TANGENTS AND NORMALS

TOPICS:

1. Geometrical interpretation of the derivative Equations of tangents and normals

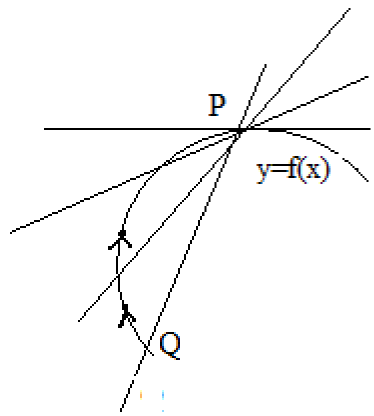
2.length of tangent, normal,sub-tangent and sub- normal.

3.Angle between two curves and orthogonality

TANGENT TO A CURVE

Definition :

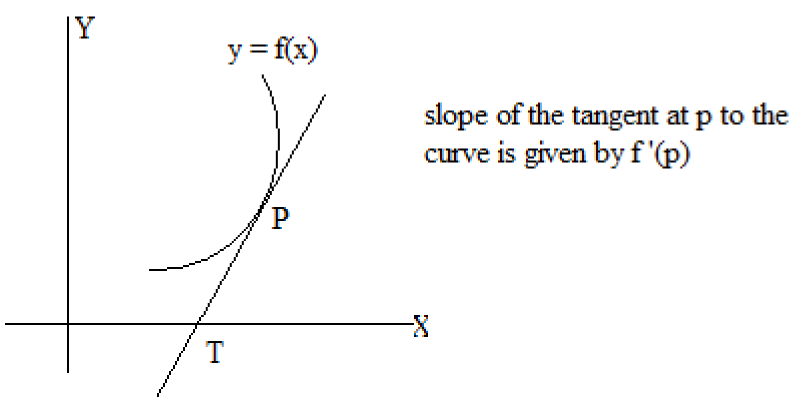
Let y = f(x) be a curve and P be a point on the curve. If Q is a point on the curve other than P, then PQ is called a secant line of the curve. If the secant line $\overleftrightarrow{PQ}$ approaches the same limiting position as Q approaches P along the curve from either side then the limiting position is called the tangent line to the curve at the point P. The point P is called the point of contact of the tangent line to the curve.



The tangent at a point to a curve, if it exists, is unique. Therefore, there exists at most one tangent at a point to a curve.

GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let P be a point on the curve y = f(x). Then the slope of the tangent to the curve at P is equal to $\left(\frac{dy}{dx}\right)_P$ i.e., $\left(f'(x)\right)_P$.



Gradent The slope of the tangent at a point to a curve is called the gradient of the curve at that point.

The gradient of the curve y = f(x) at P is $\left(\frac{dy}{dx}\right)_P$.

Note 1: If $\left(\frac{dy}{dx}\right)_P$ =o then the tangent to the curve at P is parallel to the x - axis. The tangent, in this case, is called a horizontal tangent.

Note 2: If $\left(\frac{dy}{dx}\right)_P = +\infty or -\infty$ i.e., if $\left(\frac{dx}{dy}\right)_P = 0$ then the tangent to the curve at P is perpendicular to x - axis. The tangent, in this case, is called a vertical tangent.

Note 3: If $\left(\frac{dy}{dx}\right)_P$ does not exist then there exists no tangent to the curve at P.

EQUATION OF TANGENT

The equation of the tangent at the point P(x_1, y_1) to the curve y = f(x) is $y - y_1 = m(x - x_1)$ where $m = \left(\frac{dy}{dx}\right)_P$

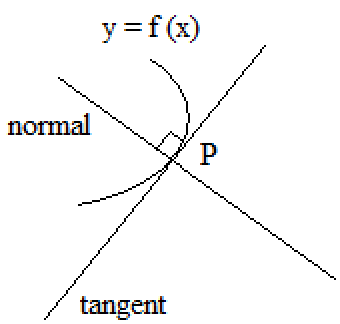
Note:

1. x - intercept of the tangent = $x_1 - \frac{y_1}{m} = x_1 - y_1 m$

   y - intercept of the tangent = . $y_1 - mx_1 = y_1 - x_1 m$

NORMAL TO A CURVE

Let P be a point in the curve y = f(x). The line passing through P and perpendicular to the tangent at P to the curve is called the normal to the curve at P.



The slope of the normal to the curve y = f(x) at P is $m' = -\frac{1}{m} = -\left(\frac{dx}{dy}\right)_P$ where $m = \left(\frac{dy}{dx}\right)_P \neq 0$

The equation of the normal at $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = \frac{-1}{m}(x - x_1) \text{ where } m = \left(\frac{dy}{dx}\right)_p \quad \text{i.e.,. } y - y_1 = -\left(\frac{dx}{dy}\right)_P (x - x_1)$$

GEOMETRICAL APPLICATIONS EXERCISE

I **1. Find the slope of the tangent to the curve $y = 3x^4 – 4x$ at $x = 4$.**

Sol: Equation of the curve is $y = 3x^4 - 4x$

Diff. w.r.t.x => $\frac{dy}{dx} = 12x^3 - 4$

At x = 4, slope of the tangent = $\left(\frac{dy}{dx}\right)_{x=4} = 12(4)^3 - 4$

$= 12 \times 64 – 4 \quad = 786 – 4 = 764$

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$ at $x \neq 2$ and $x = 10$.

Sol: ans: $-\frac{1}{(10-2)^2} = -\frac{1}{64}$

3. Find the slope of the tangent to the curve, $y = x^3 – x + 1$ at the point whose x co-ordinate is 2.

Sol: ans: 11

4. Find the slope of the tangent to the curve, $y = x^2 - 3x + 2$ at the point whose x co-ordinates is 3.

Sol: Equation of the curve is $y = x^3 - 3x + 2$

Diff. w.r.t.x , $\frac{dy}{dx} = 3x^2 - 3$

At x = 3, slope of the tangent = $\left(\frac{dy}{dx}\right)_{x=3} = 3(3)^2 - 3$

$= 27 - 3 = 24$

5. **Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.**

Sol: $x = a\cos^3\theta$

Diff wrt θ, $\frac{dx}{d\theta} = a(3\cos^2\theta)(-\sin\theta) = -3a\cos^2\theta \cdot \sin\theta$

$y = a\sin^3\theta$

Diff wrt θ, $\frac{dy}{d\theta} = a(3\sin^2\theta)\cos\theta = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^{2}\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

At $\theta = \frac{\pi}{4}$, slope of the tangent = $\tan\frac{\pi}{4} = -1$

Slope of the normal = $-\frac{1}{m} = 1$.

6. **Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.**

Ans: $\frac{-a}{2b}$

7. **Find the points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x - axis.**

Sol: Equation of the curve is $y = x^3 - 3x^2 - 9x+7$

Diff.wrt x, $\frac{dy}{dx} = 3x^2 - 6x - x$

Since tangent is parallel to x – axis, Slope of the tangent = 0

$\Rightarrow \left(\frac{dy}{dx}\right) = 0$

⇨ $3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$

⇨ $(x-3)(x+1) = 0 \Rightarrow x = 3$ or -1

$y = x^3 - 3x^2 - 9x + 7$

$x = 3 \Rightarrow y = 27 - 27 - 27 + 7 = -20$

$x = -1, y = -1 - 3 + 9 + 7 = 12$

The points are (3, - 20), (-1, 12).

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent us parallel to the chord joining the points (2, 0) and (4, 4).

Sol: Equation of the curve is $y = (x-2)^2$

Diff wrt x, $\frac{dy}{dx} = 2(x - 2)$

Slope of the chord joining A (2, 0) and B (4,4) = $\frac{4-0}{4-2} = \frac{4}{2} = 2$.

since tangent us parallel to the chord, slope of tangent = slope of chord AB

⇨ $2(x - 2) = 2 \Rightarrow x - 2 = 1$

⇨ $x = 3$

⇨ $y = (x-2)^2 = (3-2)^2 = 1$

The required point is p (3, 1).

9. Find the point on the curvey $= x^2 - 11x + 5$ at which the tangent is $y = x - 11$.

Sol: Equation if the curve is $y = x^2 - 11x + 5$

⇨ $\frac{dy}{dx} = 3x^2 - 11$

Since tangent is y = x -1 => slope =1

⇨ $3x^2 - 11 = 1 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

y = x – 11

$x = 2 \Rightarrow y = 2 - 11 = -9$

The point on the curve is p (2,-9).

10. Find the equations of all lines having slope O which are tangent to the curve

$\mathbf{y = \frac{1}{x^2 - 2x + 3}}$.

Sol: ans:2y – 1 = 0

1. Find the equation of tangent and normal to the following curves at the points indicated against.

i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5).

Sol: $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

⇨ $\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

At x = (0,5)

Slope of the tangent $m = \left(\frac{dy}{dx}\right)_{at(0,5)} = 0 - 0 + 0 - 10 = 0 - 0 + 0 - 10 = -10$

Equation of the tangent is y – 5 = -10(x – 0)

= - 10x

10x + y– 5 =0

Slope of the normal = $-\frac{1}{m} = \frac{1}{10}$

Equation of the normal is $y - 5 = \frac{1}{10}(x-0)$

$10y - 50 = x \Rightarrow x - 10y + 50 = 0$

ii) $y = x^3$ at (1, 1). Ans: 3x – y – 2 = 0, x + 3y – 4 = 0

iii) **$y = x^2$ at (o, o). ans: x = 0, y=0**

iv) **x = Cos t, y = Sin t, at t = $\frac{\pi}{4}$.**

Sol: x = Cos t, y = Sin t

=> $\frac{dx}{dt} = -\sin t, \frac{dt}{dx} = \cos t$

Therefore $\frac{dy}{dx} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cos t}{-\sin t} = -\cot t$

$\Rightarrow m = \left(\frac{dy}{dx}\right)_{at\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$ and $x = -\cot\frac{\pi}{4} = \frac{1}{\sqrt{2}}, y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Point on the curve $p\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Equation of the tangent is $y - \frac{1}{\sqrt{2}} = -\left(x - \frac{1}{\sqrt{2}}\right) = -x + \frac{1}{\sqrt{2}}$

$\Rightarrow x + y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \Rightarrow x + y = \sqrt{2}$

Slope of the normal $= -\frac{1}{m} = \frac{-1}{-1} = 1$

Equation of the normal is $y - \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}} \Rightarrow x - y = 0$

v) **$y = x^2 - 4x + 2$ at $P(4, 2)$**

Sol: Equation of the curve is $y = x^2 - 4x + 2 \Rightarrow \frac{dy}{dx} = 2x - 4$

At $P(4,2)$, slope of the tangent $= 2.4 - 4 = 8 - 4 = 4$

Equation of the tangent at P is

$y - 2 = 4(x - 4) = 4x - 16 \Rightarrow 4x - y - 14 = 0$

Slope of the normal $= -\frac{1}{m} = -\frac{1}{4}$

Equation of the normal at P is $y - 2 = -\frac{1}{4}(x - 4)$

$\Rightarrow 4y - 8 = -x + 4 \Rightarrow x + 4y - 12 = 0$

vi) $y = \frac{1}{1 + x^2}$ at $(0, 1)$

2. Find the equations of tangent and normal to the curve xy = 10 at (2, 5).

Sol: Equation of the curve is xy = 10

$$y = \frac{10}{x}$$

$$\frac{dy}{dx} = \frac{-10}{x^2}$$

$$m = \left(\frac{dy}{dx}\right)_{P(2,5)} = \frac{-10}{4} = \frac{-5}{2}$$

Equation of the tangent is $y - 5 = -\frac{5}{2}(x - 2)$

$\Rightarrow 2y - 10 = -5x + 10 \Rightarrow 5x + 2y - 20 = 0$

Slope of normal = 2/5

Equation of the normal is $y - 5 = \frac{2}{5}(x - 2)$

$\Rightarrow 5y - 25 = 2x - 4$

$\Rightarrow 2x - 5y + 21 = 0$

3. Find the equation of tangent and normal to the curve $y = x^3 + 4x^2$ at $(-1, 3)$.

$5x + y + 2 = 0;\ x - 5y + 16 = 0$

4. If the slope of the tangent to the curve $x^2 - 2xy + 4y = 0$ at a point on it is $-\frac{3}{2}$ then find the tangent and normal at that point.

Sol: Equation of the curve is $x^2 - 2xy + 4y = 0$ --- (1)

Diff . w.r.to x, $2x - 2x.\frac{dy}{dx} - 2y + 4\frac{dy}{dx} = 0$

$2(x-y)=2(x-2)\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{2(x-y)}{2(x-2)}=\frac{x-y}{x-2}$

But given $\frac{dy}{dx}=-\frac{3}{2}\Rightarrow \quad \therefore \frac{x-y}{x-2}=-\frac{3}{2}$

$\Rightarrow 2x-2y=-3x+6\Rightarrow 5x-2y=6$

$\Rightarrow 2y=5x-6$ ---(2)

Solving (1) and (2),

$x^2-x(5x-6)+2(5x-6)=0$

$\Rightarrow x^2-5x^2+6x+10x-12=0 \Rightarrow -4x^2-16x-12=0$

$\Rightarrow -4\left(x^2-4x+3\right)=0 \Rightarrow x^2-4x+3=0$

$\Rightarrow (x-1)=0$ or $x-3=0$

$\therefore x=1$ or $x=3$

Case (i) : $x = 1$ from (1),

$1-2y+4y=0 \Rightarrow 2y=-1\Rightarrow y=-\frac{1}{2}$

Therefore point is $P\left(1,-\frac{1}{2}\right)$

Equation of the tangent is $y+\frac{1}{2}=-\frac{3}{2}(x-1)$

$\Rightarrow \frac{2y+1}{2}=\frac{-3(x-1)}{2} \Rightarrow 2y+1=-3x+3 \Rightarrow 3x+2y-2=0$

Slope of normal is 2/3

Equation of the normal is $y+\frac{1}{2}=\frac{2}{3}(x-1)$

$\Rightarrow \frac{2y+1}{2}=\frac{2}{3}(x-1)\Rightarrow 6y+3=4x-4 \Rightarrow 4x-6y-7=0$

Case (ii) : $x = 3$

Substituting in (1), $9 - 6y + 4y = 0 \Rightarrow 2y = 9 \Rightarrow y = \frac{9}{2}$

$\therefore$ The point is $\left(3, \frac{9}{2}\right)$

Equation of the tangent is $y - \frac{9}{2} = -\frac{3}{2}(x - 3)$

$\Rightarrow \frac{2y - 9}{2} = \frac{-3(x - 3)}{2} \Rightarrow 2y - 9 = -3x + 9 \Rightarrow 3x + 2y - 18 = 0$

Equation of the normal is $y - \frac{9}{2} = \frac{2}{3}(x - 3)$

$\frac{2y - 9}{2} = \frac{2(x - 3)}{3} \Rightarrow 6y - 27 = 4x - 12 \Rightarrow 4x - 6y + 15 = 0$

5. If the slope of the tangent to the curve $y = x \log x$ at the point on it is $\frac{3}{2}$, then find the equations of tangents and normal at that point.

$3x - 2y - 2\sqrt{e} = 0, 4x + 6y - 7\sqrt{e} = 0$

6. Find the tangent and normal to the curves $y = 2e^{-x/3}$ at the point where the curve meets the y – axis. ?

Sol: Equation of the curve is $y = 2e^{-x/3}$ ---(1)

Equation of y-axis is $x = 0$

$\Rightarrow y = 2.e^{0} = 2.1 = 2$

The point of intersection of the curve and the y-axis is P(0, 2)

Diff (1) w.r.t. x, $\Rightarrow \frac{dy}{dx} = 2\left(-\frac{1}{3}\right).e^{-x/3}$

$$m = \left(\frac{dy}{dx}\right)_{at(0,2)} = \frac{-2}{3}e^{0} = \frac{-2}{3}$$

Equation of the tangent at P is

$y - 2 = -\frac{2}{3}(x - 0) \Rightarrow 3y - 6 = -2x \Rightarrow 2x + 3y - 6 = 0$

Slope of normal = 3/2

Equation of the normal is $y - 2 = \frac{3}{2}(x - 0)$

$\Rightarrow 2y - 4 = 3x ; 3x - 2y + 4 = 0$

III. 1. Show that the tangent at $P(x_1, y_1)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $yy_1^{-1/2} + xx_1^{-1/2} = a^{1/2}$.

Sol: Equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$ -----(1)

let $P(x_1, y_1)$ be a point on the curve. Then $\sqrt{x_1} + \sqrt{y_1} = \sqrt{a}$ ------(21)

Differente (1) w.r.to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} . \frac{dy}{dx} = 0 \quad \Rightarrow \frac{1}{2\sqrt{y}} . \frac{dy}{dx} = -\frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\frac{y^{1/2}}{x^{1/2}}$$

Slope of the tangent at $P(x_1, y_1) = -\frac{(y_1)^{1/2}}{(x_1)^{1/2}}$

Equation of the tangent at P is $y - y_1 = \frac{-y_1^{1/2}}{x_1^{1/2}}(x - x_1)$

$$\frac{y}{y_1^{1/2}} - \frac{y_1}{y_1^{1/2}} = -\frac{x}{x_1^{1/2}} + \frac{x_1}{x_1^{1/2}} \quad = x_1^{1/2} + y_1^{1/2} \quad = a^{1/2}$$

Equation of the tangent at P is

$$y.y_1^{-1/2} + x.x_2^{-1/2} = a^{1/2}$$

2. At what points on the curve $x^2 - y^2 = 2$, the slope of the tangents are equal to 2.

Sol: Equation of the curve is $x^2 - y^2 = 2$…..(1)

Differentiating w.r.to x ,$2x - 2y. \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$ but slope of the tangent = $\frac{dy}{dx} = 2$

$\therefore \frac{y}{x} = 2 \rightarrow y = 2x$

Sustituting in (1), $4y^2 - y^2 = 2 \Rightarrow \quad 3y^2 = 2$

$y^2 = \frac{2}{3} \Rightarrow y = \pm\sqrt{\frac{2}{3}} \Rightarrow \quad x = 2y = \pm 2\sqrt{\frac{2}{3}}$

$\therefore$The required points are

P $\left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ and Q $\left(-2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$.

3. Show that the curves $x^2+y^2= 2$ and $3x^2 + x^2 = 4x$ have a common tangent at the point (1, 1).

Sol: Equation of the first curve is $x^2 + y^2 = 2$

Differentiating w, r, to x

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

At p (1, 1) slope of the tangent = - $\frac{-1}{1} = -1$

Equation of the second curve is $3x^2 + y^2 = 4y$.

Differentiating w. r. to x, $6x + 2y.\frac{dy}{dx} = 4 \Rightarrow 2y.\frac{dy}{dx} = 4 - 6x$

$\Rightarrow \frac{dy}{dx} = \frac{4-6x}{2y} = \frac{2y-3x}{y}$

At p(1, 1) slope of the tangent = $\frac{2-3}{1} = -\frac{1}{1} = -1$

The slope of the tangents to both the curves at (1, 1) are same and pass through the same point (1, 1)

$\therefore$The given curves have a common tangent p (1, 1)

4. At a point (x_1, y_1) on the curve $x^2 + y^2 = 3axy$, show that the tangent is $(x_1^2 - ay_1)x + (y_1^2 - ax_1)y = ax_1y_1$.

Sol: Equation of the curve is $x^3 + y^3 = 3axy$

Let P(x_1, y_1) be a point on the curve. Then $x_1^2 + y_1^2 = 3ax_1y_1$

Differentiate $x^3 + y^3 = 3axy$ w. r. to x, $3x^2 + 3y^2.\frac{dy}{dx} = 3a\left(x.\frac{dy}{dx} + y\right)$

$$x^2 + y^2\frac{dy}{dx} = a\left(x.\frac{dy}{dx} + y\right) = ax.\frac{dy}{dx} + ay$$

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = ay - x^2 \Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} = -\frac{(x^2 - ay)}{(y^2 - ax)}$$

Slope of the tangent $p(x_1, y_1) = -\frac{(x_1^2 - ay_1)}{(y_1^2 - ax_1)}$

Equation of the tangent at $p(x_1, y_1)$ is $y - y_1 = -\frac{(x_1^2 - ay_1)}{(y_1^2 - ax_1)}(x - x_1)$

$$\Rightarrow y(y_{1-ax_1}^2) - y(y_1^2 - ax_1) = -x(x_1^2 - ay_1) + x_1(x_1^2 - ay_1)$$

$$\Rightarrow x(x_1^2 - ay_1) + y(y_1^2 - ax_1) = x_1(x_1^2 - ay_1) + y_1(y_{1-ax_1}^2)$$

$$= x_1^3 - ax_1y_1 + y_1^3 - ax_1y_1 = x_{1+}^3 y_1^3 - 2ax_1y_1 = 3ax_1y_1 - 2ax_1y_1 = ax_1y_1$$

5. Show that the tangent at the point P (2, –2) on the curve y (1– x) = x makes intercepts of equal length on the coordinate axes and the normal at P passes through the origin.

Sol: Equation of the curve is $y(1 - x) = x$

$$y = \frac{x}{1 - x}$$

Differentiation w. r. to x, $\frac{dy}{dx} = \frac{(1-x).1 - x(-1)}{(1-x)^2}$

$$\Rightarrow \frac{1 - x + x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

Slope m= $\Rightarrow \left(\frac{dy}{dx}\right)_{atP} = \frac{1}{(1-2)^2} = 1$

Equation of the tangent at P is y + 2 = + (x-2) = x -2; x – y = 4

$$\frac{x}{4} - \frac{y}{4} = 1 \Rightarrow \frac{x}{4} + \frac{y}{(-4)} = 1$$

$\therefore a = 4, b = -4$

∴The tangent makes equal intercepts on the co-ordinates axes but they are in opposite in sign.

Slope of normal is -1

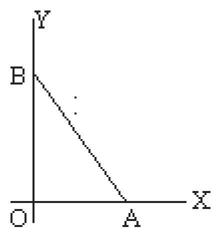
Equation of the normal at P is y + 2 = - (x-2) = - x +2

$\Rightarrow x + y = 0$

There is no constant term in the equation.

∴The normal at P (2, – 2) passes through the origin.

6. If the tangent at any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A, B show that the length AB is constant,



Sol: Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$

The parametric equations of the curve are $x = a\cos^3\theta, y = a\sin^3\theta$

$$\frac{dy}{dx} = \frac{a.3\sin^2\theta.\cos\theta}{a3\cos^2\theta(-\sin\theta)} = -\tan\theta$$

Equation of the tangent at $\left(a\cos^3\theta, a\sin^3\theta\right)$ **is**

$$y - a\sin^3\theta = -\tan\theta\left(x - a\cos^3\theta\right)$$

$$\frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a$$

X intercept = acosθ and y – intercept = a sinθ

$A = \left(a\cos\theta, 0\right)$ and $B = \left(0, a\sin\theta\right)$

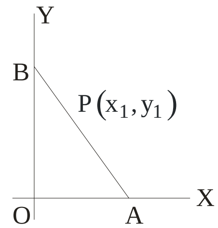
$now\ \ AB = \sqrt{(a\cos\theta)^2 + (a\sin\theta)^2}$

$= \sqrt{a^2(\cos^2\theta + \sin^2\theta)} = a$

Therefore , AB = a , constant.

7. If the tangent at a point P on the curve

$x^m y^n = a^{m+n}\ (mn \neq 0)$ meets the coordinate axes in A, B. show that AP : PB is constant.



Sol: Equation of the curve is $x^m.y^n = a^{m+n}$

Let $P(x_1, y_1)$ be a point on the curve. Then $x_1^m.y_1^n = a^{m+n}$

Differente given curve w. r. to x, $x^m.ny^{n-1}.\frac{dy}{dx} + y^n.mx^{m-1} = 0$

$nx^m.y^{n-1}\frac{dy}{dx} = -m.x^{m-1}.y^n \Rightarrow \frac{dy}{dx} = \frac{-m.x^{m-1}.y^n}{n.x^m y^{n-1}} = -\frac{my}{nx}$

Slope of the tangent at $P(x_1, y_1) = -\frac{my_1}{nx_1}$

Equation of the tangent at P is $y - y_1 = -\frac{my_1}{nx_1}(x - x_1)$

$\Rightarrow nx_1y - nx_1y_1 = -my_1x + mx_1y_1$

$\Rightarrow my_1x + nx_1y = mx_1y_1 + nx_1y_1 = (m+n)x_1y_1$

$\Rightarrow \frac{my_1}{(m+n)x_1y_1}.x + \frac{nx_1}{(m+n)x_1y_1}.y = 1$

$\Rightarrow \frac{x}{\frac{m+n}{m}.x_1} + \frac{y}{\frac{m+n}{n}.y_1} = 1$

$\Rightarrow OA = \frac{m+n}{m}.x_1 . OB = \frac{m+n}{n}$

Co-ordinates of A are $\left[\frac{m+n}{m}.x_1, 0\right]$ and B are $\left[0, \frac{m+n}{n}.y_1\right]$

The ratio in which P divides AB is

$$\frac{AP}{PB} = \frac{X - X_1}{X_1 - 0} = \frac{\frac{m+n}{m} - x_1}{x_1} = \frac{nx_1}{mx_1} = \frac{n}{m}$$

8. Show that the tangent at the point P(2, –2) on the curve y(1 – x) = x makes intercepts of equal length on the coordinate axes and the normal at P passes through the origin.

Sol. Equation of the curve is

$$y(1 - x) = x \Rightarrow y = \frac{x}{1-x}$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{(1-x)1 - x(-1)}{(1-x)^2}$$

$$= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\text{At } P(2,-2), f'(x_1) = \frac{1}{(1-2)^2} = 1 = m$$

Equation of the tangent at P is

$$y + 2 = 1(x-2) = x - 2;\ x - y = 4$$

$$\frac{x}{4} - \frac{y}{4} = 1 \Rightarrow \frac{x}{4} + \frac{y}{(-4)} = 1$$

$$\therefore a = 4,\ b = -4$$

∴ The tangent makes equal intercepts on the coordinate axes but they are in opposite in sign. Equation of the normal at P is

$$y - y_1 = \frac{1}{f'(x_1)}(x - x_1)$$

$$y + 2 = -(x-2) = -x + 2$$

$$x + y = 0$$

There is no constant term in the equation.

∴ The normal at P(2, –2) passes through the origin.