$= \sin^{m-1} x.\cos^{n-1} x.\left(m\cos^2 x - n\sin^2 x\right)$

For max or min $f'(x) = 0$

$\Rightarrow \sin^{m-1} x.\cos^{n-1} x\left(m\cos^2 x - n\sin^2 x\right) = 0$

$\Rightarrow \ m\cos^2 x.n\sin^2 x = 0 \Rightarrow \tan^2 x = \frac{m}{n}$

$\Rightarrow \ x = \tan^{-1}\sqrt{\frac{m}{n}}$

$f''(x) = \sin^{m-1} x.\cos^{n-1} x(-2m\cos x\sin x - 2n\sin x\cos x)$

$+\left(m\cos^2 x - n\sin^2 x\right).\frac{d}{dx}\left(\sin^{m-1} x.\cos^{n-1} x\right)$

$= -\sin^{m-1} x.\cos^{n-1} x(m+n)$

$\sin 2x + \left(m\cos^2 x - n\sin^2 n\right)\frac{d}{dx}$

$\left(\sin^{m-1} x.\cos^{n-1} x\right)$

when $x=\tan^{-1}\sqrt{\frac{m}{n}}, m\cos^2 x - n\sin^2 x = 0$

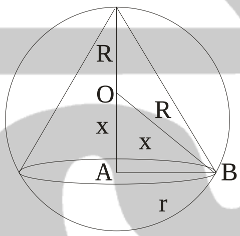
$f''(x) = -(m+n)\sin 2x.\sin^{m-1} x.\cos^{n-1} x < 0$

$f(x)$ is maximum at $x=\tan^{-1}\sqrt{\frac{m}{n}}$

1. **Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.**

Sol: Let R be the base radius, x be the distance of the centre of the sphere from the base and V be the volume of the cone.

Height h of the cone R + x



From ΔOAB, $r^2 = R^2 - x^2$

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(R+x)\left(R^2-x^2\right) = \frac{\pi}{3}\left(R^3 + R^2x - Rx^2 - x^3\right)$$

$$\frac{dv}{dx} = \frac{\pi}{3}\left(R^2 - 2Rx\right) - 3x^2$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R-6x)$$

For maximum volume, $\frac{dV}{dx}=0$

$R^2 - 2Rx - 3x^2 = 0$

$(R+x)(R-3x) = 0$

$R = -x, \frac{x}{3}$

When $x = \frac{R}{3}, \frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R - 2\pi) < 0$

V is maximum where $x = \frac{R}{3}$

Max. volume $= \frac{\pi}{3}\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right)$

$= \frac{\pi}{3}.8\frac{R^2}{9}\frac{4R}{3} = \frac{8}{27}\left(\frac{4}{3}\pi r^2\right)$

$= \frac{8}{27}$ (Volume of the sphere)

2. Find the intervals in which $f(x) = -3 + 12x - 9x^2 + 2x^3$ is increasing and the intervals in which $f(x)$ is decreasing.

3. Show that for all values of a and b, $f(x) = x^3 + 3ax^2 + 3a^2x + 3a^3 + b$ is increasing.

Sol: $f(x) = x^3 + 3ax^2 + 3a^2x + 3a^3 + b$

$\Rightarrow f'(x) = 3x^2 + 6ax + 3a^2$

$= 3(x^2 + 2ax + a^2)$

$= 3(x + a)^2 \geq 0$

$\therefore$ For all values of 0, the function f is increasing.

4. Show that $x - \frac{x^2}{x} \leq ln(1+x) \leq x - \frac{x^2}{2(1+x)}$ for all $x \geq 0$.

5. Show that $x^3 - 6x^2 + 15x \geq 0$ for all $x \geq 0$.

6. If $\frac{\pi}{2} \geq x \geq 0$ show that $x \geq \sin x$.

7. Find all the local maxima and local minima of the function $f(x) = x^2 - 12x$.

8. Find al he points as local maxima and local minima of the function $f(x) = x^4 - 8x^2$.

9. Find all the local maxima and local minima of the sine function.

10. Find the absolute maximum and value of $x^{40} - x^{20}$ on the interval $[0,1]$. Find also its absolute minimum value on this interval.

Sol: Let $f(x) = x^{40} - x^{20}$

$$f'(x) = 40x^{39} - 20x^{19} = 20x^{19}(2x^{20} - 1)$$

$$f'(x) = 0 \Rightarrow 20x^{19}(2x^{20} - 1) =$$

$$x = 0 \text{ or } x^{20} = \frac{1}{2} \text{ i.e., } x = \left(\frac{1}{2}\right)^{20}$$

$x = 0$ and $x = \left(\frac{1}{2}\right)^{20}$ are the two points locate f' take the value zero with these two points, we consider the end points 0 and 1 abc of interested. Totally we get only there points. At these three points, we calculate the value of 0.

$$f(0) = 0, \ f(1) = 0$$

$$f\left(\frac{1}{2}\right)^{20} = \left(\frac{1}{2}\right)^{\frac{40}{20}} - \left(\frac{1}{2}\right)^{\frac{20}{40}} = \left(\frac{1}{2}\right)^{2} - \frac{1}{2}$$

$$= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

The maximum value of $f(0,1)$

$$= \max\left\{f(0) f\left(\frac{1}{2}\right)^{20}, f(x)\right\}$$

$= \max\left\{0, \frac{1}{4}, 0\right\} = 0$ attained at x = 0 and

x = 1

The maximum value of f over $(0,1)$

$= \min\left\{f(0), f\left\{\left(\frac{1}{2}\right)^{20}\right\}, f(1)\right\}$

$= \min\left\{0, -\frac{1}{4}, 0\right\}$

$= -\frac{1}{4}$ attained at $x = \left(\frac{1}{2}\right)^{20}$

11. Find the maximum and minimum values of $2\sin x + \sin 2x$ over $[0, 2\pi]$

Sol: Let $f(x) = 2\sin x + \sin 2x$

Then $f'(x) = 2\cos x + 2\cos 2x$

$f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos x + 2\cos^2 x - 1 = 0$

$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2} \text{ or } \cos x = -1$

The solutions of $\cos x \frac{1}{2}$ in $[0, 2\pi]$ are $x = \pi$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

The solution of cosx = -1 in $[0, 2\pi]$ is $x = \pi$

End points of the interval are $0, 2\pi$

Hence, we have to find the values of the given functions at $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ and 2π

$f(0) = 0, f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2},$

$f(\pi) = 0, f\left(\frac{5\pi}{3}\right) = \frac{-3\sqrt{3}}{2}$

and $f(2\pi) = 0$.

$\therefore$ Maximum value of over $[0, 2\pi]$

$= \max.\left\{f(0), f\left(\frac{\pi}{3}\right) f\left(\frac{5\pi}{3}\right), f(2\pi)\right\}$

$= \max.\left\{0, \frac{3\sqrt{3}}{2}, \frac{-3\sqrt{3}}{2}\right\} = \frac{3\sqrt{3}}{2}$

Similarly, the minimum value of f over $[0, 2\pi]$ $= \text{Min.}\left\{0, \frac{3\sqrt{3}}{2}, \frac{-3\sqrt{3}}{2}\right\} = \frac{-3\sqrt{3}}{2}$

12. **Find the position numbers whose sum is 24 and whose product as large as possible.**

13. **The sum of the height and diameter of the base of a right circular cylinder in given as 3 units. Find the radius of the base and height of the cylinder so that the volume is maximum.**

14. **Show that xe^{-x} has the maximum value at**

    x = 1.

15. **A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point $(3, 2)$. What is the nearest distance between the soldier and the jet?**

Sol: For each value of x, the position of the jet is $\left(x, x^2+2\right)$. Let $f(x)$ be the square of the distance between their position and the soldier.

$f(x)=(x-3)^2+\left(x^2+2-2\right)^2$

$=(x-3)^2+x^4$

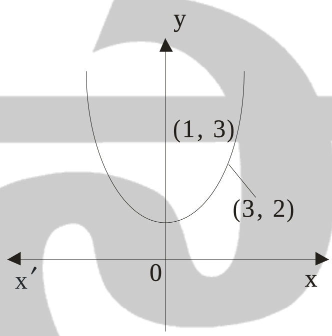
$f'(x)=2(x-3)+4x^3$

$f'(x)=2+12x^2$

$f'(x)=0$ only if $2(x-3)+4x^3=0$

$x-3+2x^3=0$

$2(x-1)\left(2x^2+2x+3\right)=0$



The only point at which $f'(x)$ takes the value zero is x = 1.

Since there are no real rots of $2x^2+2x+3=0$

$f''(1)2+12=14>0$

Hence, at x =1 the function has local minimum

These are the least value of

$f=f(1)=(1-3)^3+(1+2-2)^2=4+1=5$

$\therefore$ Required minimum distance $\sqrt{f(x)}=\sqrt{5}$.

Observe that there is no maximum value in this problem.