RATE OF CHANGE

Let y = f(x) be defined on an interval (a,b). let δy be change in y corresponding to a change δx in x.

Then , $\frac{\Delta y}{\Delta x}$ is called average rate of change of y. if $\underset{\delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$ exists finitely, the this limit (i.e., $\frac{dy}{dx}$) is called rate of change of y with respect to x .

Note : $\left(\frac{dy}{dx}\right)_{at\ x=c}$ represents the rate of change of y with respect to x at x=c.

Rectilinear Motion

Let a particle start at a point *O* on a line *L* and move along the line. After a time of *t* units, let the particle be at *P* and *OP*=*s*. Since *s* is dependent on time *t*, we write *s*=*s*(*t*). *s* is called the *displacement* of the particle during time *t*.

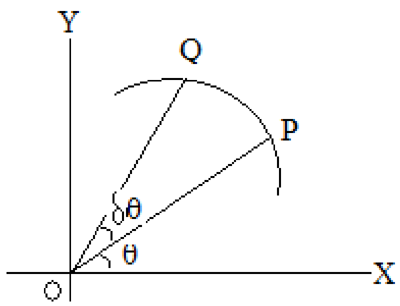
The rate of change of displacement s is called the velocity of the particle and it is denoted by v=$\frac{ds}{dt}$.

The rate of change of velocity is called acceleration. It is denoted by *a*.

$$\therefore a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

Angular Velocity

At time t, let P be the position of a moving point, Q be the position of the point after an interval δt. Let $\underline{|xop} = \theta$ and $\underline{|POQ} = \delta\theta$



The angular velocity of P at O is $\underset{\delta t \to 0}{Lt} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt}$ rad/sec.

Angular velocity w$\frac{d\theta}{dt}$

The angular acceleration of P at O is $\frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$

SAQ'S

1. If y = x⁴ then find the average rate of change of y between x = 2 and x = 4.

Sol. $y = x^4 \Rightarrow \frac{dy}{dt} = 4x^3$.

$$\left(\frac{dy}{dt}\right)_{x=2} = 32$$

$$\left(\frac{dy}{dt}\right)_{x=4} = 256$$

Average rate of change = $\frac{256+32}{2} = 144$.

2. The volume of a cube is increasing at the rate of 8 cm³/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

Sol. Suppose 'a' is the edge of the cube and v be the volume of the cube.

$v = a^3$ …(1)

given $\frac{dv}{dt} = 8cm^3/sec$

a = 12 cm

Surface area of cube S = $6a^2$

$\frac{ds}{dt} = 12a\frac{da}{dt}$ …(2)

From (1), $\frac{dv}{dt} = 3a^2\frac{da}{dt}$

$$8 = 3(144)\frac{da}{dt}$$

$$\frac{da}{dt} = \frac{8}{3(144)}cm/s$$

$$\frac{ds}{dt} = 12a\frac{da}{dt}$$

$$= 12(12)\frac{8}{3(144)} = 144\times\frac{8}{3(144)} = \frac{8}{3}cm^2/s$$

3. A stone is dropped into a quit lake and ripples move in circles at the speed of 5 cm/s. At the instant when the radius of circular ripple is 8 cm, how fast is the enclosed area increases?

Sol. Suppose r is the value of the outer ripple and A be its area.

Area of circle A = πr^2

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Given $r = 8, \frac{dr}{dt} = 5$ $\frac{dA}{dt} = 2\pi(8)(5) = 80\pi\, cm^2/s$

4. A balloon which always remains spherical on inflation is being inflated by pumping on 900 cubic centimeters of gas per second. Find the rate at which the radius of balloon increases when the radius in 15 cm.

Sol. $\frac{dv}{dt} = 900\,\text{c.c.}/\text{sec}$

r = 15 cm

Volume of the sphere $v = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} \Rightarrow 900 = 4\pi(15)^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{900}{4\times 225\pi} = \frac{dr}{dt} \Rightarrow \frac{900}{900\pi} = \frac{dr}{dt}$$

$$\frac{1}{\pi} = \frac{dr}{dt} \qquad \therefore \frac{dr}{dt} = \frac{1}{\pi}\,\text{cm}/\text{s}$$

5. Suppose we have a rectangular aquarium with dimensions of length 8 m, width 4 m and height 3 m. Suppose we are filling the tank with water at the rate of 0.4 m^3/sec. How fast is the height of water changing when the water level is 2.5 m?

Sol. Length of aquarium l = 8 m

Width of aquarium b = 4 m

Height of aquarium h = 3

$\frac{dv}{dt} = 0.4\,m^3/s$

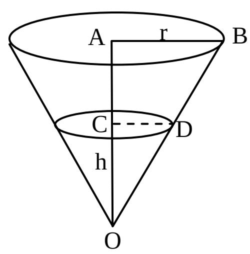
$v = lbh$

$$\Rightarrow \frac{dv}{dt} = lb\frac{dh}{dt}$$

$$\Rightarrow 0.4 = 8X4X\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{80}$$

6. A container is in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of 2m^3/minute, how fast is the height of water changing when the level is 4 m?



Sol. h = 8 m = OC

r = 6 m = AB

$\frac{dv}{dt} = 2\,m^3$/minute

ΔOAB and OCD are similar angle then

$$\frac{CD}{AB} = \frac{OC}{OA}$$

$$\frac{r}{6} = \frac{h}{8} \Rightarrow r = h\frac{3}{4}$$

Volume of cone $v = \frac{1}{3}\pi r^2 h$

$v = \frac{1}{3}\pi h^2 \frac{9}{16} h$

$v = \frac{3}{16}\pi h^3$

$\frac{dv}{dt} = \frac{3}{16}\pi 3h^2 \frac{dh}{dt} (\because h = 16)$

$2 = \frac{3}{16}\pi 3(16)\frac{dh}{dt} \Rightarrow \frac{2}{9\pi} = \frac{dh}{dt}$

7. The total revenue in rupees received from the sale of x units of a produce is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when x = 7.

Sol. Let m denotes the margin revenue. Then

$M = \frac{dR}{dx}$

Similar $R(x) = 13x^2 + 26x + 15$

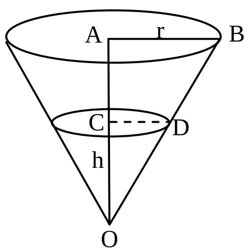
$\therefore$ m = 26x + 26

The marginal revenue at x = 7

$(M)_{x=7} = 26(7) + 26 = 208.$

8. A container in the shape of an inverted cone has height 12 cm and radius 6 cm at the top. If it is filled with water at the rate of 12 cm^3/sec., what is the rate of change in the height of water level when the tank is filled 8 cm?

Sol. Let OC be height to water level at t sec. The triangles OAB and OCD are similar triangles.



$\therefore \frac{CD}{AB} = \frac{OC}{OA}$

Let OC = h and CD = r.

Given that AB = 6 cm, OA = 12 cm

$\frac{r}{6} = \frac{h}{12} \Rightarrow r = \frac{h}{12}$ ...(1)

Volume of the cone V is given by

$V = \frac{\pi r^2 h}{3}$ …(2)

Using (1), we have $V = \frac{\pi h^3}{12}$ …(3)

Differentiating (3) w.r.t. t, we get

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

Hence, $\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$

When h = 8 cm, the rate of rise of the water level (height) is $\left(\frac{dh}{dt}\right)_{h=8}$

i.e. $\left(\frac{1}{\pi}\right)\frac{4}{8^2}(12) = \frac{3}{4\pi}\,cm/s$

Hence, the rate of change of water level is $\frac{3}{4\pi}\,cm/s$ when the water level of the tank is 8 cm.

9. **At time t, the distance s of a particle moving in a straight line given by $s = -4t^2 + 2t$. Find the velocity and acceleration when $t = \frac{1}{2}$ seconds.**

Sol: displacement $s = -4t^2 + 2t$

Velocity $v = \frac{ds}{dt} = \frac{d}{dt}\left(-4t^2 + 2t\right) = -8t + 2$

$\therefore$ Velocity at $t = \frac{1}{2}$ is $v = \left(\frac{ds}{dt}\right)_{t=\frac{1}{2}} = -8\left(\frac{1}{2}\right) + 2 = -4 + 2 = -2$ units/sec.

Acc. $a = \frac{dv}{dt} = \frac{d}{dt}(-8t + 2) = -8$

Acceleration $= a = \left(\frac{dv}{dt}\right)_{t=\frac{1}{2}} = -8$ units.sec^2.

10 .**The displacement s of a particle travelling in a straight line in t seconds is given by $s = 45t + 11t^2 - t^3$. Find the time when the particle comes to rest.**

Sol: displacement $s = 45t + 11t^2 - t^3$

Velocity $v = \frac{ds}{dt} = 45 + 22t - 3t^2$

particle becomes to rest if v= 0.

$\Rightarrow 45 + 22t - 3t^3 = 0 \quad \Rightarrow 3t^2 - 22t - 45 = 0 \Rightarrow 3t^2 - 27t + 5t - 45 = 0$

$\Rightarrow (3t + 5)(t - 9) = 0 \quad \therefore t = 9$ or $t = -\frac{5}{3}$

$\therefore$ t = 9

$\therefore$ The particle becomes to rest at t = 9 seconds.

11. **The distance – time formula for the motion of a particle along a straight line is $s = t^3 - 9t^2 + 24t - 18$. Find when and where the velocity is zero.**

Sol: Given $s = t^3 - 9t^2 + 24t - 18$

$$v = \frac{ds}{dt} = 3t^2 - 18t + 24$$

$$v = 0 \Rightarrow 3\left(t^2 - 6t + 8\right) = 0$$

$$\therefore (t-2)(t-4) = 0$$

$$\therefore t = 2 \text{ or } 4$$

The velocity is zero after 2 and 4 seconds.

Case (i):

$$t = 2$$

$$s = t^3 - 9t^2 + 24t - 18$$

$$= 8 - 36 + 48 - 18 = 56 - 54 = 2$$

Case (ii):

$$t = 4;\ s = t^3 - 9t^2 + 24t - 18$$

$$= 64 - 144 + 96 - 18$$

$$= 160 - 162 = -2$$

The particle is at a distance of 2 units from the starting point ‘O’ on either side.

12 . **A particle moving along a straight line has the relation $s = t^2 + 2t + 3$ connecting the distance s described by the particle in time t. Find the velocity and acceleration of the particle at time t = 3 seconds.**

Try your self.

13 .**A point P is moving on the curve $y = 2x^2$. The x co-ordinate of P is increasing at the rate of 4 units per second. Find the rate at which y co-ordinate is increasing when the point is (2, 8).**

Sol: Equation of the curve $y = 2x^2$

Diff .w.r.t.t, $\frac{dy}{dt} = 4x.\frac{dx}{dt}$

Given $x = 2$ and $\frac{dx}{dt} = 4.$

$$\frac{dy}{dt} = 4(2).4 = 32$$

y co-ordinate is increasing at the rate of 32 units/sec.

14. The radius of a circular plate is increasing in length at 0.7 cm/sec. What is the rate of increase in circumference.

Sol: Let r be the radius and P be the area of the circular plate. Then $P = 2\pi r$

Diff w.r.t.t,

Given $\frac{\text{dr}}{\text{dt}} = 0.7$

$P = 2\pi r$

$\frac{dp}{dt} = 2\pi \frac{dr}{dt} = 2\pi(0.7) = 1.4\pi$

14. **The total cost c(x) in rupees associated with the production of x units of an item is given by** $c(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$ **find the marginal cost when 17 units are produced.**

Sol. Let m be the marginal cost.

Then $m = \frac{dc}{dt} = 0.007(3x^2) - 0.003(x) + 15$

Marginal cost at x =17 is $\left(\frac{dc}{dt}\right)_{at\ x=17} = 0.007(867) - 0.003(34) + 15 = 20.967$