PARAMETRIC DIFFERENTIATION

If x =g (t) and y =h (t) are two functions of t then y may be expressed as a function of x, say y = f(x), by eliminating the real variable t. Such a variable t is called a parameter. Also, x and y are said to be defined parametrically. The equations x =g (t), y =h (t) are called the parametric equations of y = f(x).

The process of finding $\frac{dy}{dx}$ when x and y are defined parametrically, is called parametric differentiation.

THEOREM

Let x = g(t) and y =h (t) be two differentiable functions of t such that g^{-1} exists. Then $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$.

IMPLICIT DIFFERENTIATION

Let y be an implicit function of x defined by a relation of the type f(x, y) = 0. Then the method of finding the derivative $\frac{dy}{dx}$ of such an implicitly defined function is called implicit differentiation.

In this method, we differentiate f(x, y) = 0 w.r.t x (treating y as a function of x) to get an equation of the type F(x,y, $\frac{dy}{dx}$) = 0.

By solving this equation we find $\frac{dy}{dx}$.

LOGARITHMIC DIFFERENTIATION

If a given function is of the form y = $f(x)^{g(x)}$ where f(x) and g(x) are differentiable functions of x, we apply logarithms to transform it into a suitable form so as to find . This method of finding $\frac{dy}{dx}$ is called logarithmic differentiation.

This method is also useful in finding the derivatives of functions which are products of a number of functions.

THEOREM

If y = f(x) $^{g(x)}$ where f(x) and g(x) are differentiable functions

then $\frac{dy}{dx} = f(x)^{g(x)}\left[\frac{g(x)}{f(x)} f'(x) + g'(x)\log f(x)\right]$

Proof:

y = f(x)$^{g(x)}$, f(x) > 0

logy = g(x) log f(x)

Differentiating both sides w.r.t x,

$$\frac{1}{y}\frac{dy}{dx} = g'(x).\log f(x) + g(x)\frac{1}{f(x)} f'(x) \Rightarrow \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)} f'(x) + g'(x)\log f(x)\right]$$

$$\Rightarrow \frac{dy}{dx} = f(x)^{g(x)}\left[\frac{g(x)}{f(x)} f'(x) + g'(x)\log f(x)\right]$$

METHOD OF SUBSTITUTION

In the case of inverse trigonometric functions with typical arguments direct differentiation of the function to find the derivative may become quite tedious and cumbersome. In such cases the derivative can be easily found by using proper trigonometric substitutions and transformations. The application of this method is illustrated in the following examples.

DERIVATIVE OF ONE FUNCTION W.r.t ANOTHER FUNCTION

Let f(x) and g(x) be two differentiable functions with a common domain of differentiability then the derivative of u = f(x) w.r.t v = g(x) is given by

$$\frac{du}{dv} = \left(\frac{du}{dx}\right)\Big/\left(\frac{dv}{dx}\right) = \frac{f'(x)}{g'(x)}$$

EXERCISE

1. Find the derivatives of the following functions.

i) $\sin^{-1}\left(3x - 4x^3\right)$

sol : Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$now\ y = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right) = \sin^{-1}(\sin 3\theta) = 3\,\theta = 3\sin^{-1} x.$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{3}{\sqrt{1 - x^2}}$$

ii) $\cos^{-1}\left(4x^3 - 3x\right)$

sol : Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$ and $y = \cos^{-1}\left(4\cos^3\theta - 3\cos\ \theta\right)$

$= \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1} x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{3}{\sqrt{1 - x^2}}$$

iii) $\sin^{-1}\left(\frac{2x}{1 + x^2}\right)$

sol : Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

and $y = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\tan^{-1} x;$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

iv) $\tan^{-1}\left(\frac{a-x}{1+ax}\right)$

sol : Put $a = \tan\alpha, x = \tan\theta$ then $\theta = \tan^{-1}x$ and $\alpha = \tan^{-1}a$

$$y = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}\right)$$

$$= \tan^{-1}(\tan(\alpha-\theta)) = \alpha - \theta$$

$$= \tan^{-1}a - \tan^{-1}x;$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{1+x^2} = -\frac{1}{1+x^2}$$

v) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

sol : $\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2}$

$$y = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

Differentiating w.r. to x,we get

$$\frac{dy}{dx} = \frac{1}{2}$$

vi) $\sin\left[\cos\left(x^2\right)\right]$

sol : diff . w.r.t x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\cos\left[\cos\left(x^2\right)\right]\frac{d}{dx}\left[\cos\left(x^2\right)\right]$$

$$=\cos\left[\cos\left(x^2\right)\right].\left[-\sin\left(x^2\right)\right]\frac{d}{dx}\left(x^2\right)$$

$$=\cos\left[\cos\left(x^2\right)\right]\left[-\sin\left(x^2\right)\right].2x$$

$$=-2x.\sin\left(x^2\right).\cos\left[\cos\left(x^2\right)\right]$$

vii) $\sec^{-1}\left(\frac{1}{2x^2-1}\right)\left(0<x<\frac{1}{\sqrt{2}}\right)$

sol : put $x=\cos\theta$, then $\theta=\cos^{-1}x$ and $2x^2-1=2\cos^2\theta-1=\cos 2\theta$

$$y=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)=\sec^{-1}(\sec 2\theta)=2\theta$$

$$=2\cos^{-1}x$$

diff . w.r.t x, we get

$$\frac{dy}{dx}=2\left(\frac{-1}{\sqrt{1-x^2}}\right)=\frac{-2}{\sqrt{1-x^2}}$$

viii) $\sin\left[\tan^{-1}\left(e^{-x}\right)\right]$

Diff. w.r.t. x,

$$\frac{dy}{dx}=\frac{d}{dx}\sin\left[\tan^{-1}\left(e^{-x}\right)\right]=\cos\left[\tan^{-1}\left(e^{-x}\right)\right].\left[\tan^{-1}\left(e^{-x}\right)\right]^1$$

$$= \cos\left(\tan^{-1}\left(e^{-x}\right)\right) - \frac{1}{1+\left(e^{-x}\right)^2}\left(e^{-x}\right)^1 \quad = \frac{-e^{-x}}{1+e^{-2x}}.\cos\left[\tan^{-1}\left(e^{-x}\right)\right]$$

2. Differentiate $f(x)$ with respect to $g(x)$ for the following .

i) $f(x) = e^x, g(x) = \sqrt{x}$

sol : Let $y = e^x$ *and* $z = \sqrt{x}$ then $\frac{dy}{dx} = e^x$ *and* $\frac{dz}{dx} = \frac{1}{2\sqrt{x}}$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{e^x}{\left(\frac{1}{2\sqrt{x}}\right)} = 2\sqrt{x}.e^x$$

ii) $f(x) = e^{\sin x}, g(x) = \sin x.$

sol : Let $y = e^{\sin x}$ *and* $z = \sin x.$ then $\frac{dy}{dx} = e^{\sin x}.\cos x$ and $\frac{dz}{dx} = \cos x.$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{e^{\sin x}.\cos x}{\cos x} = e^{\sin x.}$$

iii) $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), g(x) = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$

sol : Let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *and* $z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$ then $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$

$$z = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$y = z \Rightarrow \frac{dy}{dz} = 1$$

3. **If** $y = e^{a\sin^{-1}x}$ **the prove that** $\dfrac{dy}{dx} = \dfrac{ay}{\sqrt{1-x^2}}$

Sol : $y = e^{a\sin^{-1}x}$ Diff. w. r. t . x, we get

$$\frac{dy}{dx} = \frac{d}{dx}e^{a\sin^{-1}x} = e^{a\sin^{-1}x}\frac{d}{dx}\left(a\sin^{-1}x\right)$$

$$= e^{a\sin^{-1}}.a\frac{1}{\sqrt{1-x^2}} \quad = \frac{ay}{\sqrt{1-x^2}}$$

II.

1. **Find the derivatives of the following functions.**

i) **y** = $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a\left(a^2 - 3x^2\right)}\right)$

sol : Put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}(x/a)$

$$y = \tan^{-1}\frac{3a^2x - x^3}{a^3 - 3ax^2} = \tan^{-1}\frac{3a^2(a\tan\theta) - (a\tan\theta)^3}{a^3 - 3a(a\tan\theta)^2}$$

$$= \tan^{-1}\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a\left(a^2\tan^2\theta\right)} = \tan^{-1}\frac{a^3\left(3\tan\theta - \tan^3\theta\right)}{a^3\left(1 - 3\tan^2\theta\right)}$$

$$= \tan^{-1}\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan^{-1}\tan 3\theta$$

$$\Rightarrow y = 3\theta = 3.Tan^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}3.Tan^{-1}\left(\frac{x}{a}\right) = 3.\frac{1}{1 + \dfrac{x^2}{a^2}}.\left(\frac{1}{a}\right) = \frac{3a}{x^2 + a^2}$$

ii) $\tan^{-1}(\sec x + \tan x)$

sol : $\sec x + \tan x = \dfrac{1+\tan^2 x}{1-\tan^2\frac{x}{2}} + \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

$$= \frac{1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{\left(1+\tan\frac{x}{2}\right)^2}{\left(1-\tan\frac{x}{2}\right)\left(1+\tan\frac{x}{2}\right)}$$

$$= \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{x}{2}} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right) = \frac{\pi}{4}+\frac{x}{2};$$

Diff. w.r.t x, we get,

$$\frac{dy}{dx} = 0 + \frac{1}{2} = \frac{1}{2}$$

iii) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

sol : Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$ and

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} = \frac{\sec\theta - 1}{\tan\theta}$$

$$= \frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2} \quad y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x\right)=\frac{1}{2\left(1+x^2\right)}$$

iv) $(\log x)^{\tan x}$

sol : *let* $y=(\log x)^{\tan x}$

taking logs on both sides,

$$\log y=\log(\log x)^{\tan x}$$

$$=(\tan x).\log(\log x)$$

Differentiating w.r.to x

$$\frac{1}{y}.\frac{dy}{dx}=\tan x.\frac{dy}{dx}(\log(\log x))+\log(\log x)\frac{d}{dx}(\tan x)$$

$$=\tan x.\frac{1}{\log x}.\frac{1}{x}+\log(\log x).\sec^2 x$$

$$\frac{dy}{dx}=y\left(\frac{\tan x}{x\log x}+\log(\log x).\sec^2 x\right)=(\log x)^{\tan x}$$

$$\left(\frac{\tan x}{x\log x}+\log(\log x).\sec^2 x\right)$$

v) $\left(x^x\right)^x=x^{x^2}$

let $y=x^{x^2}$

Applying logs on both sides,

$$\log y=\log x^{x^2}=x^2.\log x$$

Differentiating w.r.to x

$$\frac{1}{y}.\frac{dy}{dx} = x^2.\left(\frac{\log}{x}x\right) + (\log x)\frac{d}{dx}\left(x^2\right)$$

$$= x^2.\frac{1}{x} + 2x.\log x$$

$$= x + 2x\log x = x(1 + 2\log x) = x\left(\log e + \log x^2\right)$$

$$= x.\log(e)x^2 \quad \frac{dy}{dx} = y.x.\log\left(ex^2\right) \quad = x^{x^2}.x.\log\left(ex^2\right) = x^{x^2+1} + \log\left(ex^2\right)$$

vi) $20^{\log(\tan x)}$

sol : $let\ \ y = 20^{\log(\tan x)}$

Applying logs on both sides,

$$\log y = \log(20)^{\log(\tan x)}$$

$$\mathbf{Y} = \log(\tan x)\log 20$$

Differentiating w. r. to x,

$$\frac{1}{y}.\frac{dy}{dx} = \log 20\frac{1}{\tan x}\sec^2 x \quad = \log 20\frac{\cos x}{\sin x}.\frac{1}{\cos^2 x}$$

$$= \frac{2\log 20.}{2\sin x.\cos x} = \frac{2\log 20}{\sin 2x} = (2\log 20).\cos ec\ 2x$$

$$\frac{dy}{dx} = y.(2\log 20).\cos ec\ 2x$$

$$= 20^{\log\tan x}(2\log 20).\cos ec\,2x$$

vii) $x^x + e^{e^x}$

sol : Let $u = x^x and\ v = e^{e^x}$ so that $y = u + v$

$u = x^x \Rightarrow \log u = \log x^x = x\log x$

Diff w.r.t x, we get

$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{x} + \log x = 1 + \log x$

$\frac{du}{dx} = u(1+\log x) = x^x(1+\log x)$ $v = e^{e^x} \Rightarrow \log v = \log e^{e^x} = e^x.\log e = e^x$

Diff. w.r.t x,

$\frac{1}{v}.\frac{dv}{dx} = e^x \Rightarrow \frac{dv}{dx} = v.e^x = e^{e^x}.e^x$

$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= x^x(1+\log x) + e^{e^x}.e^x$

viii) $x.\log.\log(\log x)$

sol : $y = x.\log.\log(\log x)$

diff. wrt x,

$\frac{dy}{dx} = \frac{d}{dx}x.\log.\log(\log x)$

$= x.\log x\frac{d}{dx}\left(\log.(\log x) + \log(\log x)\log x.1 + x.\log(\log x)\frac{1}{x}\right)$

$= x\log x.\frac{1}{\log x}.\frac{1}{x} + \log x.\log(\log x) + \log(\log x)$

$= 1 + \log(\log x)(1+\log x) = 1 + \log(\log x) + \log x\log(\log x)$

$= \log e + \log(\log x) + \log x.\log(\log x)$

$= \log(e\log x) + \log x.\log(\log x)$

ix) $e^{-ax^2}.\sin(x\log x)$

sol : let $y = e^{-ax^2}.\sin(x\log x)$

differtiate w.r.t. x,

$$\frac{dy}{dx} = \frac{d}{dx}e^{-ax^2}.\sin(x\log x)$$

$$= e^{-ax^2}.\frac{d}{dx}(\sin(x\log x)) + \sin(x\log x)\frac{d}{dx}\left(e^{-ax^2}\right)$$

$$= e^{-ax^2}\cos(x\log x).\left(x.\frac{1}{x} + \log x\right) + \sin(x\log x)e^{-ax^2(-2ax)}$$

$$= e^{-ax^2}(\cos(x\log x)(1+\log x)) - 2ax.\sin(x\log x)$$

$$= e^{-ax^2(\cos(x\log x)\,(\log ex) - 2ax.\sin\,(x\log x))}$$

x) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

sol : $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

Put $2^x = \tan\theta$, then $\theta = \tan^{-1}2^x$

$$y = \sin^{-1}\left(\frac{2-2^x}{1+\left(2^x\right)^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) \;\; = 2\theta = 2\tan^{-1}2^x$$

$$\frac{dy}{dx} = \frac{d}{dx}2\tan^{-1}2^x = 2\frac{1}{1+\left(2^x\right)^2}\frac{d}{dx}2^x$$

$$= \frac{2}{1+4^x}2^x\log 2$$

$$= \frac{2^{x+1}.\log 2}{\left(1+4^x\right)}$$

2. Find $\frac{dy}{dx}$ for the following functions.

i) $x = 3\cos t - 2\cos^3 t,$

$y = 3\sin t - 2\sin^3 t$

Sol : $\frac{dx}{dt} = \frac{d}{dt} 3\cos t - 2\cos^3 t = -3\sin t + 6\cos^2 t(\sin t)$

$= -3\sin t + 6\cos^2 t(\sin t) = 3\sin t\left(2\cos^2 t - 1\right)$

$= 3\sin.\cos 2t \quad y = 3\sin t - 2\sin^3 t$

$\frac{dy}{dt} = \frac{d}{dt}\left(3\sin t - 2\sin^3 t\right)$

$= 3\cos t - 2\left(3\sin^2 t\right)(-\cos t)$

$= 3\cos t\left(1 - 2\sin^2 t\right) = 3\cos t.\cos 2t$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3\cos t - \cos 2t}{3\sin t - \cos 2t} = \cot t$$

ii) $x = \frac{3at}{1+t^3}.y = \frac{3at^2}{1+t^2}$

sol : $\frac{dx}{dt} = \frac{d}{dt}\frac{3at}{1+t^3} = \frac{\left(1-t^3\right)3a - 3at\left(3t^2\right)}{\left(1+t^3\right)^2}$

$$= \frac{3a\left(1+t^3-3t^3\right)}{\left(1+t^3\right)^2} = \frac{3a\left(1-2t^3\right)}{\left(1+t^3\right)^2}$$

$$y = \frac{3at^2}{\left(1+t^3\right)}$$

$$\frac{dy}{dt} = \frac{d}{dt}\frac{3at^2}{\left(1+t^3\right)} = \frac{\left(1+t^3\right)(6at) - 3at\left(3t^2\right)}{\left(1+t^3\right)^2}$$

$$= \frac{3at\left(2+2t^3-3t^3\right)}{\left(1+t^3\right)^2} = \frac{3at\left(2-t^3\right)}{\left(1-t^3\right)^2}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3at\left(2-t^3\right)}{\left(1+t^3\right)^2}.\frac{\left(1-t^3\right)^2}{3a\left(1-2t^3\right)}$$

$$= \frac{t\left(2-t^3\right)}{1-2t^3}$$

iii) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$

sol : ans $=$ $\tan t$

iv) $x = a\frac{\left(1-t^2\right)}{1+t^2}, y = b\frac{2t}{1+t^2}$

sol : $ans = \frac{2b\left(1-t^2\right)}{\left(1+t^2\right)^2}.\frac{\left(1+t^2\right)^2}{-4at}$

3. Differentiate $f(x)$ with respect to $g(x)$ for the following.

i) $f(x) = \log_a^x,\ g(x) = a^x.$

sol : $f = \log_a^x = \dfrac{\log x}{\log_e^a}$

diff.w.r.t x, $\dfrac{df}{dx} = \dfrac{1}{x\log_e^a}$

$g = a^x \Rightarrow \dfrac{dg}{dx} = a^x.\log_e^a$

$$\frac{df}{dg} = \frac{\left(\frac{df}{dx}\right)}{\left(\frac{dg}{dx}\right)} = \frac{\frac{1}{x.\log_e^a}}{a^x\log_e^a} = \frac{1}{x.a^x\left(\log_e^a\right)^2}$$

ii) $f(x) = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right), g(x) = \sqrt{1-x^2}$

sol : Let $f = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ *and* $g = \sqrt{1-x^2}$

Put $x = \cos\theta$ **then** $\theta = \cos^{-1}x$

$$f = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x$$

$$\frac{df}{dx} = \frac{d}{dx}2\cos^{-1}x = 2.\frac{-1}{\sqrt{1-x^2}}$$

$$g = \sqrt{1-x^2} \Rightarrow \frac{dg}{dx} = \frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{2\sqrt{1-x^2}}.\frac{d}{dx}\left(1-x^2\right) = \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}}(-x)$$

$$\therefore \frac{df}{dg} = \frac{\left(\frac{df}{dx}\right)}{\left(\frac{dg}{dx}\right)} = \frac{2}{x}$$

iii) $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), g(x) = \tan^{-1} x.$

sol : **Let** $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ $and\ z = \tan^{-1} x$

put $x = \tan z$ **then z = tan** $^{-1}$**x and** $y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 z}-1}{\tan z}\right)$

$$= \tan^{-1}\left(\frac{\sec z - 1}{\tan z}\right) = \tan^{-1}\left[\frac{\frac{1}{\cos z}}{\frac{\sin z}{\cos z}}\right] = \tan^{-1}\left(\frac{1-\cos z}{\sin z}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{z}{2}}{2\sin\frac{z}{2}.\cos\frac{z}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{z}{2}\right) = \frac{z}{2} = \frac{1}{2}\tan^{-1} x$$

$$\therefore \frac{dy}{dz} = \frac{d\left(\frac{1}{2}\tan^{-1} x\right)}{d\left(\tan^{-1} x\right)} = \frac{1}{2}$$

4. Find the derivatives of the function y defined implicitly by each of the following equations.

i) $x^4 + y^4 - a^2\ xy = 0$

sol : Differentiate w. r. to x

$$\frac{d}{dx}\left(x^4 + y^4 - a^2\ xy\right) = 0$$

$$4x^3 + 4y^3.\frac{dy}{dx} - a^2\left(x.\frac{dy}{dx} + y.1\right) = 0$$

$$4x^3 + 4y^3.\frac{dy}{dx} - a^2 x\frac{dy}{dx} - a^2\ y = 0$$

$$\left(4y^3 - a^2 x\right)\frac{dy}{dx} = a^2 y - 4x^3\ \frac{dy}{dx} = \frac{a^2 y - 4x^3}{4y^3 - a^2 x}$$

ii) $y = x^y$

applying logs on both sides,

sol : $\log y = \log x^y = y \log x$

Differentiate w. r. to x

$$\frac{1}{y}.\frac{dy}{dx} = \log x.\frac{dy}{dx} + y.\frac{1}{x}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} - \log x.\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}; \frac{1 - y\log x}{y}\ \frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)} = \frac{y^2}{x(1 - \log y)}\ [by(1)]$$

iii) $y^x = x^{\sin y}$

sol : Taking logs on both sides

$$\log y^x = \log x^{\sin y} \Rightarrow x.\log y = (\sin y)\log x$$

Differentiating w. r. to x

$$x.\frac{1}{y}.\frac{dy}{dx}+\log y=\sin y.\frac{1}{x}+\log x.\cos y\frac{dy}{dx}$$

$$\Rightarrow\left(\frac{x}{y}-\log x.\cos y\right).\frac{dy}{dx}=\frac{\sin y}{x}-\log y$$

$$\Rightarrow\frac{x-y\log x.\cos y}{y}.\frac{dy}{dx}=\frac{\sin y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y(\sin y-x\log y)}{x(x-y\log x.\cos y)}$$

i) **If** $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, **then** $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

sol : Given $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Put $x=\sin\theta, y=\sin\varphi$ then $\theta=\sin^{-1}x$ *and* $\varphi=\sin^{-1}y$

$$\therefore\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\varphi}=a(\sin\theta-\sin\varphi)$$

$$\cos\theta+\cos\varphi=a(\sin\theta-\sin\varphi)$$

$$2\cos\frac{\theta+\varphi}{2}.\cos\frac{\theta-\varphi}{2}=a\left[2\cos\frac{\theta+\varphi}{2}.\sin\frac{\theta-\varphi}{2}\right]$$

$$\therefore\cos\frac{\theta-\varphi}{2}=a.\sin\frac{\theta-\varphi}{2}$$

$$\Rightarrow\tan\frac{\theta-\varphi}{2}=\frac{1}{a}\Rightarrow\frac{\theta-\varphi}{2}=\tan^{-1}\left(\frac{1}{a}\right)$$

$$\varphi=\theta-2\tan^{-1}\left(\frac{1}{a}\right)\Rightarrow\sin^{-1}y=\sin^{-1}x-2\tan^{-1}\left(\frac{1}{a}\right)$$

Differentiating w. r. to x

$$\frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

ii) **If** $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{x^2+a^2}\right)$, **show that** $\frac{dy}{dx} = 2\sqrt{a^2+x^2}$

sol : $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)$

diff. w.r.t x,

$$\frac{d}{dx}y = \frac{d}{dx}\left(x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)\right)$$

$$\Rightarrow \frac{dy}{dx} = x.\frac{1}{2\sqrt{a^2+x^2}}\sqrt{2}\,x + \sqrt{a^2+x^2}.1 + \frac{a^2}{x+\sqrt{a^2+x^2}}\left(1-\frac{1}{2\sqrt{a^2+x^2}}2x\right)$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2} + \frac{a^2}{x+\sqrt{a^2+x^2}}.\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2} + \frac{a^2}{\sqrt{a^2+x^2}}$$

$$= \frac{x^2+a^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2}$$

$$= \sqrt{a^2+x^2} + \sqrt{a^2+x^2} = 2\sqrt{a^2+x^2}$$

iii) **If** $x^{\log y} = \log x$, **show that** $\frac{x}{y}.\frac{dy}{dx} = \frac{1-\log x.\log y}{(\log x)^2}$

sol :

Given problem

$x^{\log y} = \log x$

applying logs on both sides,

$\Rightarrow \log x^{\log y} = \log \log x$

$(\log y)(\log x) = \log(\log x)$.

Differentiating w. r. to x

$$\frac{d}{dx}(\log y)(\log x) = \frac{d}{dx}\log(\log x)$$

$$\Rightarrow \log x.\frac{1}{y}.\frac{dy}{dx} + \log y.\frac{1}{x} = \frac{1}{\log x}.\frac{1}{x}$$

$$\Rightarrow \frac{\log x}{y}.\frac{dy}{dx} = \frac{1}{x\log x} - \frac{1}{x}.\log y$$

$$= \frac{1-\log x.\log y}{x\log x}; \frac{x}{y}\frac{dy}{dx} = \frac{1-\log x.\log y}{(\log x)^2}$$

iv) **If** $y = Tan^{-1}\left(\frac{2x}{1-x^2}\right) + Tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) - \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$ **show that** $\frac{dy}{dx} = \frac{1}{1+x^2}$

Sol : $y = Put x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) - \tan^{-1}\left(\frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) + \tan^{-1}(\tan 3\theta) + \tan^{-1}(\tan 4\theta)$$

$$= 2\theta + 3\theta - 4\theta = \theta = \tan^{-1}x$$

$$\frac{d}{dx}y = \frac{d}{dx} = \tan^{-1}x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

v) **If** $x^y = y^x$, **show that** $\frac{dy}{dx} = \frac{y(x\log y - y)}{x(y\log x - x)}$

sol : Given $x^y = y^x; \log x^y = \log y^x$

$y\log x = x\log y$

Differentiating w. r. to x

$$\frac{d}{dx} y\log x = \frac{d}{dx} x\log y$$

$$\Rightarrow y.\frac{1}{x}\log x.\frac{dy}{dx} = x\frac{1}{y}.\frac{dy}{dx} + \log y$$

$$\therefore \log x.\frac{dy}{dx} - \frac{x}{y}.\frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\Rightarrow \left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\Rightarrow \frac{y\log x - x}{y}.\frac{dy}{dx} = \frac{x\log y - y}{x}$$

$$\frac{dy}{dx} = \frac{y(x\log y - y)}{x(y\log x - x)}$$

vi) **If** $x^{2/3} + y^{2/3} = a^{2/3}$ then $\frac{dy}{dx} = -3\sqrt{y / x}$

sol : Given $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w. r. to x

$$\frac{d}{dx} x^{2/3} + y^{2/3} = \frac{d}{dx} a^{2/3}$$

$$\frac{2}{3}x^{2/3} + \frac{2}{3}y^{-1/3}.\frac{dy}{dx} = 0$$

$$\frac{2}{3}\cdot\frac{1}{y^{1/3}}\cdot\frac{dy}{dx} = -\frac{2}{3}\cdot\frac{1}{x^{1/3}}$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}} = -3\sqrt{y / x}$$

6. Find the derivative $\frac{dy}{dx}$ of the following functions.

i) $y = \frac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}}$

sol : applying logs on both sides,

$$\log y = \log\left\{\frac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}}\right\}$$

$$= \log(1-2x)^{2/3} + \log(1+3x)^{-3/4} - \log(1-6x)^{5/6} - \log(1+7x)^{-6/7}$$

$$= \frac{2}{3}\log(1-2x) - \frac{3}{4}.\log(1+3x) - \frac{5}{6}\log(1-6x) + \frac{6}{7}\log(1+7x)$$

Differentiating w. r. to x

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left(\frac{2}{3}\log(1-2x) - \frac{3}{4}.\log(1+3x) - \frac{5}{6}\log(1-6x) + \frac{6}{7}\log(1+7x)\right)$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{2}{3}.\frac{1(-2)}{1-2x} - \frac{3}{4}.\frac{1}{1+3x}.3 - \frac{5}{6}.\frac{1}{1-6x}.(-6) + \frac{6}{7}.\frac{1}{1+7x}.7$$

$$= \frac{-4}{3(1-3x)} - \frac{9}{4(1+3x)} + \frac{5}{1-6x} + \frac{5}{1+7x}$$

$$\frac{dy}{dx} = y\left(\frac{5}{1-6x} + \frac{6}{1+7x} - \frac{4}{3(1-2x)} - \frac{9}{4(1+2x)}\right)$$

ii) $y = \dfrac{x^4 , \sqrt[3]{x^2+4}}{\sqrt{4x^2-7}} = \dfrac{x^4 . \left(x^2+4\right)^{1/3}}{\left(4x^2-7\right)^{1/2}}$

sol : $ans = y\left(\dfrac{4}{x} + \dfrac{2x}{3\left(x^2+4\right)} - \dfrac{4x}{4x^2-7}\right)$

iii) $y = \dfrac{(a-x)^2 (b-x)^3}{(c-2x)^3}$ $ans = y\left[\dfrac{6}{c-2x} - \dfrac{3}{b-x} - \dfrac{2}{a-x}\right]$

iv) $\log y = \log \dfrac{x^3 (2+3x)^{1/2}}{(2+x)(1-x)}$ $ans = y\left[\dfrac{3}{x} + \dfrac{3}{2(2+3x)} - \dfrac{1}{2+x} + \dfrac{1}{1-x}\right]$

v) $y = \sqrt{\dfrac{(x-3)\left(x^2+4\right)}{3x^2+4x+5}}$

sol : applying logs

$$\log y = \log\left(\frac{(x-3)\left(x^2+4\right)}{3x^2+4x+5}\right)^{1/2}$$

$$= \frac{1}{2}\log \frac{(x-3)\left(x^2+4\right)}{3x^2+4x+5} = \frac{1}{2}\left(\log(x-3) + \log\left(x^2+4\right) - \log\left(3x^2+4x+5\right)\right)$$

Differentiating w.r.t x,

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right)$$

III.

1. Find the derivative of the following functions.

i) $y = (\sin x)^{\log x} + x^{\sin x}$

sol : **Let** $u = (\sin x)^{\log x}, v = x^{\sin x}$ **so that** $y = u + v$,

$u = (\sin x)^{\log x}$ applying logs,

$\Rightarrow \log u = \log\{(\sin x)^{\log x}\} = \log x.\log(\sin x)$

Differentiating w. r. to x

$\Rightarrow \frac{d}{dx}\log u = \frac{d}{dx}\log x.\log(\sin x)$

$\frac{1}{u}.\frac{dy_1}{dx} = \log x\frac{1}{\sin x}.\cos x + \log(\sin x)\frac{1}{x}$

$\frac{dy_1}{dx} = u\left[\cot x.\log x + \frac{\log(\sin x)}{x}\right]$

$= (\sin x)^{\log x}\left[\cot x.\log x. + \frac{\log \sin x}{x}\right]$

$v = x^{\sin x}$

$\log v = (\log x)^{\sin x} = \sin x.\log x$

$\frac{1}{v}.\frac{dv}{dx} = \sin x.\frac{1}{x} + (\log x.\cos x)$

$\frac{dv}{dx} = v\left[\frac{\sin x}{x}.\cos x.\log x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x.\log x\right]$

$\sin ce\ \ y = u+v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= (\sin x)^{\log x}\left(\cot x.\log x. + \frac{\log \sin x}{x}\right) + x^{\sin x}\left(\frac{\sin x}{x} + \cos x.\log x\right)$

ii) $\left(x^x\right)^x$

$let \quad y = \left(x^x\right)^x$

$\Rightarrow \log y = \log x^{\left(x^x\right)} = x^x.\log x$

Diff. w.r.t x, we get,

$\frac{1}{y}.\frac{dy}{dx} = x^x.\frac{1}{x} + (\log x)\, x^x\,(1+\log x)$

$\left[\frac{d}{dx}\left(x^x\right) = x^x\,(1+\log x)\right]$

$= x^{x-1}\,[1 + x\log x\,(\log e + \log x)]$

$= x^{x-1}\,(1 + x.\log x.\log ex)$

$\frac{dy}{dx} = y.x^{x-1}(1 + x\log x.\log ex) = x^{\left(x^x\right)}.x^{x-1}(1 + x\log x.\log ex)$

$= x^{x^x + x-1}(1 + x\log x.\log ex)$

iii) $(\sin x)^x + x^{\sin x}$

$ans = (\sin x)^x\,(x\cot x + \log(\sin x)) + x^{\sin x}\left(\frac{\sin x}{x} + \cos x.(\log x)\right)$ **try yourself**

iv) $x^x + (\cot x)^x$

sol : try yourself $ans = x^x(1+\log x) + (\cot x)^x$

2. Establish the following

i) If $x^y + y^x = a^b$, **show that** $\frac{dy}{dx} = -\left(\frac{y.x^{y-1} + y^x.\log y}{x^y.\log x + x.y^{x-1}}\right)$

sol : Let $u = x^y$ *and* $v = y^x$. **so that** $u + v = a^b$

$$u = x^y \Rightarrow \log u = \log x^y = y\log x$$

Diff. w.r.t x, we get

$$\frac{1}{y_1}.\frac{dy_1}{dx} = y.\frac{1}{x} + \log x.\frac{dy}{dx}, \frac{dy_1}{dx} = y_1\left(\frac{y}{x} + \log x.\frac{dy}{dx}\right) = \left(x^y\right)\left(\frac{y}{x} + \log x.\frac{dy}{dx}\right) \text{ ....(1)}$$

$$v = y^x \Rightarrow \log v = \log y^x = x\log y$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx} = x.\frac{1}{y}.\frac{dy}{dx} + \log y.1$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{x}{y}.\frac{dy}{dx} + \log y\right) = y^2\left(\frac{x}{y}.\frac{dy}{dx} + \log y\right) \text{ ......(2)}$$

$$but \quad u + v = a^b \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$\Rightarrow \left(x^y\right)\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + y^x\left(\frac{x}{y}.\frac{dy}{dx} + \log y\right) = 0$$

$$\Rightarrow y.x^{y-1} + x^y.\log x\frac{dy}{dx} + x.y^{x-1}\frac{dy}{dx} + y^x.\log y = 0$$

$$\Rightarrow \frac{dy}{dx}.\left(x^y.\log x + x.y^{x-1}\right) = -\left(y.x^{y-1} + y^x.\log y\right)$$

$$\therefore \frac{dy}{dx} = -\frac{\left(y.x^{y-1} + y^x.\log y\right)}{\left(x^y.\log x + x.y^{x-1}\right)}$$

ii) **If** $f(x) = \sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$ **and** $g(x) = \tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$, **then**

$f'(x) = g'(x)(\beta < x < \alpha)$

Sol : let $y = \sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$

$$\sin y = \sqrt{\frac{x-\beta}{\alpha-\beta}} \Rightarrow \sin^2 y = \frac{x-\beta}{\alpha-\beta}$$

$$\cos^2 y = 1 - \sin^2 y = 1 - \frac{x-\beta}{\alpha-\beta}$$

$$= \frac{\alpha-\beta-x+\beta}{\alpha-\beta} = \frac{\alpha-x}{\alpha-\beta} \Rightarrow \cos y = \sqrt{\frac{\alpha-x}{\alpha-\beta}}$$

$$\therefore \quad \tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{\frac{x-\beta}{\alpha-\beta}}}{\sqrt{\frac{\alpha-x}{\alpha-\beta}}} = \sqrt{\frac{x-\beta}{\alpha-\beta}.\frac{\alpha-\beta}{\alpha-x}} = \sqrt{\frac{x-\beta}{\alpha-x}};$$

$y = \tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$ **i.e.,** $\sin^{-1}\sqrt{\frac{x-\beta}{\alpha-x}} = \tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$

$\therefore$ **The derivatives of** $\sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$ **and** $\tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$ **are same.**

iii) $f(x) = \left(a^2 - b^2\right)^{-1/2}.\cos^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right)$ $a > b > 0$ *and* $0 < x < \pi$; **then**

$f'(x) = (a + b\cos x)^{-1}$

Sol : **Let** $u = \cos^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right)$

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-\frac{(a\cos x+b)^2}{(a+b\cos x)^2}}}\cdot\frac{d}{dx}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

$$\frac{(a+b\cos x)(-a\sin x)-(a\cos x+b)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{a+b\cos x}{\sqrt{(a+b\cos x)^2-(a\cos x+b)^2}}\frac{(a+b\cos x)(-a\sin x)-(a\cos x+b)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{\left(a^2-b^2\right)\sin x}{(a+b\cos x)\sqrt{\left(a^2-b^2\right)\left(1-\cos^2 x\right)}}$$

$$= \frac{\left(a^2-b^2\right)\sin x}{(a+b\cos x)\sqrt{\left(a^2-b^2\right)}.\sin x} = \frac{\sqrt{a^2-b^2}}{a+b\cos x}$$

$$f'(x) = \left(a^2-b^2\right)^{-1/2}\frac{\sqrt{a^2-b^2}}{a+b\cos x} = \frac{1}{a+b\cos x} = (a+b\cos x)^{-1}$$