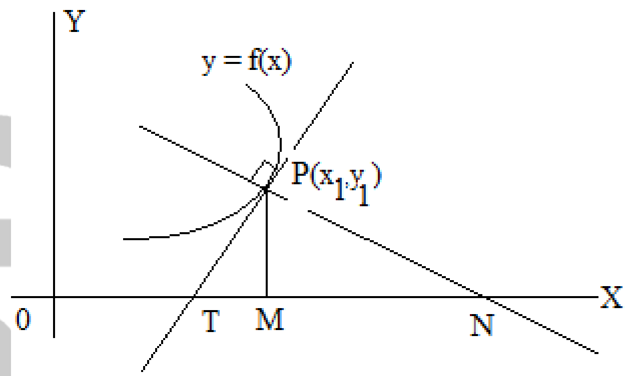
LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUB NORMAL

Definition :

Let $y = f(x)$ be a differentiable curve and P be a point on the curve.

Let the tangent and normal at P to the curve meet the x - axis in T and N respectively. Let M be the projection of P on the x - axis. Then

(i) PT is called the length of the tangent, (ii) PN is called the length of the normal, (iii) TM is called the length of the subtangent, (iv) MN is called and length of the subnormal at the point P.



Let $P(x_1, y_1)$ be a point on the curve $y = f(x)$. Then

(i) the length of the tangent to the curve at P is $\left|\frac{y_1}{m}\sqrt{1+m^2}\right|$.

(ii) the length of the normal to the curve at P is $\left|y_1\sqrt{1+m^2}\right|$.

(iii) the length of the subtangent to the curve at P is $\left|\frac{y_1}{m}\right|$.

(iv) the length of the subnormal to the curve at P is $|y_1 m|$ where $m = \left(\frac{dy}{dx}\right)_P$.

Exercise

1. Find the length of sub-tangent and sub normal at a point of the curve $y = b.\sin\frac{x}{a}$.

Sol: Equation of the curve is $y = b.\sin\frac{x}{a} \Rightarrow \frac{dy}{dx} = b.\cos\frac{x}{a}.\frac{1}{a} = \frac{b}{a}.\cos\frac{x}{a}$ =m

Let P(x,y) be a point on the curve.

Length of the sub-tangent = $\left|\frac{y_1}{m}\right| = \frac{b.\sin\frac{x}{a}}{\frac{b}{a}.\cos\frac{x}{a}} = \left|a.\tan\frac{x}{a}\right|$

Length of the sub-normal = $|y_1 m| = \left|b.\sin\frac{x}{a}.\left(\frac{b}{a}\cos\frac{x}{a}\right)\right| = \left|\frac{b^2}{2a}.\sin\frac{2x}{a}\right|$.

2. Show that the length of sub-normal at any point on the curve xy = a^2 varies as the cube of the ordinate of the point.

Sol: Equation of the curve is xy = a^2.

$\Rightarrow y = \frac{a^2}{x} \Rightarrow \quad \frac{dy}{dx} = \frac{-a^2}{x^2}$ = m

Let P(x,y) be a point on the curve.

Length of the sub-normal = $|y_1 m| = \left|y_1\left(\frac{(-a)^2}{x_1^2}\right)\right| = \left|-a^2 y_1 \frac{y_1^2}{a^4}\right|$

$= \frac{y_1^3}{a^2} \propto y_1^3 =$ cube of the ordinate.

3. Show that at any point (x, y) on the curve $y = be^{x/3}$, the length of the sub-tangent is constant and the length of the sub-normal is $\frac{y^2}{a}$.

Sol: Let P(x,y) be a point on the curve.

Equation of the curve is $y = be^{x/3} \Rightarrow \frac{dy}{dx} = b.e^{x/3}.\frac{1}{a} = \frac{y}{a}$ =m

Length of the sub-tangent $\left|\frac{y_1}{m}\right| = \frac{y}{\left(\frac{y}{a}\right)}$ = a = constant

Length of the sub-normal $|y_1 m|$=y.y/a=y^2/a

4. Find the value of k so that the length of the sub-normal at any point on the curve $x.y^k = a^{k+1}$ is constant. ?

Sol: Let P(x_1,y_1) be a point on the curve.

Equation of the curve is $x.y^k = a^{k+1}$.

Differentiating w.r.to x

$$x.k.y^{k-1}\frac{dy}{dx} + y^k.1 = 0$$

$$\Rightarrow k.xy^{k-1}.\frac{dy}{dx} = -y^k$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^k}{k.x.y^{k-1}} = -\frac{y}{kx}$$

Length of the sub-normal = $|y_1 m|$ $= \left|y_1 \frac{-(y_1)}{kx_1}\right| = \frac{y_1^2}{kx_1}$

$$= \frac{y_1^2}{k}.\frac{y_1^k}{a^{k+1}} = \frac{y_1^{k+2}}{k.a^{k+1}}$$

Length of the sub-normal is constant at any point on the curve is independent of x_1

and $\frac{y_1^{k+2}}{k.a^{k+1}}$ is independent of $y_1 \Rightarrow k+2=0 \Rightarrow k=-2$

5. At any point t on the curve x = a (t + sin t), y = a(1 – cos t). Find the lengths of the tangent, normal, sub-tangent and sub-normal.

Sol: Equation of the curve is x = a (t + sin t), y = a(1 – cos t)

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{a\sin t}{a(1+\cos t)} \qquad =\frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}=\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}=m$$

Length of the tangent $\left|\frac{y_1}{m}\sqrt{1+m^2}\right| \quad =\left|a(1-\cos t)\sqrt{1+\cot^2\frac{t}{2}}\right|$

$$=\left|2a.\sin^2\frac{t}{2}.\operatorname{cosec}\frac{t}{2}\right| \;=\left|2a.\sin^2\frac{t}{2}.\frac{1}{\sin\frac{t}{2}}\right| \;=\left|2a\sin\frac{t}{2}\right|$$

Length of the normal $\left|y_1\sqrt{1+m^2}\right|=\left|a(1-\cos t)\sqrt{1+\tan^2\frac{t}{2}}\right|$

$$=\left|a\left(a\sin^2\frac{t}{2}\right).\sec\frac{t}{2}\right| \;=\left|2a.\sin\frac{t}{2}.\frac{\sin\frac{t}{2}}{\cot\frac{t}{2}}\right| \;=\left|2a.\sin\frac{t}{2}.\tan\frac{t}{2}\right|$$

Length of the sub-tangent $\left|\frac{y_1}{m}\right|=\frac{a(1-\cos t)}{\frac{\sin t/2}{\cos t/2}} \;=\left|a.2\sin^2 t/2.\frac{\cos t/2}{\sin t/2}\right|$

$$=\left|a.(2\sin t/2.\cos t/2)\right|=\left|a.\sin t\right|$$

Length of the sub-normal $\left|y_1 m\right|=a(1-\cos t)\frac{\sin t/2}{\cos t/2}$

$$=\left|2a\sin^2 t/2.\tan t/2\right|=\left|2a\sin^2 t/2.\tan t/2\right|$$

6. Find the length of normal and sub-normal at a point on the curve $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$.

Sol: Equation of the curve is $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) \quad = a.\cosh\left(\frac{x}{a}\right)$

$$\Rightarrow \frac{dy}{dx} = a.\sinh\left(\frac{x}{a}\right)\frac{1}{a} = \sinh\frac{x}{a} \text{ =slope of tantent at any point =m}$$

Length of the normal $\left|y_1\sqrt{1+m^2}\right| \quad = \left|a.\cosh\frac{x}{a}\right|\sqrt{1+\sinh^2\frac{x}{a}}$

$$= a.\cosh\frac{x}{a}.\cosh\frac{x}{a} = a.\cosh^2\frac{x}{a}$$

Length of the sub-normal $|y_1 m| \quad = \left|a.\cosh\left(\frac{x}{a}\right).\sinh\left(\frac{x}{a}\right)\right|$

$$= \left|\frac{a}{2}\left(2\sinh\frac{x}{a}.\cosh\frac{x}{a}\right)\right| \quad = \left|\frac{a}{2}.\sinh\frac{2x}{a}\right|$$

7. Find the length of the sub-tangent and sub-normal of a point t o the curve $x = a(\cos t + t\sin t)\, y = a(\sin t - t\cos t)$.

Sol: Equation of the curve are $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Length of the sub-tangent $= \left|\frac{y_1}{f'(x_1)}\right| = \left|\frac{a(\sin t - t\cos t)}{\tan t}\right|$

$$= \left|a\cot t(\sin t - t\cos t)\right|$$

Length of the sub-normal $= \left|y_1.f'(x_1)\right|$

$$= \left|a(\sin t - t\cos t)\tan t\right| = \left|a\tan t(\sin t - t\cos t)\right|$$