DERIVATIVES OF INVERSE TRIGNOMETRIC FUNCTIONS

If y = Sin^{-1} x, x∈ (–1, 1) then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Proof:

$y = \mathrm{Sin}^{-1}x \Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, |x| \le 1$

THEOREM

$y = \mathrm{Cos}^{-1}x$ x∈ (–1, 1) then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

THEOREM

$y = \mathrm{Tan}^{-1}x$,x∈ R then $\frac{dy}{dx} = \frac{1}{1+x^2}$

proof;

$y = \mathrm{Tan}^{-1} x \Rightarrow x = \tan y$ where x∈ R and $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$x = \tan y \Rightarrow \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2 > 0.$

$\therefore \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{1+x^2}, x \in R$

Theorem

$y = \mathrm{Cot}^{-1}x$ x∈ R then $\frac{dy}{dx} = \frac{-1}{1+x^2}$

Theorem

$y = \mathrm{Sec}^{-1} x$, x∈ $(-\infty,-1)\cup(1,\infty)$ then $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

Theorem

$$\frac{d}{dx}(\text{Cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}},|x|>1$$

DERIVATIVES OF HYPERBOLIC FUNCTIONS

1. $\frac{d}{dx}(\sinh x)=\cosh x. x\in R$

$$\frac{d}{dx}(\sinh x)=\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)$$

$$=\frac{1}{2}\left[\frac{d}{dx}(e^x)-\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}\left[e^x+e^{-x}\right]$$

= coshx, exists for all x.

2. $\frac{d}{dx}(\cosh x)=\sinh x$,x€ R

$$\frac{d}{dx}(\cosh x)=\frac{d}{dx}\left[\frac{1}{2}(e^x+e^{-x})\right]$$

$$=\frac{1}{2}\left[\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{-x})\right]$$

$$=\frac{1}{2}(e^x-e^{-x})=\sinh x,\forall x$$

3. $\frac{d}{dx}(\tanh x)=\text{sech}^2 x, x\in R$

$$\frac{d}{dx}(\tanh x)=\frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$

$$=\frac{d}{dx}\frac{\cosh x.\frac{d}{dx}(\sinh x)-\sinh x.\frac{d}{dx}(\cosh x)}{\cosh^2 x}$$

$$=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}=\frac{1}{\cosh^2 x}=\text{sech}^2\text{x}$$

4. $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x, x \in$ **R- {0}.**

$\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right)$

$= \frac{\sinh x.\sinh x - \cosh x.\cosh x}{\sinh^2 x}$

$= \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x}$

$= \frac{-1}{\sinh^2 x} =$ - cosech^2x exists for all $x \neq 0$

5. $\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$,x€ R

$\frac{d}{dx}(\text{sech } x) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right)$

$= \frac{-1}{\cosh^2 x}\sinh x$

= – sech x. tanh x exists for all x

6. $\frac{d}{dx}(\text{cosech } x) = -\text{cosech } x.\coth x \quad x \in$ **R- {0}**

$\frac{d}{dx}(\text{cosech } x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{-1}{\sinh^2 x}.\cosh x$

= – cosech x. coth x, exists for all x≠0.

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

1. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}, x \in R$

Let y = $\sinh^{-1}$ x, $x, y \in R$

Then x = sinh y, sinh y is differentiable for all y €R

$\frac{dx}{dy} = \cosh y \text{ and } \cosh y > 0 \,\forall x \in R$

$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} . = \frac{1}{\sqrt{1+x^2}}$ exists $\forall x \in$ R.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}, x \in R$$

2. $\mathbf{\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x \in (1, \infty)}$

3. $\mathbf{\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1}$

Let y = Tanh^{-1} x, x€(–1, 1), y€R

Then x = tanhy. Tanh y is differetniable on R

$$\therefore \frac{dx}{dy} = \text{sech}^2 y \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y} \quad = \frac{1}{1-x^2} \text{ exists for all } x \in (-1, 1).$$

4. $\mathbf{\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, |x| > 1}$

5. $\mathbf{\frac{d}{dx}(\text{sech}^{-1} x) == \frac{-1}{x\sqrt{1-x^2}}, x \in (0, 1)}$

6. $\mathbf{\frac{d}{dx}(\text{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}, x \neq 0}$

Let y = cosech^{-1} x, x€R – {0}, y€R– {0}.

Then x = cosech y, cosech y is differentiable on R – {0}.

$\therefore \frac{dx}{dy}$=-cosech y. coth y

cosech y. coth y > 0 y€R – {0}

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{-1}{\text{cosech } y.\coth y} = \frac{-1}{|\text{cosech } y|.\sqrt{1+\text{cosech}^2 y}}$$

$$= \frac{-1}{|x|\sqrt{1+x^2}}, \forall x \in \text{R -}\{0\}.$$

$$\frac{d}{dx}\text{cos}\,ech^{-1}x = \frac{-1}{|x|\sqrt{1+x^2}}, x \neq 0$$

EXERCISE

1. Find the derivatives of the following function.

i) $y = \cot^n x$

Sol : $\frac{dy}{dx} = \frac{d\left(\cot^n x\right)}{dx} = n.\cot^{n-1} x.\frac{d}{dx}(\cot x)$

$= n.\cot^{n-1} x.\left(-\cos ec^2 x\right)$

$= -n.\cot^{n-1} x.\cos ec^2 x$

ii) $y = \cos ec^4 x$

sol : $\frac{dy}{dx} = \frac{d}{dx}(\cos ec\ x)^4 = 4.\cos ec^3 x.\frac{d}{dx}(\cos ec\ x)$

$= 4.\cos ec^3 x(-\cos ec\, x.\cot x)$

$= -4.\cos ec^4 x.\cot x$

iii) $y = \tan\left(e^x\right)$

sol : $\frac{dy}{dx} = \sec^2\left(e^x\right).\left(e^x\right)^1 = e^x.\sec^2\left(e^x\right)$

iv) $y = \frac{1-\cos 2x}{1+\cos 2x} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\tan^2 x\right) = 2\tan x.\sec^2 x$

v) $y = \sin^m x.\cos^n x$

sol :

$$\frac{dy}{dx} = \frac{d}{dx}\sin^m x\cos^n x = \sin^m x.\frac{d}{dx}\left(\cos^n x\right) + \left(\cos^n x\right)\frac{d}{dx}\left(\sin^m x\right)$$

$$= \sin^m xn\cos^{n-1} x(-\sin x) + \cos^n x.m\sin^{m-1} x.\cos x$$

$$= m.\cos^{n+1} x.\sin^{m-1} x - n.\sin^{m+1} x.\cos^{n-1} x.$$

vi) $y = \sin mx.\cos nx$

sol : $\frac{dy}{dx} = \sin mx\frac{d}{dx}(\cos nx) + (\cos nx)\frac{d}{dx}(\sin mx)$

$$= \sin mx(-n\sin nx) + \cos nx(m\cos mx)$$

$$= m.\cos mx.\cos nx - n.\sin mx.\sin nx$$

vii) $y = x\tan^{-1} x$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(x\tan^{-1} x\right) = x.\frac{d}{dx}\left(\tan^{-1} x\right) + \left(\tan^{-1} x\right)\frac{d}{dx}(x)$

$$= \frac{x}{1+x^2} + \tan^{-1} x. = \frac{x}{1+x^2} + \tan^{-1} x.$$

viii) $y = \sin^{-1}(\cos x) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \frac{\pi}{2} - x$

Sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - x\right) = 0 - 1 = -1$

ix) $y = \log(\tan 5x)$

sol :

$$\frac{dy}{dx} = \frac{d}{dx}(\log\tan 5x) = \frac{1}{\tan 5x}\frac{d}{dx}(\tan 5x)$$

$$= \frac{5\sec^2 5x}{\tan 5x} = 5.\frac{1}{\cos^2 5x.\frac{\sin 5x}{\cos 5x}} = \frac{10}{2\sin 5x.\cos 5x}$$

$$= \frac{10}{\sin 10x} = 10.\cos ec10x$$

x) $y = \sin h^{-1}\left(\frac{3x}{4}\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\sinh^{-1}\frac{3x}{4}\right) = \frac{1}{\sqrt{1+\left(\frac{3x}{4}\right)^2}}\frac{d}{dx}\left(\frac{3x}{4}\right)$

$$= \frac{3}{4}\frac{1}{\sqrt{1+\frac{9x^2}{16}}} = \frac{3}{4.\sqrt{\frac{16+9x^2}{16}}} = \frac{3}{\sqrt{9x^2+16}}$$

xi) $y = \tan^{-1}(\log x)$

sol : $\frac{dy}{dx} = \frac{1}{1+(\log x)^2}.\frac{d}{dx}(\log x) = \frac{1}{x\left(1+(\log x)^2\right)}$

xii) $y = \log\left(\frac{x^2+x+2}{x^2-x+2}\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\log\left(x^2+x+2\right) - \log\left(x^2-x+2\right)\right)$

$$= \frac{1}{x^2+x+2} - \frac{d}{dx}\left(x^2+x+2\right) - \frac{1}{x^2-x+2}\frac{d}{dx}\left(x^2-x+2\right)$$

$$= \frac{2x+1}{x^2+x+2} = \frac{2x-1}{x^2-x+2}$$

$$= \frac{(2x+1)\left(x^2-x+2\right)-(2x-1)\left(x^2+x+2\right)}{\left(x^2+x+2\right)\left(x^2-x+2\right)} 2x^3-2x^2+4x+x^2-x+2$$

$$= \frac{-2x^3-2x^2-4x+x^2+x+2}{\left(x^2+2\right)^2-x^2} = \frac{4-2x^2}{x^4+4x^2+4-x^2} = \frac{4-2x^2}{x^4+3x^2+4}$$

xiii) $y = \log\left(\sin^{-1}\left(e^x\right)\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\left(\log\sin^{-1}e^x\right) = \frac{1}{\sin^{-1}\left(e^x\right)}\frac{d}{dx}\left(\sin^{-1}\left(e^x\right)\right)$

$$= \frac{1}{\sin^{-1}\left(e^x\right)}.\frac{1}{\sqrt{1-\left(e^x\right)^2}}\left(e^x\right) = \frac{e^x}{\sin^{-1}\left(e^x\right)\sqrt{1-\left(e^2\right)^x}}$$

xiv) $y = (\sin x)^2\left(\sin^{-1}x\right)^2$

sol : $\frac{dy}{dx} = \frac{d}{dx}(\sin x)^2\left(\sin^{-1}x\right)^2$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^2\frac{d}{dx}\left(\sin^{-1}x\right)^2 + \left(\sin^{-1}x\right)^2\frac{d}{dx}(\sin x)^2$$

$$= \sin^2 x.\frac{2\sin^{-1}x}{\sqrt{1-x^2}} + \left(\sin^{-1}x\right)^2\ (2\sin x.\cos x)$$

$$= \left(\sin^{-1}x\right)^2(\sin 2x) + 2\sin^2\times\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

xv) $y = \frac{\cos x}{\sin x + \cos x}$

sol : $\frac{dy}{dx} = \frac{d}{dx}\frac{\cos x}{\sin x + \cos x}$

$$\frac{dy}{dx} = \frac{(\sin x + \cos x)\frac{d}{dx}(\cos x) - \cos \times \frac{d}{dx}(\sin x + \cos x)}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x)(-\sin x) - \cos \times (\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= -\frac{\sin^2 x - \sin x \cos x - \cos^2 x + \sin x \cos x}{(\sin x + \cos x)^2}$$

$$= -\frac{1}{\sin^2 x + \cos^2 x + 2\sin x \cos x} = -\frac{1}{1 + \sin 2x}$$

xvi) $y = \frac{x\left(1 + x^2\right)}{\sqrt{1 - x^2}} = \frac{x + x^3}{\sqrt{1 - x^2}}$

sol : $\frac{dy}{dx} = \frac{d}{dx}\frac{x + x^3}{\sqrt{1 - x^2}} = \frac{\sqrt{1 - x^2}\frac{d}{dx}\left(x + x^3\right) - \left(x + x^3\right)\frac{d}{dx}\left(\sqrt{1 - x^2}\right)}{\left(1 - x^2\right)}$

$$= \frac{\sqrt{1 - x^2}\left(1 + 3x^2\right) - \left(x + x^3\right)\frac{1}{2\sqrt{1 - x^2}}(-2x)}{\left(1 - x^2\right)}$$

$$= \frac{\left(1 - x^2\right)\left(1 + 3x^2\right) + x\left(x + x^3\right)}{\left(1 - x^2\right)^{\frac{3}{2}}} = \frac{1 + 3x^2 - x^2 - 3x^4 + x^2 + x^4}{\left(1 - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{1 + 3x^2 - 2x^4}{\left(1 - x^2\right)^{\frac{3}{2}}}$$

xvii) $y = e^{\sin^{-1} x}$

Ans $= \dfrac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$

xviii) $y = \cos\left(\log x + e^x\right)$

sol : $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos\left(\log x + e^x\right) = -\sin\left(\log x + e^x\right)\dfrac{d}{dx}\left(\log x + e^x\right)$

$= -\sin\left(\log x + e^x\right)\left(\dfrac{1}{x} + e^x\right)$

xix) $y = \dfrac{\sin(x+a)}{\cos x}$

sol : $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{\sin(x+a)}{\cos x} = \dfrac{\cos x.\dfrac{d}{dx}[\sin(x+a)] - \sin(x+a)\dfrac{d}{dx}(\cos x)}{\cos^2 x}$

$= \dfrac{\cos x.\cos(x+a) + \sin(x+a).\sin x}{\cos^2 x} = \dfrac{\cos(x+a-x)}{\cos^2 x} = \dfrac{\cos a}{\cos^2 x}$

xx) $y = \cot^{-1}(\operatorname{cosec} 3x)$

sol : $\dfrac{dy}{dx} = \dfrac{d}{dx}\cot^{-1}(\operatorname{cosec} 3x) = -\dfrac{1}{1+\operatorname{cosec}^2}.\dfrac{d}{dx}(\operatorname{cosec} 3x)$

$= -\dfrac{1}{1+\operatorname{cosec}^2 3x}(-\operatorname{cosec} 3x.\cot 3x)\dfrac{d}{dx}(3x) = \dfrac{3.\operatorname{cosec} 3x.\cot 3x}{1+\operatorname{cosec}^2 3x}$

2. Find the derivatives of the following functions with respect to x. ?

i) $x = \sinh^2 y$

sol : differentiating w. r. t y, we get

$$\frac{dx}{dy} = \frac{d}{dy}\sin h^2 y = 2\sin h\, y.\cos h\, y$$

we know that $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{2\sin h\, y \cosh y} = \frac{1}{2\sin h\, y\sqrt{1+\sin h^2 y}}$

$$= \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x+x^2}}$$

ii) $x = \tan h^2\, y$

sol : differentiating w. r. t y, we get

$$\frac{dx}{dy} = \frac{d}{dy}\tan h^2\, y = 2\tan hy.\sec h^2\, y$$

$$\frac{dy}{dx} = \frac{1}{2\tan h\, y.\sec h^2 y}$$

$$y > 0 \Rightarrow \sqrt{x} = \tan h\, y$$

$$\sec h^2 y = 1 - \tan h^2 y = 1 - x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

iii) $x = e^{\sin h\, y}$

Sol : differentiating w.r.t x, we get

$$\frac{dx}{dy} = \frac{d}{dy}e^{\sin h\, y} = e^{\sin h\, y}\frac{d}{dx}(\sin h\, y) = e^{\sin h\, y}.\cos h\, y \quad = x.\cos h\, y$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{dx}{dy}}} = \frac{1}{x.\cos h\, y}$$

iv) $x = \tan\left(e^{-y}\right)$

Ans. $-\frac{e^{y}}{1+x^{2}}$

v) $x = \log\left(1+\sin^{2} y\right)$

sol : differentiating w. r. t y, we get

$$\frac{dx}{dy} = \frac{d}{dy}\log\left(1+\sin^{2} y\right) = \frac{1}{1+\sin^{2} y}\left(\sin^{2} y\right)^{1}$$

$$= \frac{2\sin y - \cos y}{1+\sin^{2} y} = \frac{\sin 2y}{1+\sin^{2} y} = \frac{\sin 2y}{e^{x}}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{e^{x}}{\sin 2y}$$

vi) $x = \log\left(1+\sqrt{y}\right)$ ans. $2\left(y+\sqrt{y}\right)$

II.

1. Find the derivativies of the following functions.

i) $y = \cos[\log(\cot x)]$

sol : $\frac{dy}{dx} = \frac{d}{du}\cos[\log(\cot x)]$

$$= -\sin[\log(\cot x)].\frac{1}{\cot x}\left(-\cos ec^{2} x\right)$$

$$= \frac{1}{\sin^{2} x}.\frac{1}{\cos x}.\sin[\log(\cot\ x)]$$

$$= \frac{\sin[\log(\cot x)].\cos ecx}{\cos x}$$

ii) $y = \sin h^{-1}\left(\frac{1-x}{1+x}\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin h^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{\sqrt{1+\left(\frac{1-x}{1+x}\right)^2}}$

$$\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{1}{\sqrt{\frac{(1+x)^2+(1-x)^2}{1+x}}}.\frac{(1+x)(-1)-(1-x)}{(1+x)^2}$$

$$= \frac{-1-x-1+x}{\sqrt{2\left(1+x^2\right)(1+x)}} = \frac{-2}{\sqrt{2}\,(1+x)\sqrt{1+x^2}}$$

$$= \frac{-\sqrt{2}}{(1+x)\sqrt{1+x^2}}$$

iii) $y = \log\left(\cot\left(1-x^2\right)\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\log\left(\cot\left(1-x^2\right)\right) = \frac{1}{\cot\left(1-x^2\right)}\frac{d}{dx}\left(\cot\left(1-x^2\right)\right)$

$$= \frac{-\cos ec^2\left(1-x^2\right)\frac{d}{dx}\left(1-x^2\right)}{\cot\left(1-x^2\right)} = \frac{-\cos ec^2\left(1-x^2\right)(-2x)}{\cot\left(1-x^2\right)}$$

$$= \frac{2x.\cos ec^2\left(1-x^2\right)}{\cot\left(1-x^2\right)} = 2x.\frac{1}{\sin^2\left(1-x^2\right).\frac{\cos\left(1-x^2\right)}{\sin\left(1-x^2\right)}}$$

$$= \frac{4x}{2\sin\left(1-x^2\right).\cos\left(1-x^2\right)} = \frac{4x}{\sin 2\left(1-x^2\right)} \quad = 4x.\cos ec\left(2\left(1-x^2\right)\right)$$

iv) $y = \sin\left[\cos\left(x^2\right)\right]$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin\left[\cos\left(x^2\right)\right] = \cos\left[\cos\left(x^2\right)\right].\frac{d}{dx}\left[\cos\left(x^2\right)\right]$

$= \cos\left[\cos\left(x^2\right)\right]\left(-\sin\left(x^2\right)\right).\frac{d}{dx}\left(x^2\right)$

$= -2x.\sin\left(x^2\right).\cos\left[\cos\left(x^2\right)\right]$

v) $y = \sin\left[\tan^{-1}\left(e^x\right)\right]$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin\left[\tan^{-1}\left(e^x\right)\right] = \cos\left[\tan^{-1}\left(e^x\right)\right].\frac{d}{dx}\left[\tan^{-1}\left(e^x\right)\right]$

$= \cos\left(\tan^{-1}\left(e^x\right)\right)\left[\frac{1}{1+\left(e^x\right)^2}\right]\frac{d}{dx}\left(e^x\right)$

$= \frac{e^x}{1+e^{2x}}.\cos\left[\tan^{-1}\left(e^x\right)\right]$

vi) $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

sol : $\frac{dy}{dx} = \frac{d}{dx}\frac{\sin(ax+b)}{\cos(cx+d)} = \frac{\cos(cx+d)\frac{d}{dx}[\sin(ax+b)] - \sin(ax+b)\frac{d}{dx}[\cos(cx+d)]}{\cos^2(cx+d)}$

$= \frac{\cos(cx+d)\cos(ax+b).a - \sin(ax+b)[-\sin(cx+d)].c}{\cos^2(cx+d)}$

$= \frac{a.\cos(ax+b).\cos(cx+d) + c.\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$

vii) $y = \tan^{-1}\left(\tan h\frac{x}{2}\right)$

sol : $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left(\tan h\frac{x}{2}\right) = \frac{1}{1+\tan h^2\frac{x}{2}}.\frac{d}{dx}\left(\tan h\frac{x}{2}\right)$

$$= \frac{\sec h^2\frac{x}{2} - \frac{1}{2}}{1+\tan h^2\frac{x}{2}} = \frac{\sec h^2\frac{x}{2}}{2\left(1+\tan h^2\frac{x}{2}\right)}$$

viii) $y = \sin x.\left(Tan^{-1}x\right)^2$

sol : $\frac{dy}{dx} = \frac{d}{dx}\sin x.\left(Tan^{-1}x\right)^2 = \sin x.\frac{d}{dx}\left(Tan^{-1}x\right)^2 + \left(Tan^{-1}x\right)^2.\frac{d}{dx}(\sin x)$

$$= \sin x.\ 2Tan^{-1}x.\frac{1}{1+x^2} + \left(Tan^{-1}x\right)^2.\cos\ x$$

$$= \frac{2\sin xTan^{-1}x}{1+x^2} + \cos\ x.\left(Tan^{-1}x\right)^2$$

III. Find the derivatives of the following functions.

1. $y = \sin^{-1}\left(\frac{b+a\sin x}{a+b\sin x}\right)(a > 0, b > 0)$

Sol : $Let\ u = \frac{b+a\sin x}{a+b\sin x}$

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{b+a\sin x}{a+b\sin x}\right) = \frac{(a+b\sin x)(a\cos x)-(b+a\sin x)(b\cos x)}{(a+b\sin x)^2}$$

$$= \frac{a^2\cos x + ab\sin x\cos x - b^2\cos\ x - ab\sin x\cos x}{(a+b\sin x)^2}$$

$$= \frac{\left(a^2-b^2\right)\cos x}{(a+b\sin x)^2}$$

$now\ \ y = \sin^{-1} u.$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\ \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{b+a\sin x}{a+b\sin x}\right)^2}}\left\{\frac{\left(a^2-b^2\right)\cos x}{(a+b\sin x)^2}\right\}$$

$$= \frac{a+b\sin x}{\sqrt{(a+b\sin x)^2-(b+a\sin x)^2}}\left\{\frac{\left(a^2-b^2\right)\cos x}{(a+b\sin x)^2}\right\}$$

$$= \frac{\left(a^2-b^2\right)\cos x}{\sqrt{a^2+b^2\sin^2 x-b^2-a^2\sin^2 x}}\frac{1}{(a+b\sin x)}$$

$$= \frac{\left(a^2-b^2\right)\cos x}{\sqrt{\left(a^2-b^2\right)-\left(a^2-b^2\right)\sin^2 x}}\ \frac{1}{(a+b\sin x)} = \frac{\sqrt{a^2-b^2}}{(a+b\sin x)}$$

2. $\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)(a>0, b>0)$

Sol : Let $u = \frac{b+a\cos x}{a+b\cos x}$ *then* $y = \sin^{-1}(u)$

$$\frac{du}{dx} = \frac{(a+b\cos x)(-a\sin x)+(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{\sin x\left\{a^2+ab\cos x-b^2-ab\cos x\right\}}{(a+b\cos x)^2} = \frac{\sin x\left(a^2-b^2\right)}{(a+b\cos x)^2}$$

$$\because y = \cos^{-1}(u) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}}.\frac{d}{dx}(u)$$

$$= -\frac{1}{\sqrt{1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2}}.\left\{\frac{\sin x.\left(a^2-b^2\right)}{(a+b\cos x)^2}\right\}$$

$$= \frac{-\sin x.(a^2 - b^2)}{\sqrt{(a + b\cos x)^2 - (b + a\cos x)^2}} \cdot \frac{1}{(a + b\cos x)}$$

$$= \frac{-(a^2 - b^2)\sin x}{\sqrt{a^2 + b^2\cos^2 x - b^2 - a^2\cos^2 x}} \cdot \frac{1}{(a + b\cos x)} = -\frac{\sqrt{a^2 - b^2}}{a + b\cos x}$$

3. $Tan^{-1}\left[\frac{\cos x}{1 + \cos x}\right]$

Sol : Let $u = \frac{\cos x}{1 + \cos x} \Rightarrow \frac{du}{dx} = \frac{(1 + \cos x)(-\sin x) - \cos x(-\sin x)}{(1 + \cos x)^2}$

$$= \frac{-\sin x - \sin x\cos x + \sin x.\cos x}{(1 + \cos x)^2} = \frac{-\sin x}{(1 + \cos x)^2} \quad y = \tan^{-1} u \Rightarrow \frac{dy}{du} = \frac{1}{1 + u^2}\frac{du}{dx}$$

$$= \frac{1}{1 + \frac{\cos^2 x}{(1 + \cos x)^2}} \cdot \frac{-\sin x}{(1 + \cos x)^2} = \frac{(1 + \cos x^2)}{(1 + \cos x)^2 + \cos^2 x} \cdot \frac{-\sin x}{(1 + \cos x)^2}$$

$$= \frac{(1 + \cos x)^2}{2\cos^2 x + 2\cos x + 1} \cdot \frac{-\sin x}{(1 + \cos x)^2}$$

$$= \frac{(1 + \cos x)^2}{2\cos^2 x + 2\cos x + 1} \times \frac{-\sin x}{(1 + \cos x)^2}$$

$$= \frac{-\sin x}{2\cos^2 x + 2\cos x + 1}$$