CHAPTER 10

MAXIMA AND MINIMA - 1

TOPICS:

1.Increasing and decreasing functions

2. Maxima and minima of functions

3.Mensuration

MAXIMA AND MINIMA

MONOTONIC FUNCTIONS OVER AN INTERNAL

Definition: A function $f:[a, b] \to R$ is said to be

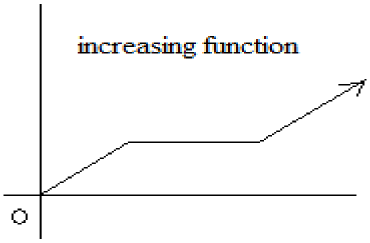
(i) Monotonically increasing (or non - decreasing) on [a, b] if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)\ \forall x_1, x_2 \in [a, b]$

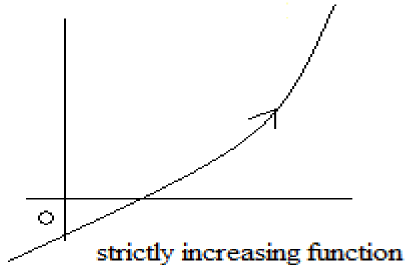
(ii) Monotonically decreasing (or non - increasing) on [a, b] if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ $\forall x_1, x_2 \in [a, b]$

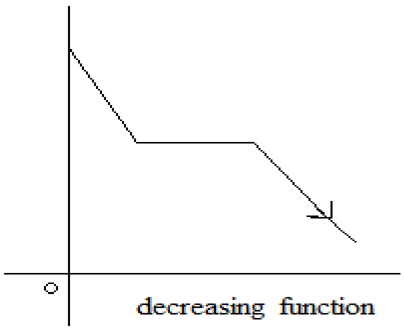
(iii) strictly increasing on [a, b] if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall x_1, x_2 \in [a, b]$

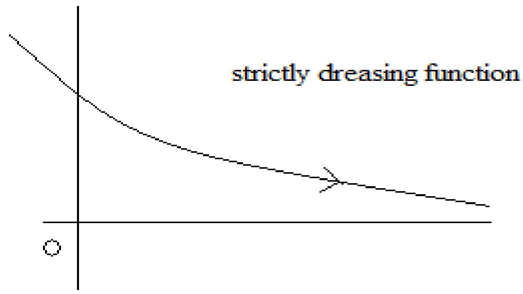
(iv) strictly decreasing on [a, b] if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall x_1, x_2 \in [a, b]$

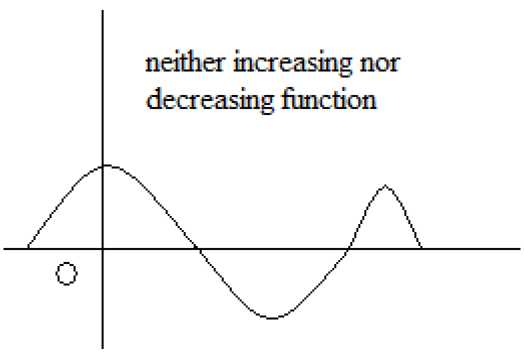
(v) a monotonic function on [a, b] if f is either monotonically increasing or monotonically decreasing on [a, b].











THEOREM:

Let f be a function defined in a nbd of a point a and f be differentiable at a. Then

(i) $f'(a) > 0 \Rightarrow$ f is locally increasing at a.

(ii) $f'(a) < 0 \Rightarrow$ f is locally decreasing at a.

Note:

Let f be continuous on [a, b] and differentiable on (a, b). Then

(i) $f'(x) \geq 0 \ \forall x \in (a,b) \Rightarrow f(x)$ is increasing on [a, b].

(ii) $f'(x) \leq 0 \ \forall x \in (a,b) \Rightarrow f(x)$ is decreasing on [a, b].

(iii) $f'(x) > 0 \ \forall x \in (a,b) \Rightarrow f(x)$ is strictly increasing on [a, b].

(iv) $f'(x) < 0\ \forall x \in (a,b) \Rightarrow f(x)$ is strictly decreasing on [a, b].

(iv) $f'(x) = 0\ \forall x \in (a,b) \Rightarrow f(x)$ is a constant function on [a,b].

INCREASING AND DECREASING FUNCTIONS

EXERCISE

1. Without using the derivative, show that $f(x) = \left(\frac{1}{2}\right)^x$ **is strictly decreasing on R.**

Sol. $f(x) = \left(\frac{1}{2}\right)^x$, let $x_1, x_2 \in R$

let $x_1 < x_2 \Rightarrow \left(\frac{1}{2}\right)^{x_1} > \left(\frac{1}{2}\right)^{x_2}$

$\Rightarrow f(x_1) > f(x_2)$

Therefore, f is strictly decreasing on R.

2. Without using the derivative, show that $f(x) = 3x + 7$ **is strictly increasing on R.**

Sol. $f(x) = 3x + 7$

let $x_1, x_2 \in R$

let $x_1 < x_2 \Rightarrow 3x_1 < 3x_2$

$\Rightarrow 3x_1 + 7 < 3x_2 + 7$

$\Rightarrow f(x_1) < f(x_2)$

Therefore, given function is strictly increasing on R

3. Show that $x + \frac{1}{x}$ **is increasing on [1,∞).**

Sol.

$$f(x) = x + \frac{1}{x}$$

$$f^1(x) = 1 - \frac{1}{x^2}$$

$$\text{since} \quad x \in [1, \infty),\ 1 - \frac{1}{x^2} > 0$$

$$\Rightarrow f^1(x) > 0$$

Therefore, f(x) is increasing function.

4. Show that $f(x) = \cos^2 x$ **is strictly decreasing on** $(0, \pi/2)$

Sol.

$$f(x) = \cos^2 x$$

$$f'(x) = 2\cos x(-\sin x) = -\sin 2x$$

$$x \in (0, \pi/2) \Rightarrow 0 < 2x < \pi$$

$$\Rightarrow f'(x) = -(\text{positive value}) = -\text{ve.}$$

$$\Rightarrow f'(x) < 0$$

Therefore, $f(x) = \cos^2 x$ is strictly decreasing on $(0, \pi/2)$

I. State the points at which the following functions (from 1 to 4) are increasing and the points at which they are decreasing.

1. $\mathbf{f(x) = x^3 - 3x^2}$

Sol: Given function $\mathbf{f(x) = x^3 - 3x^2}$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

f(x) is increasing if $f'(x) > 0 \Rightarrow 3x(x-2) > 0$

$\therefore$ x does not lie between 0 and 2

$\therefore f(x)$ is increasing in $(-\alpha, 0) \cup (2, \alpha)$

f(x) is decreasing if $f'(x)<0$

$x(x-2)<0$

x lies between 0 and 2

f(x) is decreasing in (0, 2)

2. $\mathbf{f(x)=x^3(x-2)^2}$.

Sol: Given function $\mathbf{f(x)=x^3(x-2)^2}$

$\Rightarrow f'(x)=x^3.2(x-2)+(x-2)^2.3x^2 \quad =x^2(x-2)(2x+3(x-2))$

$=x^2(x-2)(2x+3x-6)=x^2(x-2)(5x-6)\forall x\in R, x^2\geq 0$

If f(x) is increasing $f'(x)>0$

$\Rightarrow x^2(x-2)(5x-6)>0$

$\Rightarrow (x-2)(5x-6)>0, x\neq 0$

$\Rightarrow x\in\left(-\infty,\frac{6}{5}\right)\cup(2,\infty)-\{0\}$

$\Rightarrow f(x)$ is increasing if $x\in\left(-\infty,\frac{6}{5}\right)\cup(2,\infty)-\{0\}$

and f(x) is decreasing if $x\in\left(\frac{6}{5},2\right)$

3. $\mathbf{f(x)=x^3-3x^2-6x+12}$

Ans: $f(x)$ is decreasing in $\left(1-\sqrt{3},\sqrt{3}+1\right)$ and $x<1-\sqrt{3}$ and $x>\sqrt{3}+1$

4. $\mathbf{f(x)=xe^x}$

Sol: given function $\mathbf{f(x)=xe^x}$

$\Rightarrow f'(x)=x.e^x+e^x.1=e^x(x+1)$

e^x is positive for all real values of x

$f'(x) > 0 \Rightarrow x+1 > 0 \Rightarrow x > -1$

$f(x)$ is increasing when $x > -1$

$f'(x) < 0 \Rightarrow x+1 < 0 \Rightarrow x < -1$

f(x) is decreasing when x < – 1

Determine the intervals in which the functions (from 5 to 8) are increasing and the interval in which they are decreasing.

5. $f(t) = \frac{\ln t}{t}$

Sol: Given function $f(t) = \frac{\ln t}{t}$

$$\Rightarrow f'(t) = \frac{t\frac{1}{t} - \ln t 1}{t^2} = \frac{1-\ln t}{t^2}$$

If f(x) is increasing $f'(x) > 0$

$$f'(t) = \frac{1-\ln t}{t^2} > 0$$

$\Rightarrow 1 - \ln t > 0 \Rightarrow \ln t < 1 \Rightarrow t < e$

$f(x)$ is increasing when t < e

f(x) is decreasing if $f'(x) < 0$

$$\Rightarrow f'(t) < 0 \Rightarrow \frac{1-\ln t}{t^2} < 0$$

$\Rightarrow 1 - \ln t < 0$

$\Rightarrow \ln t > 1 \Rightarrow t > e$

$f(t)$ is decreasing when $t > e$

6. $f(x) = 1 - \frac{\ln(1+x)}{x}, (x > 0)$

Sol: $f(x) = 1 - \frac{\ln(1+x)}{x}, (x > 0)$

$$\Rightarrow f'(x) = 1 - \frac{x\frac{1}{1+x} - \ln(1+x).1}{x^2}$$

$$= 1 - \frac{x - (1+x).\ln(1+x)}{x^2(1+x)}$$

$$= \frac{x^2 + x^3 - x + (1+x).\ln(1+x)}{x^2(1+x)} > 0 \quad \text{for } x > 0$$

f(x) is increasing $(0,\infty)$

7. $f(x) = \sqrt{25 - 4x^2}$

Sol: f(x) is defined when $25 - 4x^2 \geq 0$

$\Rightarrow -(4x^2 - 25) \geq 0$

$\Rightarrow -(2x+5)(2x-5) \geq 0$

$\therefore$ x lies between $-\frac{5}{2}$ and $\frac{5}{2}$

Domain of $f = \left(-\frac{5}{2}, \frac{5}{2}\right)$

$$f'(x) = \frac{1}{2\sqrt{25-4x^2}}(-8x) \quad = -\frac{4x}{\sqrt{25-4x^2}}$$

f(x) is increasing when $f'(x) > 0$

$$\Rightarrow \frac{-4x}{\sqrt{25-4x^2}} > 0$$

i.e., $x < 0$

$f(x)$ is increasing when $\left(-\frac{5}{2}, 0\right)$

f(x) is decreasing when $f'(x) < 0$

$$\Rightarrow -\frac{4x}{\sqrt{25-4x^2}} < 0$$

$\therefore \; x > 0$

f(x) is decreasing when $\left(0, \frac{5}{2}\right)$.

8. f(x) = l n (l n x); x > 1.

Sol: given function **f(x) = l n (l n x); x > 1.**

Diff, w.r.t x, $f^1(x) = \frac{1}{1nx} \cdot \frac{1}{x}$

f(x) is decreasing when $f^1(x) > 0$

$$\frac{1}{x.1n} x > 0$$

$\Rightarrow$ x .1n x > 0

ln x is real only when x > 0

$\therefore$ *ln* x < 0 = l n 1

i.e., x < 1

f(x) is increasing when x > 1

i.e., in $(1, \infty)$

f(x) is decreasing when $f^1(x) < 0$

$\Rightarrow l\,n\,x > o = l\,n$

i.e., $x < 1$

f(x) is decreasing in (0, 1)

9. Show that $\frac{x}{x+1} < l\,n\,(1 + x) < x$ when $x > 0$.

Sol: Let $f(x) = l\,n\,(1 + x) - \frac{x}{1+x}$

$= l\,n\,(1 + x) - \frac{1 + x - 1}{1 + x}$

$= l\,n\,(1 + x) - 1 + \frac{1}{1 + x}$

$\Rightarrow f^1(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{1+x-1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0 \ (\because x > 0)$

f(x) is increasing when $x > 0$.

$\Rightarrow f(0) = l\,n\,1 - \frac{0}{1+0} = 0 - 0 = 0$

$x>0 \Rightarrow f(x) > f(0) \Rightarrow l\,n\,(1+x) - \frac{x}{1+x} > 0$

$\Rightarrow l\,n\,(1+x) > \frac{x}{1+x}$ -------(1)

Let $g(x) = x - l\,n\,(1+x)$

$\Rightarrow g^1(x) = 1 - \frac{1}{1+x} = \frac{1 + x - 1}{(1 + x)}$

$= \frac{1}{1+x} > 0 \ (\because x > 0)$

g(x) is increasing when x > 0

now $\quad g(0) = 0 - l\,n = 0 - 0 = 0$

$x > 0 \Rightarrow g(x) > g(0) \Rightarrow x - l\,n\,(1 + x) > 0$

$\Rightarrow x > l\,n\,(1 + x)$ ------(1)

From (1), (2) we get

$$\frac{x}{x+1} < l\,n\,(1 + x) < x \text{ for } x > 0$$

10. Show that $\dfrac{x}{(1+x)^2} < \tan^{-1} x < x$ when x > 0.

Ans. Same as above.

11. Show that tan x > x for every $x \in \left(0, \dfrac{\pi}{2}\right)$

Sol:` Let f(x) = tan x – x

$f^1(x) = \sec^2 x - 1 > 0$ for every $x \in \left(0, \dfrac{\pi}{2}\right)$

$\therefore$ f(x) is increasing for every $x \in \left(0, \dfrac{\pi}{2}\right)$

Now f(0) = tan 0 – 0 = 0 – 0 = 0

$\therefore \tan x - x > 0$ as f(x) is increasing.

$\Rightarrow \tan x > x$ for every $x \in \left(0, \dfrac{\pi}{2}\right)$

12. If $x \in \left(0, \dfrac{\pi}{2}\right)$, then show that $\dfrac{2x}{\pi} < \sin x < x$.

13. Show that $2x < \ln\dfrac{(1+x)}{1-x} < 2x\left(1+\dfrac{x^2}{2(1-x)^2}\right)$ **for all** $x \in (0, 1)$**.**

14. For the function $f(x) = x^x$ **find dthe points at which it is**

i) Increasing and ii) deceasing. Hence determine which of e^{π}, π^{e} **is greater**

Sol: Let $f(x) = x^x$

$\Rightarrow f^1(x) = x^x(1+\log x)$

For f to be increasing $f^1(x) > 0$

$\Rightarrow x^x(1+\log x) > 0 \Rightarrow 1 + \log x > 0 \Rightarrow \log x > -1 \Rightarrow x > e^{-1} = \dfrac{1}{e}$

$\therefore f(x)$ is increasing in $\left(\dfrac{1}{e}, \infty\right)$

For f to be dereasing $f^1(x) < 0$

$\Rightarrow x^x(1+\log x) < 0 \Rightarrow 1 + \text{lo } x < 0$

$\Rightarrow \log x < -1 \Rightarrow x < e^{-1} = \dfrac{1}{e}$

But $x > 0$

f(x) is decreasing in $\left(0, \dfrac{1}{e}\right)$

Consider the function $f(x) = x^{1/x}$

$\log f(x) = \log x^{1/x} = \dfrac{1}{x}\log x$

Differentiating w. r. to x

$\dfrac{1}{f(x)}.f^1(x) = \dfrac{1}{x}.\dfrac{1}{x}\log x.\dfrac{1}{x^2} = \dfrac{1}{x^2}(1-\log x) < 0$ when $x > e$

$$\Rightarrow \frac{f^1(x)}{f(x)} < 0 \Rightarrow f^1(x) < 0$$

f(x) is decreasing if x > e

But $\pi > e \;\Rightarrow\; f(\pi) < f(e)$

$$\pi^{1/\pi} < e^{1/e} \Rightarrow (\pi^{1/\pi})^{\pi e} < (e^{1/e})^{\pi e} \qquad \Rightarrow \pi^e < e\pi$$

$\therefore e\pi$ is greater than π^e.

15. State the points at which the following Functions are increasing and the points At which they are decreasing.

i) f(x) = sin h (sin x)

Sol: f(x) = sin h (sin x)

$$\Rightarrow f^1(x) = \cosh(\cos x)\ \cos x$$

f(x) is increasing if $f^1(x) > 0$

$$\Rightarrow \cosh\ (\sin x).\ \cos x > o$$

$$\Rightarrow \cos x > 0 \quad (\because \cosh(\sin x) > 0 \forall x \in R)$$

$$\Rightarrow x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(or) $x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right), n \in z$

f(x) is decreasing if $f^1(x) < 0$

$$\Rightarrow \cos x < 0$$

$$\Rightarrow x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right), n \in z$$

ii) f(x) =Cos h (cos x)

Sol: f(x) = cos h (cos x)

$$\Rightarrow f^1(x) = \sinh(\cos x)(-\sin x)$$

f(x) is increasing $\Rightarrow f^1(x) > 0$

i.e., $-\sin h(\cos x), \sin x > o$

case (1) : $\sin x > 0$, $\sin h(\cos x) < 0$

$\Rightarrow \sin x > 0, \cos x < 0$

$\Rightarrow x \in \left(\frac{\pi}{2}, \pi\right)$ (or) $x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \pi\right), n \in z$

Case (2) : $\sin x < 0$, $\sin h(\cos x) > 0$

$\Rightarrow \sin x < 0, \cos x > 0 \Rightarrow x \in \left(\frac{3\pi}{2}, 2\pi\right)$ (or)

$x \in \left(2n\pi \in \frac{3\pi}{2}, 2n\pi + 2\pi\right) n \in z$

f(x)is decreasing if $f(x) < 0$

$-\sin h(\cos x), \sin x < 0)$

$\sin h(\cos x), \sin x > 0$

Case (3) : $\sin h(\cos x) > 0, \sin x > 0$

$\Rightarrow \cos x > 0, \sin x > 0 \Rightarrow x \in \left(0, \frac{\pi}{2}\right)$ i.e., $\left(2x\pi + 0, 2x\pi + \frac{3\pi}{2}\right), x \in z$

Case (4) : $\sin h(\cos x) < 0$ and $\sin x < o$

$\cos x > 0, \sin x < 0$

$\Rightarrow x \in \left(\pi, \frac{3\pi}{2}\right)$ a

$x \in \left(2x\pi + \pi, 2x\pi + \frac{3\pi}{2}\right)$

$\therefore$f(x) is increasing if

$$x \in \left(2x\pi + \frac{\pi}{2}, 2x\pi + \pi\right) \cup \left(2x\pi + \frac{3\pi}{2}, 2x\pi + 2\pi\right)$$

f(x) is decreasing if

$x \in \left(2x\pi + \pi, 2x\pi + \frac{3\pi}{2}\right) \cup \left(2x\pi + \pi, 2x\pi + \frac{\pi}{2}\right)$

iii) f(x) = sin (tanh x)

Ans : f increasing on R.

iv) f(x) = $\tan^{-1}(\sin x)$

Ans. f(x) is increasing in $\left(2x\pi - \frac{\pi}{2}, 2x\pi + \frac{\pi}{2}\right), x \in z$

f(x) is decreasing in $\left(2x\pi + \frac{\pi}{2}, 2x\pi + \frac{3\pi}{2}\right), x \in z$

16. At what point by the slope of $y = \frac{x^3}{6} - \frac{3}{2}x^2 + \frac{11x}{2} + 12$ increasing?

Sol. Equation of the curve is

$$y = \frac{x^3}{6} - \frac{3}{2}x^2 + \frac{11x}{2} + 12 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{6} - \frac{3}{2}.2x + \frac{11}{2} = \frac{x^2}{2} - 3x + \frac{11}{2}$$

Slope $m = \frac{x^2}{2} - 3x + \frac{11}{2} \Rightarrow \frac{dm}{dx} = \frac{2x}{2} - 3 = x - 3$

Slope m increasing if $m > 0 \quad x - 3 > 0$

$x > 3$

The Slope increasing in$(3, \infty)$

17. Show that the function $\frac{ln(1+x)}{x}$, $\frac{x}{(1+x)ln(1+x)}$ decreasing in $(0, \infty)$.

i) **sol**: let $f(x) = \frac{ln(1+x)}{x}$

$$\Rightarrow f'(x) = \frac{x\frac{1}{1+x} - ln(1+x)}{x^2} = \frac{x - (1+x), ln(1+x)}{x^2(1+x)} < 0$$

$\therefore f(x)$ is a decreasing for $x \in (0, \infty)$

ii) $f(x) = \dfrac{x}{(1+x)\,ln(1+x)}$

Sol: $f(x) = \dfrac{x}{(1+x)\,ln(1+x)}$

$$\Rightarrow f'(x) = \frac{(1+x)\,l\,n(1+x) - x\left[(1+x).\frac{1}{1+x} + l\,n(1+x)\right]}{\left(1+x^2\right).\left[l\,n(1+x)\right]^2}$$

$$= \frac{(1+x).\ln(1+x) - x - x\,l\,n(1+x)}{(1+x)^2.\left[l\,n(1+x)\right]^2}$$

$$= \frac{l\,n(1+x) + x.l\,n(1+x) - x. - x.l\,n(1+x)}{(1+x)^2\,.lx(1+x)^2}$$

$$= \frac{-\left[x - l\,n(1+x)\right]}{(1+x)^2\left[l\,n(1+x)^2\right]} < 0$$

$\therefore f(x)$ is decreasing for $x \in (0, \infty)$