

## SECOND ORDER DERIVATIVES

If  $y = f(x)$  is a differentiable function of  $x$  then its derivative  $f'(x)$  is a function of  $x$ .

If  $f'(x)$  is a differentiable function then its derivative is called the second order

derivative of  $f(x)$ . It is denoted by  $f''(x)$  and  $f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$

.

$f''(x)$  is also denoted by  $f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$  or  $\frac{d^2 f}{dx^2}$  or  $D^2 y$  or  $y''$  or  $y_2$ .

### EXERCISE

I. If  $y = \frac{2x+3}{4x+5}$  then find  $y''$

Sol :  $y = \frac{2x+3}{4x+5}$

Differentiating w. r. to  $x$

$$\frac{dy}{dx} = \frac{d}{dx} \frac{2x+3}{4x+5}$$

$$\frac{dy}{dx} = \frac{(4x+5).2 - (2x+3).4}{(4x+5)^2} = \frac{8x+10 - 8x-12}{(4x+5)^2}$$

$$= \frac{-2}{(4x+5)^2} = 2(4x+5)^{-2}$$

Again diff w.r.t  $x$ ,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} 2(4x+5)^{-2}$$

$$\frac{d^2 y}{dx^2} = (-2)(-2)(4x+5)^{-3} . 4$$

$$y'' = \frac{16}{(4x+5)^3}$$

2.  $y = ae^{nx} + be^{-nx}$  then prove that  $y'' = n^2 y$

**Sol :**  $y = ae^{nx} + be^{-nx}$

**Diff with respect to x**

$$y_1 = nae^{nx} - nb e^{-nx}$$

**Diff with respect to x**

$$y_2 = n^2 .ae^{nx} + n^2 .be^{-nx}$$

$$y_2 = n^2 (ae^{nx} + b.e^{-nx}) = n^2 y$$

**II.**

1. Find the second order derivatives of the following functions  $f(x)$

i)  $\cos^3 x$

$$\text{sol : } y = \cos^3 x = \frac{1}{4} [\cos 3x + 3\cos x]$$

**diff w.r.t x, we get**

$$y_1 = \frac{1}{4} [-3\sin 3x - 3\sin x]$$

**Diff w.r.t x, we get**

$$y_2 = \frac{1}{4} (-9\cos 3x - 3\cos x)$$

$$= -\frac{1}{4} (\cos x + 3\cos 3x)$$

ii)  $y = \sin^4 x$

**sol :**  $y = \sin^4 x = (\sin^2 x)^2 = \left( \frac{(1 - \cos 2x)^2}{2} \right)$

$$= \frac{1}{4} [1 - 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4} \left[ 1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right]$$

$$= \frac{1}{8} [2 - 4\cos 2x + 1 + \cos 4x]$$

$$= \frac{1}{8} [3 - 4\cos 2x + \cos 4x]$$

**Diff w.r.t. x , then**

$$y_1 = \frac{1}{8} (8\cos 2x - 4\sin 4x)$$

$$y_2 = \frac{1}{8} (16\cos 2x - 16\cos 4x) = 2(\cos 2x - \cos 4x)$$

iii)  $y = \log(4x^2 - 9)$

$$= \log(2x - 3)(2x + 3) = \log(2x - 3) + \log(2x + 3)$$

**Diff with respect to x, then**

$$y_1 = \frac{2}{2x - 3} + \frac{2}{2x + 3}$$

**Again diff w.r.t. x, then**

$$y_2 = \frac{2(-1)^{-2}}{(2x - 3)^2} + \frac{2(-1)^2}{(2x + 3)^2}$$

$$= -4 \frac{((2x+3)^2 + (2x+3)^2)}{(4x^2 - 9)^2} = \frac{-4(2\{4x^2 + 9\})}{(4x^2 - 9)^2}$$

$$= -8 \frac{(4x^2 + 9)}{(4x^2 - 9)^2}$$

**iv)**  $y = e^{-2x} \cdot \sin^3 x$

**sol :**  $y = e^{-2x} \cdot \sin^3 x$

**Diff with respect to x**

$$\begin{aligned} y' &= e^{-2x} (3\sin^2 x \cdot \cos x) + \sin^3 x (e^{-2x})(-2) \\ &= e^{-2x} [3\sin^2 x \cdot \cos x - 2\sin^3 x] \end{aligned}$$

**Diff with respect to x**

$$\frac{d^2y}{dx^2} = e^{-2x} [3\sin^2 x(-\sin x) + 3\cos x(2\sin x)\cos x - 6\sin^2 x \cos x] - 2e^{-2x}$$

$$[3\sin^2 x \cdot \cos x - 2\sin^3 x]$$

$$= e^{-2x} [6\sin x \cdot \cos^2 x - 6\sin^2 x \cdot \cos x - 3\sin^3 x] - 6\sin^2 x \cdot \cos x + 4\sin^3 x$$

$$= e^{-2x} [\sin^3 x - 12\sin^2 x \cdot \cos x + 6\sin x \cdot \cos^2 x]$$

**v)**  $e^x \cdot \sin x \cdot \cos 2x$

**sol :**  $y = e^x \cdot \sin x \cdot \cos 2x = \frac{e^x}{2} (2\cos 2x \cdot \sin x)$

$$= \frac{e^x}{2} (\sin 3x - \sin x)$$

**diff w.r.t. x, then**

$$y_1 = \frac{1}{2} [e^x (3\cos 3x - \cos x) + e^x (\sin 3x - \sin x)]$$

**Again diff w.r.t. x, then**

$$y_2 = \frac{1}{2} [e^x (-9\sin 3x + \sin x) + e^x (3\cos 3x - \cos x)] + e^x [3\cos 3x - \cos x] + e^x (\sin 3x - \sin x)$$

$$= \frac{e^x}{2} [-9\sin 3x + \sin x + 3\cos 3x - \cos x + 3\cos 3x - \cos x + \sin x] \\ = \frac{e^x}{2} [6\cos 3x - 8\sin 3x - 2\cos x]$$

$$= e^x [3\cos 3x - 4\sin 3x - \cos x]$$

vi)  $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

sol :  $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

**Put  $x = \tan \theta$  then  $\theta = \tan^{-1} x$**

$$y = \tan^{-1}\left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \cdot \tan \theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right) = \frac{\pi}{4} + \theta$$

$$\therefore f(x) = \frac{\pi}{4} + \tan^{-1}(x)$$

**Diff. w. r. to x**

$$f'(x) = 0 + \frac{1}{1+x^2}$$

Again diff w.r.t. x,

$$f''(x) = (-1)(1+x^2)^{-2}(2x)$$

$$\therefore f''(x) = \frac{-2x}{(1+x^2)^2}$$

vii)  $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

sol :  $f(x) = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Put  $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$

$$f(x) = \tan^{-1}\left(\frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$\therefore f(x) = 3\tan^{-1}(x)$$

Diff. w. r. to x

$$f'(x) = 3\left(\frac{1}{1+x^2}\right) = \frac{3}{1+x^2}$$

Again Diff. w. r. to x

$$f''(x) = (3)(-1)(1+x^2)^{-2}(2x) \Rightarrow f''(x) = \frac{-6x}{(1+x^2)^2}$$

**II. Prove the following .**

**1.** If  $y = ax^{n+1} + bx^{-n}$ , then  $x^2 y'' = n(n+1)y$ .

**Sol :**  $y = ax^{n+1} + bx^{-n}$

$$\text{Diff w.r.t } x, \quad y_1 = (n+1).ax^n - nbx^{-n-1}$$

Again diff with respect x,

$$y_2 = n(n+1).ax^{n-1} + n(n+1)bx^{-n-2}$$

$$\therefore x^2 y_2 = n(n+1)ax^{n+1} + n(n+1)bx^{-n}$$

$$= n(n+1)(ax^{n+1} + bx^{-n}) = n(n+1)y$$

$$\therefore x^2 y'' = n(n+1)y$$

**ii)** If  $y = a \cos x + (b + 2x) \sin x$ , then show that  $y'' + y = 4 \cos x$

**iii)** If  $y = 6(x+1) + (a+bx)e^{3x}$  then  $y'' = 6y' + 9y = 54x + 18$

**iv)** If  $ay^4 = (a+b)^5$ , then  $5y y'' = (y')^2$

$$\text{sol : } ay^4 = (a+b)^5 \Rightarrow y^4 = \frac{(a+b)^5}{a}$$

$$\Rightarrow y = \frac{(a+b)^{5/4}}{a^{1/4}}$$

Diff w.r.t x, we get

$$y_1 = \frac{1}{a^{1/4}} \cdot \frac{5}{4} (a+b)^{\left(\frac{5}{4}-1\right)} = \frac{5}{4a^{1/4}} (a+b)^{1/4} \cdot 1 \quad \text{Again diff with respect x,}$$

$$y_2 = \frac{5}{4a^{1/4}} \cdot \frac{1}{4} (x+b)^{\frac{1}{4}-1} = \frac{5}{16a^{1/4}} (x+b)^{-3/4}$$

$$\text{L.H.S.} = 5y y'' = 5 \cdot \frac{(x+b)^{5/4}}{a^{1/4}} \cdot \frac{5}{16a^{1/4}} (a+b)^{-3/4}$$

$$= \frac{25}{16a^{1/2}} (x+b)^{2/4} = \left[ \frac{5}{4a^{1/4}} (x+b)^{1/4} \right] = (y')^2$$

v) If  $y = a \cos(\sin x) + b \sin(\sin x)$ , then  $y'' + (\tan x)y' + y \cos^2 x = 0$  try yourself.

### III.

1. i) If  $y = 128 \sin^3 x \cos^4 x$ , then find  $y''$ .

Sol :  $y = 128 \sin^3 x \cos^4 x$

D.w.r. to x

$$\begin{aligned} y_1 &= 128 \left[ \sin^3 x \{4 \cos^3 x (-\sin x)\} \right] + \cos^4 x \{3 \sin^2 x \cos x\} \\ &= 128 \left[ 3 \sin^2 x \cos^5 x - 4 \sin^4 x \cos^3 x \right] \end{aligned}$$

Again D. w. r to x

$$y_2 = 128 \left\{ 3[\sin^2 x 5 \cos^4 x.(-\sin x) + \cos^5 x] 2 \sin x \cos x - 4[\sin^4 x.3.\cos^2 x(-\sin x) \right.$$

$$\left. + \cos^3 x.4 \sin^3 x \cos x] \right\}$$

$$= 128[-15 \sin^3 x \cos^4 x + 6 \sin x \cos^6 x + 12 \sin^5 x \cos^2 x - 16 \sin^3 x \cos^4 x]$$

$$f''(x) = 128[6 \sin x \cos^6 x + 12 \sin^5 x \cos^2 x - 31 \sin^3 x \cos^4 x]$$

ii) If  $y = \sin 2x \sin 3x \sin 4x$ , then find  $y''$ .

**sol :**  $y = \sin 2x \sin 3x \sin 4x$

$$= \frac{1}{2} \sin 2x [2 \sin 4x \cdot \sin 3x]$$

$$= \frac{1}{2} \sin 2x [\cos x - \cos 7x]$$

$$= \frac{1}{2} [\sin 2x \cdot \cos x - \cos 7x \cdot \sin 2x]$$

$$= \frac{1}{2} \times \frac{1}{2} [2 \sin 2x \cdot \cos x - 2 \cos 7x \cdot \sin 2x]$$

$$= \frac{1}{4} [(\sin 3x + \sin x) - (\sin 9x - \sin 5x)]$$

$$= \frac{1}{4} [-\sin 9x + \sin 5x + \sin 3x + \sin x]$$

**D.w.r to x**

$$y_1 = \frac{1}{4} [-9 \cos 9x + 5 \cos 5x + 3 \cos 3x + \cos x]$$

**D. w. r. to x**

$$y_2 = \frac{1}{4} [81 \sin 9x - 25 \sin 5x - 9 \sin 3x - \sin x]$$

iii) If  $ax^2 + 2hxy + by^2 = 1$ , then prove that  $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$

**sol :** Given  $ax^2 + 2hxy + by^2 = 1$

**Differentiating w. r. to x**

$$\frac{d}{dx}(ax^2 + 2hxy + by^2) = 0$$

$$\Rightarrow a.2x + 2h\left(x \cdot \frac{dy}{dx} + y\right) + b.2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow 2ax + 2hx \cdot \frac{dy}{dx} + 2hy + 2by \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow 2(hx + by) \cdot \frac{dy}{dx} = -2(ax + hy)$$

$$\frac{dy}{dx} = \frac{-2(ax + hy)}{2(hx + by)} = -\frac{(ax + hy)}{(hx + by)} \quad \dots\dots(1)$$

**Differentiating again w. r. to x,**

$$\frac{d^2y}{dx^2} = -\frac{d}{dx} \frac{(ax + hy)}{(hx + by)}$$

$$\frac{d^2y}{dx^2} = \frac{\left[(hx + by)\left(a + h \frac{dy}{dx}\right) - (ax + hy)\left(h + b \cdot \frac{dy}{dx}\right)\right]}{(hx + by)^2}$$

$$= \frac{(ax + hy)\left[h - b \cdot \frac{(ax + hy)}{hx + by}\right] - (hx + by)\left[\frac{(ax + hy)}{hx + by}\right]}{(hx + by)^2} (ax + hy)(h^2x + bhy - abx - bhy)$$

$$= \frac{-(hx + by)(ahx + aby - ahx - h^2y)}{(hx + by)^3}$$

$$= \frac{(h^2 - ab)[x(ax + hy)] + (h^2 - ab)(y(hx + by))}{(hx + by)^3} = \frac{(h^2 - ab)[x(ax + hy) + y(hx + by)]}{(hx + by)^3}$$

$$= \frac{(h^2 - ab)[ax^2 - 2hxy + by^2]}{(hx + by)^3}$$

$$= \frac{h^2 - ab}{(hx + by)^3} \left[ \because ax^2 + 2hxy + by^2 = 1 \right]$$

iv) If  $y = ae^{-bx} \cos(cx + d)$  then prove that  $y'' + 2by' + (b^2 + c^2)y = 0$ .

v) If  $y = e^{\frac{-k}{2}x} (a \cos nx + b \sin nx)$ . Then prove that  $y'' + ky' + \left(n^2 + \frac{k^2}{4}\right)y = 0$

sol : Given  $y = e^{\frac{-k}{2}x} (a \cos nx + b \sin nx)$  .....(1)

diff w.r.t. x, we get,

$$y_1 = e^{\frac{-k}{2}x} (-n.a \sin nx + n.b \cos nx) + (a \cos nx + b \sin nx).e^{\frac{-k}{2}x} \left(-\frac{k}{2}\right)$$

$$= -\frac{k}{2}.y - n.e^{-kx/2} (a \sin nx + b \cos nx)$$

$$\therefore y_1 + \frac{k}{2}y = -n.e^{-kx/2} (a \sin nx + b \cos nx) \quad \dots\dots(2)$$

Differentiating w. r. to x

$$y_2 + \frac{k}{2}y_1 = -n \left[ \left\{ e^{-kx/2} (-na \cos nx - nb \sin nx) \right\} \right] + \left\{ (a \sin nx + b \cos nx).e^{-kx/2} \left(-\frac{k}{2}\right) \right\}$$

$$= -n^2 - e^{-kx/2} (a \cos nx + b \sin nx) - \frac{k}{2} \left\{ -n.e^{-kx/2} (a \sin nx + b \cos nx) \right\}$$

$$= -n^2.y - \frac{k}{2} \left[ y_1 + \frac{k}{2}y \right] \text{by (1),(2)}$$

$$= -n^2y - \frac{k}{2}y_1 - \frac{k^2}{4}y$$

$$= -\frac{k}{2}y_1 - \left( n^2 + \frac{k^2}{4} \right)y$$

$$\therefore y_2 + k \cdot y_1 + \left( n^2 + \frac{k^2}{4} \right) y = 0$$

### PROBLEMS FOR PRACTICE

1. If  $f(x) = x^2$  ( $x \in R$ ), Prove that f is differentiable on R and find its derivative.
2. Suppose  $f(x) = \sqrt{x}$  ( $x > 0$ ), prove that f is differentiable on  $(0, \infty)$  and find  $f'(x)$ .
3. If  $f(x) = \frac{1}{x^2 + 1}$  ( $x \in R$ ), prove that f is differentiable and find  $f'(x)$
4. If  $f(x) = \sin x$  ( $x \in R$ ), then show that f is differentiable on R and  $f'(x) = \cos x$
5. Show that  $f(x) = |x|$  ( $x \in R$ ) is not differentiable at zero and is differentiable at any  $x \neq 0$
6. Check whether the following function is differentiable at zero

$$f(x) = \begin{cases} 3+x & \text{if } x \geq 0 \\ 3-x & \text{if } x < 0 \end{cases}$$

**Sol :** Suppose  $h > 0$  and  $\frac{f(0+h) - f(0)}{h} = \frac{f(h) - 3}{h}$

$$= \frac{3+h-3}{h} = \frac{h}{h} = 1$$

$$Lt_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 1$$

f has the right hand derivative zero and  $f'(0^+) = 1$

$$\underset{h \rightarrow 0}{Lt} \frac{f(0+h) - f(0)}{h} = \underset{h \rightarrow x}{Lt} \frac{3-h-3}{h} = \underset{h \rightarrow 0}{Lt} -\frac{h}{h} = -1$$

F has the left hand derivative at zero and  $f'(0^-) = -1$

$$\therefore f'(0^+) \neq f'(0^-)$$

$f(x)$  is not differentiable at zero

**7. Show that the derivative of a constant function on an interval is zero.**

**8. Suppose for all  $x, y \in R$   $f(x+y) = f(x)$ .  $f(y)$  and  $f'(0)$  exists. Then show that  $f'(x)$  exists and equal to  $f(x)f'(0)$  for all  $x \in R$ .**

**Sol :** Let  $x \in R$ , for  $h \neq 0$ , we have

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \frac{[f(h) - 1]}{h} \quad \dots\dots(1) \end{aligned}$$

$$f(0) = f(0+0) = f(0)f(0) \Rightarrow f(0)(1-f(0)) = 0$$

$$\therefore f(0) = 0 \quad f(0) = 1$$

**Case (i) :** Suppose  $f(0) = 0$

$$f(x) = f(x+0) = f(x)f(0) = 0 \quad \forall x \in R$$

$$\therefore f(x) \text{ is a constant function} \Rightarrow f'(x) = 0$$

For all  $x \in R$

$$\therefore f'(x) = 0 = f(x).f'(0)$$

**Case (ii) :** Suppose  $f(0) = 1$

$$\underset{h \rightarrow 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \rightarrow 0}{\text{Lt}} \frac{f(x)f(h) - f(x)}{h} = f(x) \underset{h \rightarrow 0}{\text{Lt}} \frac{f(h) - 1}{h}$$

$$f(x) \underset{h \rightarrow 0}{\text{Lt}} \frac{f(h) - f(0)}{h} (\because f(0) = 0)$$

$$= f(x)f'(0)$$

$\therefore f$  is differential and  $f'(x) = f(x)f'(0)$

**9.** If  $f(x) = (ax + b)^n$  ( $x > -\frac{b}{a}$ ), then find  $f'(x)$

**10.** Find the derivative of  $f(x) = ex(x^2 + 1)$ ,

**11.** If  $y = \frac{a-x}{a+x}$  ( $x \neq -a$ ), find  $\frac{dy}{dx}$

**12.** If  $f(x) = e^{2x} \cdot \log x$  ( $x > 0$ ), then find  $f'(x)$

**13.** If  $f(x) = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$  ( $|x| < 1$ ), then find  $f'(x)$

**14.** If  $f(x) = x^2, 2^x \log x$  ( $x > 0$ ), find  $f'(x)$ .

**15.** if  $y = \begin{vmatrix} f(x) & g(x) \\ \varphi(x) & \psi(x) \end{vmatrix}$  then show that  $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) \\ \varphi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix}$

**Sol :** Given is  $= \begin{vmatrix} f(x) & g(x) \\ \varphi(x) & \psi(x) \end{vmatrix} = f(x)\psi'(x) + \psi(x)g(x)$

$$\frac{dy}{dx} = f(x)\psi'(x) + \psi(x)f'(x) - [\varphi(x).g'(x) + g(x).\varphi'(x)]$$

$$= [f(x)\psi'(x) - g(x)\varphi'(x)] + [f'(x)\psi(x) - \psi(x)\varphi(x).g'(x)]$$

$$= \begin{vmatrix} f(x) & g(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) \\ \varphi(x) & \psi(x) \end{vmatrix}$$

16. If  $f(x) = 7^{x^3+3x}$  ( $x > 0$ ), then find  $f'(x)$ .

17. If  $f(x) = x.e^x \sin x$ , then find  $f'(x)$ .

18. If  $f(x) = \sin(\log x)$  ( $x > 0$ ), find  $f'(x)$ .

19. If  $f(x) = (x^3 + 6x^2 + 12x - 13)^{100}$ ; find  $f'(x)$ .

20. Find the derivative of  $f(x) = \frac{x \cos x}{\sqrt{1+x^2}}$ .

21. If  $f(x) = \log(\sec x + \tan x)$  find  $f'(x)$

22. If  $y = \sin^{-1} \sqrt{x}$ , find  $\frac{dy}{dx}$ .

23. If  $y = \sec(\sqrt{\tan x})$ , find  $\frac{dy}{dx}$

24. If  $y = \frac{x \sin^{-1} x}{\sqrt{1+x^2}}$ , find  $\frac{dy}{dx}$

25. If  $y = \log(\cosh 2x)$ , find  $\frac{dy}{dx}$ .

26. If  $y = \log(\sin(\log x))$ , find  $\frac{dy}{dx}$ .

27. If  $y = (\cot^{-1} x^3)^2$ , find  $\frac{dy}{dx}$ .

28. If  $y = \cos ec^{-1}(e^{2x+1})$  find  $\frac{dy}{dx}$ .

29. If  $y = \tan^{-1}(\cos \sqrt{x})$ , find  $\frac{dy}{dx}$

Sol :  $v = \sqrt{x}$  and  $u = \cos v$ ,  $y = \tan^{-1} u$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \frac{du}{dv} = -\sin u \frac{dy}{du} = \frac{1}{1+u^2}$$

$$= -\sin \sqrt{x} = \frac{1}{1+\cos^2(\sqrt{x})}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{du}{dx}$$

$$= \frac{1}{1+\cos^2(\sqrt{x})} - \sin \sqrt{x} \frac{1}{2\sqrt{x}}$$

$$= \frac{-\sin \sqrt{x}}{2\sqrt{x}[1+\cos^2(\sqrt{x})]}$$

30. If  $y = \tan^{-1} \left( \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right)$  for  $0 < |x| < 1$ , find  $\frac{dy}{dx}$ .

31. If  $y = x^2 e^x \sin x$ , find  $\frac{dy}{dx}$ .

32. If  $y = x^{\tan x} + (\sin x)^{\cos x}$ , find  $\frac{dy}{dx}$

Sol : Let  $u = x^{\tan x}$  and  $v = (\sin x)^{\cos x}$

$$\log u = \log x^{\tan x} = (\tan x) \log x$$

$$\frac{1}{u} \cdot \frac{dy}{dx} = \tan x \frac{1}{x} + (\log x) \sec^2 x.$$

$$\frac{du}{dx} = u \left( \frac{\tan x}{x} + (\log x) \cdot \sec^2 x \right)$$

$$= x^{\tan x} \left( \frac{\tan x}{x} + (\log x) \cdot \sec^2 x \right)$$

$$\log v = \log(\sin x) \cos x] = \cos x \cdot \log \sin x$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = \cos x \cdot \frac{1}{\sin x} \cos x + (\log \sin x)(-\sin x)$$

$$= \frac{\cos^2 x}{\sin x} - \sin x \log(\sin x)^{\cos x}$$

$$\frac{dv}{dx} = v \left( \frac{\cos^2 x}{\sin x} - \sin x \log \sin x \right)$$

$$= (\sin x)^{\cos x} \left( \frac{\cos^2 x}{\sin x} - \sin x \log(\sin x) \right)$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\tan x}$$

$$\left( \frac{\tan x}{x} + (\log x) (\sec^2 x) \right) + (\sin x)^{\cos x} \left( \frac{\cos^2 x}{\sin x} - \sin x \log(\sin x) \right)$$

33. If  $x = a \left( \cos t + \log \tan \left( \frac{t}{2} \right) \right)$ ,  $y = a \sin t$ , find  $\frac{dy}{dx}$

34. If  $x^y = e^{x-y}$ , then show that  $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

**Sol :**  $x^y = e^{x-y} \Rightarrow \log x^y = \log e^{x-y}$

$$\Rightarrow y \log x = x - y (\log e = 1)$$

$$y(1 + \log x) = x, y = \frac{x}{1 + \log x} \mathbf{c}$$

$$\frac{dy}{dx} = \frac{(1 + \log x).1 - x \cdot \frac{1}{x}}{(1 + \log x)^2}$$

$$= \frac{1 + \log x - 1}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

35. If  $\sin y = x \sin(a + y)$ , then show that  $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$  (a is not a multiple of  $\pi$ )

Sol :  $x = \frac{\sin y}{\sin(a + y)}$

$$\frac{dy}{dx} = \frac{\sin(a + y)\cos y - \sin y\cos(a + y)}{\sin^2(a + y)} \cdot \frac{dy}{dx}$$

$$= \frac{\sin(a + y - y)}{\sin^2(a + y)} \frac{\sin a}{\sin^2(a + y)}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{\sin^2(a + y)}{\sin a}$$

36. If  $y = x^4 + \tan x$ , then find y”.

37. If  $f(x) = \sin x, \sin 2x, \sin 3x$  find  $f''(x)$ .

Sol :

$$f(x) = \frac{1}{2} \sin 2x (2 \sin 3x \sin x) = \frac{1}{2} (\sin 2x)(\cos 2x - \cos 4x)$$

$$= \frac{1}{4} (2 \sin 2x \cos 2x - 2 \sin 2x \cos 4x)$$

$$= \frac{1}{4} (\sin 2x + \sin 4x - \sin 6x)$$

**Therefore,**

$$f'(x) = \frac{1}{4}[2\cos 2x + 4\cos 4x - 6\cos 6x]$$

**Hence,**

$$f''(x) = \frac{1}{4}(-4\sin 2x - 16\sin 4x + 36\sin 6x) = 9\sin 6x - 4\sin 4x - \sin 2x.$$

**38. Show that**  $y = x + \tan x$  **satisfies**  $\cos^2 x \frac{dy^2}{dx^2} + 2x = 2y$ .

**39. If**  $x = a(t - \sin t)$ ,  $y = a(1 + \cos t)$ , **find**  $\frac{d^2y}{dx^2}$ .

**40. Find the second order derivative of**  $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

**41. If**  $y = \sin(\sin x)$ , **show that**  $y'' + (\tan x)y' + y\cos^2 x = 0$ .

$$y'' + ky^1 + \left(n^2 + \frac{k^2}{4}\right)y = 0$$