CHAPTER 10

ERRORS AND APPROXIMATIONS

TOPICS:

1.ERRORS

2.APPROXIMATIONS.

INFINITESIMALS

Let x be a finite variable quantity and be a minute change in x. Such a quanitity , which is very small when compared to x and which is smaller than any pre-assigned small quantity, is called an infinitesimal or an infinitesimal of first order. If δx is an infinitesimal then $(\delta x)^2$, $(\delta x)^3$, ... are called infinitesimals respectively of 2nd order, 3rd order....

If A is a finite quantity and is an infinitesimal then $A.\delta x$, $A.(\delta x)^2$, $A.(\delta x)^3$, ... are also infinitesimals and they are infinitesimals respectively of first order, second order, third order

Definition: A quantity α= α(x) is called an infinitesimal as $x \to a$ if $\underset{x \to a}{Lt}\, \alpha(x) = 0$

THEOREM

Let $y = f(x)$ be a differentiable function at x and be a small change in x. Then $f'(x)$ and $\frac{\delta y}{\delta x}$ differ by an infinitesimal $\alpha(\delta x)$ as $\delta x \to 0$, where $\delta y = f(x+\delta x) - f(x)$.

DIFFERENTIAL

Definition: If $y = f(x)$ is a differentiable function of x then $f'(x).\delta x$ is called the differential of f. It is denoted by df or dy.

$\therefore\ dy = f'(x)\delta x$ or $df = f'(x)\delta x$.

Note: $\delta f \cong df$ i.e., error in f is approximately equal to differential of f

APPROXIMATIONS

We have $\delta f = f(x+\delta x) - f(x)$ -------------(1)

$$\Rightarrow df \cong f(x+\delta x) - f(x)$$

$$\Rightarrow f^1(x).\delta x \cong f(x+\delta x) - f(x)$$

$$\Rightarrow f(x+\delta x) \cong f(x) + f^1(x).\delta x$$

If we know the value of f at a point x, then the approximate value of f at a very nearby point x + δx can be calculated with the help of above formula.

ERRORS

Definition: Let y=f(x) be a function defined in a nbd of a point x. Let δx be a small change in x and δy be the corresponding change in y.

If δx is considered as an error in x, then

(i) δy is called the absolute error or error in y,

(ii) $\frac{\delta y}{y}$ **is** called the relative error (or proportionate error) in y,

(iii) $\frac{\delta y}{y}\times 100$ is called the percentage error in y corresponding to the error δx in x.

EXERCISE

I. Find Δy, dy for the following functions.

1. $y = x^2 + 3x + 6, x = 10, \Delta x = 0.01$. (Mar. '5)

Sol: $\Delta y = f(x+\Delta x) - f(x) = (x+\Delta x)^2 + 3(x+\Delta x) + 6 - (x^2 + 3x + 6) = (\Delta x)^2 + 2x.\Delta x + 3\Delta x$

Put x=10and Δx=0.01

$\Rightarrow \Delta y = (0.01)^2 + 2.10.(0.01) + 3(0.01)$
$= 0.0001 + 0.2 + 0.03 = 0.2301$

$y = x^2 + 3x + 6$

$dy = f^1(x)\,\delta x$

$dy = (2x + 3)\,\delta x = (2.\ 10 + 3)\ (0.01) = 0.23$

2. $y = e^x$, x = 0, Δx = 0.1.

Sol: $\Delta y = f(x + \Delta x) - f(x)$

$= e^{(x+\delta x)} - e^x$ put x =0 and Δ **x = 0.1**

$\Delta y = e^{0.1} - e^0 = e^{0.1} - 1.$

$dy = f^1(x).\ \delta x = e^x.\Delta x = e^0(0.1) = 0.1$

3. $y = \frac{1}{x}$, x = 2, Δx = 0.002.

Ans: $-\frac{1}{2000}$

4. y = log x, x = 3, Δx = 0.003.

Ans: 0.001

5. $y = x^2 + 2x$, $x = 5$, $\Delta x = -0.1$

Sol: $\Delta y = -1.19$ ans dy = -1.2

6. If the increase in the side of a square is 1%, find the percentage of change in the area of the square.

Sol: Let x be the side and Abe the area of the

Square Percentage error in x is $\frac{\delta x}{x} \times 100 = 1$

Area A= x^2

Applying logs on both sides

Log A = 2 logx

Taking differentials on both sides

$\frac{1}{A}\delta A = 2.\frac{1}{x}\delta x \Rightarrow \frac{\delta A}{A} \times 100 = 2.\frac{\delta x}{x} \times 100$ = 2 x 1

Therefore, percentage error in A is 2%

7. Area of ΔABC is measured, by the measure of a, b, c. If Δc is the error in measuring c, then what is the percentage error in the area?

Sol: area of the triangle is A = $\frac{1}{2}$ab sin c

Applying logs on both sides , Log A = log ($\frac{1}{2}$ab sin c)

Log A = log($\frac{1}{2}$ab) + log sinC

Taking differentials on both sides

$\frac{1}{A}\delta A = 0 + \frac{1}{\sin C}\cos C\ \delta C \Rightarrow \frac{\delta A}{A} \times 100 = \delta C \cot C \times 100$ Percentage error in A = 100 cot C. Δc

8. The diameter of a sphere us measured to be 20 cms. If an error of 0.02 cm occurs in this, find the error in volume and surface area of the sphere.

Sol: let d be the diameter of the sphere.

Volume of the sphere is $V = \frac{4}{3}\pi r^3 = \frac{4\pi}{3}\left(\frac{d}{2}\right)^3$

$= \frac{4\pi d^3}{3\times 8} = \frac{\pi d^3}{6} = \frac{\pi d^3}{6}$

$\Delta V = \frac{\pi}{6}\left(3d^2\right).\Delta d = \frac{\pi}{2} d^2. \Delta d$

Given d= 20, $\Delta d = 0.02$

$\Delta V = \frac{\pi}{2}(20)^2 (0.02) = \pi(400)(0.01) = 4\pi\ cm^3$

$\therefore$ Error in volume = 4π cms^3

Let S be the surface area of the sphere.

Then S = $4\pi r^2 = 4\pi\left(\frac{d}{2}\right)^2 = 4\pi\frac{d^2}{4} = \pi d^2$

$\Delta S = \pi(2d).\Delta d = 2\pi d\ .\ \Delta d$

Put d = 20, $\Delta d = 0.02$

$\Delta S = 2\pi(2o)(0.02) = 0.8\pi$ cm^2

$\therefore$ Error un surface area = 0.8π $sq.cms^2$.

9. The time t of a complete oscillation of a simple pendulum of length l is given by the equation t = $2\pi\sqrt{\frac{1}{g}}$ where g gravitational constant. Find the approximate percentage error in the calculated g, corresponding to an error of 0.01 percent is the value of t.

Sol: percentage error in t is $\frac{\Delta t}{t}\times 100 = 0.01$

Given t= $2\pi\sqrt{\frac{l}{g}}$

Taking logs on both sides $\log t = \log(2\pi) + \frac{1}{2}\{(\log(l) - \log g\}$

Taking differentials on both sides, $\frac{1}{t}(\Delta t) = 0 + \frac{1}{2}\left\{o - \frac{1}{g}.(\Delta g)\right\}$

Multiplying with 100, $\frac{\Delta t}{t}\times 100 = -\frac{1}{2}\frac{\Delta g}{g}\times 100$

$\Rightarrow 0.001 = -\frac{1}{2}\frac{\Delta g}{g}\times 100$

$\Rightarrow \frac{\Delta g}{g}\times 100 = -0.02$

$\therefore$ Percentage error in g = -0.02

II. Find the approximate value of

1) $\sqrt{82}$ **2)** $\sqrt[3]{63}$ **3)** $\sqrt{25.2}$ **4)** $\sqrt[3]{7.8}$

5) Sin 60°1' $\left(\frac{\pi}{180} \approx 0.0175\right)$ **6) cos 45°6' 7)** $(x-1)^3 (x-2)^2 (x-3)$ **when x = 0.001.**

1. $\sqrt{82}$

Sol: let f(x) = $\sqrt{x}$,x = 81, Δx = 1

Now

$f(x+\delta x) = f(x) + f^1(x)\,\delta x$

$= \sqrt{x} + \frac{1}{2\sqrt{x}}.\Delta x$, put x = 81, Δx = 1

$= \sqrt{81} + \frac{1}{2\sqrt{81}}.1 = 9 + \frac{1}{2.9} = 9 + \frac{1}{18} = 9 + 0.056 = 9.056$

2. $\sqrt[3]{63}$

Sol: let $f(x) = \sqrt[3]{x}, x = 64, \Delta x = -1$

Follow above method.

3. $\sqrt{25.2}$

Sol: Let x = 25, $\Delta x = 0.2, f(x) = \sqrt{x}$

Follow above method

$\therefore \sqrt{25.2} = 5.02$

4. $\sqrt[3]{7.8}$

Sol: Let x = 8, $\Delta x = -0.2, f(x) = \sqrt[3]{x}$

$\text{ans}: \sqrt[3]{7.8} = 1.9834$

5. **Sin 60º1’, = $\frac{\pi}{80}$ = 0.0175.**

Sol: Let $f(x) = \sin x, x = 60^0 = \frac{\pi}{3} \text{and} \Delta x = 1' = \frac{\pi}{60 \times 180} \text{radians}$ $f(x + \delta x) = f(x) + f^1(x)\ \delta x$

$= \sin x + \cos x . \Delta x = \sin 60 + \cos 60 . \frac{\pi}{60 \times 180}$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} . \frac{0.01745}{60}$ = 0.86605 + 0.00013 = 0.86618

$\therefore$ Sun 60°1' = 0.86618

6. **Cos 45º6’**

Ans; Cos 45º 6’ = 0.7059

7. **$(x-1)^3 (x-2)^2 (x-3)$ at x = 0.001.**

Sol: $f(x) = (x-1)^3 (x-2)^2 (x-3)$

$f^1(x) = (x-1)^3 (x-2)^2 . 1 + (x+1)^3 (x-3) + 2(x-2) + (x-2)^2 (x-3)\, 3(x-1)^2$

$= (x-1)^2 (x-2) [(x-1)(x-2) + 2(x-1)(x-3) + 3(x-2)(x-3)]$

$= (x-1)^2 (x-2) [x^2 - 3x + 2 + 2x^2 - 8x + 6 + 3x^2 - 15x + 18]$

$= (x-1)^2 (x-2)(6x^2 - 26x + 26)$

$dy = f^1(x).\ \Delta x$ put $x = 0,\ \Delta x = 0.001$

$dy = [(-1)^2 (-2)(0 - 0 + 26)](0.001)$

$= -52\ (0.001)$

$= -0.052$

$f(x + \delta x) = f(x) + f^1(x)\ \delta x$

i.e., $f(x + \delta x) \simeq f(x) + dy$

$= f(0) + dy = (-1)^3 (-2)(-3) + (-0.052)$

$= 12 - 0.052 = 11.948$

8. $y = \cos(x)$, $x = 60^\circ$ and $\Delta x = 1^\circ$.

Sol. $\Delta y = f(x + \Delta x) - f(x)$

$= \cos(x + \Delta x) - \cos x$

$= \cos(60^\circ + 1^\circ) - \cos 60^\circ$

$= \cos 61^\circ - \cos 60^\circ$

$= 0.4848 - \frac{1}{2} = 0.4848 - 0.5 = -0.0152$

$dy = f'(x)\Delta x$

$= -\sin x \Delta x$

$= -\sin 60^\circ (1^\circ) = \frac{-\sqrt{3}}{2}(0.0174)$

$= -(0.8660)(0.0174) = -0.0151$