CHAPTER 9
DIFFERENTIATION

TOPICS

DERIVATIVE OF A FUNCTION

Let f be a function defined in a nbd of a point a. If $\underset{x \to a}{Lt} \dfrac{f(x) - f(a)}{x - a}$ exists finitely, then f is said to be differentiable or derivable at a. In this case, the limit is called the derivative or differential coefficient of f. It is denoted by f '(a). f '(a) = $\underset{x \to a}{Lt} \dfrac{f(x) - f(a)}{x - a}$.

Note: In the above definition, by the substitution x – a=h, we get another equivalent definition for f '(a). i.e., f '(a) = $\underset{x \to a}{Lt} \dfrac{f(x) - f(a)}{x - a}$

Left hand derivative:- Let f be a function defined in a nbd of a point a. If $\underset{x \to a-}{Lt} \dfrac{f(x) - f(a)}{x - a}$ exists finitely then f is said to be differentiable from the left at a. This limit is called the left hand derivative (LHD) or left derivative at a . It is denoted by L f '(a) or f '(a–) or f '(a–0).

L f '(a) f '(a–) = $\underset{x \to a-}{Lt} \dfrac{f(x) - f(a)}{x - a}$

Right hand derivative:- Let f be a function defined in a nbd of a point a. If $\underset{x \to a+}{Lt} \dfrac{f(x) - f(a)}{x - a}$ exists finitely then f is said to be differentiable from the right at a. This limit is called the right hand derivative (RHD) or right derivative at a . It is denoted by R f '(a) or f '(a+) or f '(a + 0).

R f '(a) f '(a+) = $\underset{x \to a+}{Lt} \dfrac{f(x) - f(a)}{x - a}$

Note: f is derivable at a iff f is both left and right derivable at a and f '(a+) = f '(a–)

Derivability of a function in (a,b):

Let f(x) be a function defined on [a,b]. If

f is derivable for all c ∈ (a,b) ,then we say that f is derivable on (a,b).

Derivability of a function in [a,b]:

A function f, defined on [a, b] is said to be differentiable on [a, b] if f is differentiable on (a, b), f is right differentiable at a and f is left differentiable at b.

Note: the process of finding derivative of a function using the definition is called derivative from the first principles.

2. Every differentiable function is continuous but every continuous function is not differentiable.

THEOREM

If a function f is a differentiable at a then it is continuous at a.

Proof:

f is a differentiable at a

=> $f'(a) = \underset{x\to a}{Lt} \frac{f(x)-f(a)}{x-a}$

$\underset{x\to a}{Lt}\,(f(x)-f(a)) = \underset{x\to a}{Lt} \frac{f(x)-f(a)}{x-a}(x-a)\,(\because x \neq a)$

$= \underset{x\to a}{Lt}\left(\frac{f(x)-f(a)}{x-a}\right) \underset{x\to a}{Lt}(x-a)$

$= f'(a).0 = 0$

=> $\underset{x\to a}{Lt} f(x) = f(a)$

$\because$ f is continuous at a.

The converse of the above theorem need not be true. That is, if a function f is continuous at a then it need not be differentiable at a.

e.g., The function f(x) = | x | is continuous but not differentiable at x = 0. For,

$\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt} |x| = 0 = f(0)$

f(x) is continuous at x = 0

Now, $Lf'(0) = \underset{x\to 0-}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x\to 0-}{Lt} \frac{|x|-0}{x-0} = \underset{x\to 0-}{Lt} \frac{-x}{x} = -1$

$Rf'(0) = \underset{x\to 0+}{Lt} \frac{f(x)-f(0)}{x-0} = \underset{x\to 0+}{Lt} \frac{|x|-0}{x-0} = \underset{x\to 0+}{Lt} \frac{+x}{x} = 1$

$\because Lf'(0) = -1\ 1 = Rf'(0)$

$\because$ f(x) is not differentiable at x = 0.

DERIVATIVES OF SOME REAL FUNCTIONS FROM IST PRINCIPLE

1. DERIVATIVE OF CONSTANT

A constant function is differentiable on R and its derivative is equal to zero for all x$\in$ R.

Proof:

Let f (x) = c (c is constant),for all x

$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{c-c}{h} = 0$

$\therefore$ f (x) = c is differentiable and is zero.

Hence the derivative of a constant function is 0.

2. If f(x) = x ,x ∈ R then f '(x) = 1

Proof:

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{x+h-x}{h} = \underset{h\to 0}{Lt} \frac{h}{h} = 1$$

3. If f(x) = x^n, x∈ R then f '(x) = nx^{n-1}

Proof

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{(x+h)^n - x^n}{h}$$

$$= \underset{h\to 0}{Lt} \frac{(x+h)^n - x^n}{(x+h)-x} = n\,.\,x^{n-1}$$

4. If f(x)= e^x ,x∈ Rthen f '(x) = e^x

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{e^{x+h}-e^x}{h}$$

$$= e^x \underset{h\to 0}{Lt} \left(\frac{e^h-1}{h}\right) == e^x\,.1 = e^x$$

5. If f(x) = a^x (a > 0, a≠ 1) x∈ R then

f '(x) = a^x.ln a.

i.e., $\frac{d}{dx}(a^x) = a^x \ln a, \ (x \in R, a > 0, a \neq 1)$

6. If f(x) = ln x ll x > 0 then

$$f'(x) = \frac{\mathbf{1}}{\boldsymbol{x}}$$

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{ln(x+h)-\ln(x)}{h} = \underset{h\to 0}{Lt} \frac{1}{h} ln\left(\frac{x+h}{x}\right)$$

$$= \underset{h\to 0}{Lt}\, \ell n\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \left[\underset{h\to 0}{Lt}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}}$$

$= \ln (e^{1/x}) = \mathbf{\frac{1}{x}}$

7. If f(x) = sin x x ∈ R, then f '(x) = cos x.

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\sin(x+h)-\sin x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{2\cos\left[x+\frac{h}{2}\right]\sin\left(\frac{h}{2}\right)}{h}$$

$$= \underset{h\to 0}{Lt} \cos\left(x+\frac{h}{2}\right) . \underset{h\to 0}{Lt} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

= cos (x + 0) .1 = cos x.

Therefore, $\frac{d}{dx}(\sin x) = \cos x$

8. If f(x) = cos x x ∈ R,then f '(x) = –sin x.

$$\mathbf{f'(x) =} \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\cos(x+h)-\cos x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= -\underset{h\to 0}{Lt} \sin\left(x+\frac{h}{2}\right) \underset{h\to 0}{Lt} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

= – sin (x + 0) . 1= = – sin x.

$\frac{d}{dx}(\cos x) =$– sin x

9. If f(x) = tan x x ∈ R –{(2n + 1), nZ} then f '(x) = sec²x.

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\tan(x+h)-\tan x}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right)$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{\sin(x+h-x)}{\cos x.\cos(x+h)}\right)$$

$$= \underset{h\to 0}{Lt}\frac{\sinh}{h}\underset{h\to 0}{Lt}\frac{1}{\cos x.\cos(x+h)}$$

$$=1.\frac{1}{\cos^2 x} = \sec^2 x.$$

Therefore, $\frac{d}{dx}(\tan x) = \sec^2 x$.

10.f(x) = cot x is differentiable on R –{n, nZ} and f '(x) = – cosec² x.

i.e., $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

11. If f(x) = sec x x ∈ R – {(2n + 1) , nZ} then f '(x) = secx. tanx.

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\sec(x+h)-\sec x}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x.\cos(x+h)}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{2.\sin\left(x+\frac{h}{2}\right).\sin\left(\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{\cos x.\cos(x+h)}\underset{h\to 0}{Lt}\sin\left(x+\frac{h}{2}\right)\underset{h\to 0}{Lt}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$= \frac{1}{\cos^2 x}$ sin x. 1 = sec x. tan x.

$\frac{d}{dx}$(sec x) = sec x. tan x

12. If f(x) = cosec x then f '(x) = – cosec x. cot x.

i.e., $\frac{d}{dx}(\text{cosec}\, x) = -\cos ecx.\cot x$

Theorem

THE DERIVATIVE OF SUM AND DIFFERENCE OF TWO FUNCTIONS

If f and g are two differentiable functions at x, then f + g is differentiable at x and (f + g)' (x) = f '(x) + g'(x).

Proof:

Since f and g are differentiable at x,therefore

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$$

and $\underset{h\to 0}{Lt} \frac{g(x+h)-g(x)}{h} = g'(x)$

Let = f + g in a sufficiently small neighborhood of x. Then

$$\underset{h\to 0}{Lt} \frac{\phi(x+h)-\phi(x)}{h} = \underset{h\to 0}{Lt} \frac{(f+g)(x+h)-(f+g)(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \underset{h\to 0}{Lt} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$$

$$= \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} + \underset{h\to 0}{Lt} \frac{g(x+h)-g(x)}{h}$$

= f '(x) + g'(x) which exists and is finite is differentiable at x and

$$\phi'(x) = \underset{h\to 0}{Lt} \frac{\phi(x+h)-\phi(x)}{h}$$

Hence φ = f + g is differentiable at x and (f + g)' (x) = f '(x) + g'(x).

Similarly,

If f and g are two differentiable functions of x then f – g is differentiable and (f – g)' (x) = f '(x) – g'(x).

Note : 1.If u and v are two differentiable functions of x then u + v is a differentiable and

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

2. $\frac{d}{dx}(u_1+u_2+...+u_n)=\frac{du_1}{dx}+\frac{du_2}{dx}+....+\frac{du_n}{dx}$

THEOREM

If f is a differentiable function at x and k is a constant then kf is differentiable and (kf)'(x) = k.f '(x)

Proof:

Since f is differentiable at x,therefore,

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{(kf)(x+h)-(kf)(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{kf(x+h)-kf(x)}{h}$$

$$= k\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = k.f'(x)$$

THEOREM

If f and g are two differentiable functions at x then the product function f.g is differentiable at x and (fg)' (x) = f '(x). g(x) + f(x) . g'(x)

Proof:

Since f and g are differentiable at x,therefore

f '(x) and g'(x) exist and

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

and $\underset{h\to 0}{Lt}\frac{g(x+h)-g(x)}{h} = g'(x)$

$$\underset{h\to 0}{Lt}\frac{(fg)(x+h)-(fg)(x)}{h} =$$

$$= \underset{h\to 0}{Lt}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}[f(x+h).g(x+h)-f(x)g(x+h)] + \underset{h\to 0}{Lt}\frac{1}{h}[f(x)g(x+h)-f(x)g(x)]$$

$$= \underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}\right]\underset{h\to 0}{Lt}\,g(x+h) + f(x).\ \underset{h\to 0}{Lt}\left[\frac{g(x+h)-g(h)}{h}\right]$$

$\therefore$ (fg)'(x)= f '(x) g(x) + f(x) g'(x)

Note :

Above formula can be taken as follows which is known as Product rule or uv rule :

If u and v are two differentiable function of x then their product uv is a differentiable function of x and $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{du}{dx}$

Quotient Rule

THEOREM

If f and g are differentiable functions at x and g(x)≠0 then the quotient function $\frac{f}{g}$ is differentiable at x and $\left(\frac{f}{g}\right)'(x) = \frac{g(x).f'(x) - f(x)g'(x)}{[g(x)]^2}$

Proof:

Since f and g are differentiable at x,therefore

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

and $\underset{h\to 0}{Lt}\frac{g(x+h)-g(x)}{h} = g'(x)$

$$\left(\frac{f}{g}\right)^1 = \underset{h\to 0}{Lt}\frac{(f/g)(x+h)-(f/g)(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{f(x+h)g(x) - g(x+h)f(x)}{g(x)g(x+h)}\right]$$

$$= \cdot\underset{h\to 0}{Lt}\frac{1}{g(x).g(x+h)}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}\right]$$

$$= \frac{1}{[g(x)]^2}\cdot\left\{g(x)\underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\underset{h\to 0}{Lt}\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \frac{1}{[g(x)]^2}\left\{g(x)\underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\underset{h\to 0}{Lt}\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \frac{1}{[g(x)]^2}\{g(x)f'(x) - f(x)g'(x)\}$$

$$\therefore \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Above formula can be taken as follows.

If u and v are two differentiable functions of x then $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

which is known as $\frac{u}{v}$ rule .

Derivative of the reciprocal of a function :

If f(x) is a differentiable function at x and

f(x)$\neq$ 0 then $\frac{1}{f}$ is differentiable at x and $\left(\frac{1}{f}\right)'(x) = \frac{-f(x)}{[f(x)]^2}$

DERIVATIVE OF A COMPOSITE FUNCTION

If f is a differentiable function at x and g is a differentiable function at f(x) then gof is differentiable at x and $(gof)'(x)=g'(f(x))\,f'(x)$.

This is also known as chain rule. This rule can be further extended.

For example, if y is a function of u and u is a function x then y will be a function of x and therefore, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

DERIVATIVE OF THE INVERSE OF A FUNCTION

Let f : [a, b]$\rightarrow$ [c, d] be a bijection and g be the inverse function of f. If f is differentiable at x $\in$ (a, b),$f^1(x)\neq 0$ and g is continuous at f(x) then g is differentiable at f(x) and

$$g'(f(x))=\frac{1}{f'(x)}.$$

Note : If y =f(x) $\Leftrightarrow$ x = $f^{-1}(y)$ then

$$\frac{dx}{dy}=(f^{-1})'(y)=(f^{-1})'(f(x))=\frac{1}{f'(x)}=\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\therefore \frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)} \text{ and } \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}$$

DIFFERENTIATION OF DETERMINANTS

THEOREM:

If $y = \begin{vmatrix} f(x) & g(x) \\ \phi(x) & \psi(x) \end{vmatrix}$ then $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) \\ \phi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \phi'(x) & \psi'(x) \end{vmatrix}$

Proof: Given that y = $\begin{vmatrix} f(x) & g(x) \\ \phi(x) & \psi(x) \end{vmatrix}$

$\Rightarrow y = f(x)\psi(x) - g(x)\phi(x)$

$\therefore \frac{dy}{dx} = \frac{d}{dx}[f(x)\psi(x) - g(x)\phi(x)]$

$= [f'(x)\psi(x) + f(x)\psi'(x)] - [g'(x).\psi(x) + g(x)\psi'(x)]$

$= [f'(x)\psi(x) - g'(x)\phi(x)] + [f(x)\psi'(x) - g(x)\phi'(x)]$

$= \begin{vmatrix} f'(x) & g'(x) \\ \phi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \phi'(x) & \psi'(x) \end{vmatrix}$

EXERCISE

1. Find the derivatives of the following functions.

i) $y = \left(4 + x^2\right).e^{2x}$

Sol : $y = \left(4 + x^2\right).e^{2x}$

Differentiating w.r. to x

$$\frac{dy}{dx} = \left(4 + x^2\right)\frac{d}{dx}\left(e^{2x}\right) + e^{2x}\frac{d}{dx}\left(4 + x^2\right)$$

$$= \left(4 + x^2\right).2e^{2x} + e^{2x}(0 + 2x)$$

$$= 2e^{2x}\left[4 + x^2 + x\right] = 2e^{2x}\left(x^2 + x + 4\right)$$

ii) $y=\left(\sqrt{x}+1\right)\left(x^2-4x+2\right)(x>0)$

Sol: $y=\left(\sqrt{x}+1\right)\left(x^2-4x+2\right)(x>0)$

Differentiating w.r. to x

$$\frac{dy}{dx}=\left(\sqrt{x}+1\right)\frac{d}{dx}\left(x^2-4x+2\right)+$$

$$\left(x^2-4x+2\right)\frac{d}{dx}\left(\sqrt{x}+1\right)=\left(\sqrt{x}+1\right)(2x-4)+\frac{x^2-4x+2}{2\sqrt{x}}$$

iii) $y=a^x.e^{x^2}$

sol : $y=a^x.e^{x^2}$ Differentiating w.r. to x

$$\frac{dy}{dx}=\left(a^x\right)\frac{d}{dx}\left(e^{x^2}\right)+\left(e^{x^2}\right)\frac{d}{dx}\left(a^x\right)=a^x.e^{x^2}.2x+e^{x^2}.a^x.\log a$$

$$=a^x.e^{x^2}(2x+\log a)=y(2x+\log a)$$

iv) $y=\frac{ax+b}{cx+d}[|c|+|d|\neq 0]$

sol : $y=\frac{ax+b}{cx+d}[|c|+|d|\neq 0]$

Differentiating w .r. to x

$$\frac{dy}{dx}=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}=\frac{(cx+d)a-(ax+b).c}{(cx+d)^2}$$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}=\frac{ad-bc}{(cx+d)^2}$$

v) $y = e^{2x}.\log(3x+4)\left(x > \frac{-4}{3}\right)$

sol : $y = e^{2x}.\log(3x+4)\left(x > \frac{-4}{3}\right)$ Differentiating w.r. to x

$$\frac{dy}{dx} = e^{2x}\frac{d}{dx}\left[\log(3x+4) + \log(3x+4)\left(e^{2x}\right)\right] = e^{2x}.\frac{1}{3x+4}.3 + \log(3x+4).e^{2x}.2$$

$$= e^{2x}\left(\frac{3}{3x+4} + 2\log(3x+4)\right)$$

vi) $y = \sqrt{x} + 2x^{\frac{3}{4}} + 3x^{\frac{5}{6}}\ (x > 0)$

sol : $y = \sqrt{x} + 2x^{\frac{3}{4}} + 3x^{\frac{5}{6}}\ (x > 0)$

$$\frac{dy}{dx} = \frac{1}{2}.x^{-1/2} + 2.\ \frac{3}{4}.\ x^{-1/4} + 3.\ \frac{5}{6}\ .\ x^{-1/6}$$

$$= \frac{1}{2}\left[x^{-1/2} + 3.x^{-1/4} + 5.\ x^{-1/6}\right]$$

vii) $y = \sqrt{2x-3} + \sqrt{7-3x}$

sol : $y = \sqrt{2x-3} + \sqrt{7-3x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x-3}}.2 + \frac{1}{2\sqrt{7-3x}}(-3)$$

$$= \frac{1}{\sqrt{2x-3}} - \frac{3}{2\sqrt{7-3x}}$$

viii) $y = e^{x} + \sin x.\cos x$

sol : $y = e^{x} + \sin x.\cos x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

$$= e^x + \frac{1}{2}\cos 2x \quad \text{X } 2$$

$$= e^x + \cos\ 2x$$

ix) $y = \left(x^2 - 3\right)\left(4x^3 + 1\right)$

sol : $y = \left(x^2 - 3\right)\left(4x^3 + 1\right)$

$$\frac{dy}{dx} = \left(x^2 - 3\right)\frac{d}{dx}\left(4x^3 + 1\right) + \left(4x^3 + 1\right)\frac{d}{dx}\left(x^2 - 3\right)$$

$$= \left(x^2 - 3\right)\left(12x^2\right) + \left(4x^3 + 1\right)(2x)$$

$$= 12x^4 - 36x^2 + 8x^4 + 2x = 20x^4 - 36x^2 + 2x$$

x) $y = \left(\sqrt{x} - 3x\right)\left(x + \frac{1}{x}\right)$

sol : $y = \left(\sqrt{x} - 3x\right)\left(x + \frac{1}{x}\right)$

$$\frac{dy}{dx} = \left(\sqrt{x} - 3x\right)\left(1 - \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}} - 3\right)$$

$$= \sqrt{x} - \frac{1}{x\sqrt{x}} - 3x + \frac{3}{x} + \frac{\sqrt{x}}{2} \quad + \frac{1}{2x\sqrt{x}} - 3x - \frac{3}{x}$$

$$= \frac{3}{2}\sqrt{x} + \frac{1}{x\sqrt{x}} - 6x$$

xi) $y = 5\sin x + e^x.\log\ x$

sol : $y = 5\sin x + e^x.\log\ x$

$$\frac{dy}{dx} = 5\cos x + e^x.\frac{d}{dx}(\log x) + \log\, x\frac{d}{dx}\left(e^x\right)$$

$$= 5\cos x + e^x.\frac{1}{x} + (\log x)\left(e^x\right)$$

xii) $y = 5^x + \log x + x^3 e^x$

sol : $y = 5^x + \log x + x^3 e^x$

$$\frac{dy}{dx} = 5^x + \log 5 + \frac{1}{x} + x^3.e^x + e^x.3x^2$$

$$= 5^x.\log 5 + \frac{1}{x} + x^3 e^x + 3x^2 e^x$$

xiii) $y = \dfrac{px^2 + qx + r}{ax + b}(|a| + |b| \neq 0)$

sol : $y = \dfrac{px^2 + qx + r}{ax + b}(|a| + |b| \neq 0)$

$$(ax+b)\frac{d}{dx}\left(px^2 + qx + r\right)\frac{dy}{dx} = \frac{-\left(px^2 + qx + r\right)\frac{d}{dx}(ax + b)}{(ax+b)^2}$$

$$= \frac{(ax + b)(2px + q) - \left(px^2 + qx + r\right).a}{(ax + b)^2}$$

$$= \frac{2apx^2 + 2bpx + aqx + bq - apx^2 - aqx - ar}{(ax + b)^2}$$

$$= \frac{apx^2 + 2bpx + (bq - ar)}{(ax + b)^2}$$

xiv) $y = (ax+b)^n.(cx+d)^m$.

sol : $y = (ax+b)^n.(cx+d)^m$.

$$\frac{dy}{dx} = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n$$

$$= (ax+b)^n \left[m(cx+d)^{m-1}.c\right] + (cx+d)^m \left[n(3x+b)^{n-1}\right]$$

$$= (ax+b)^{n-1}(cx+d)^{m-1}[cm(ax+b)+an(cx+d)] = (ax+b)^n (cx+d)^m \left[\frac{an}{ax+b} + \frac{cm}{cx+d}\right]$$

xv) $y = \frac{1}{ax^2+bx+c}(|a|+|b|+|c| \neq 0)$

sol : $y = \frac{1}{ax^2+bx+c}(|a|+|b|+|c| \neq 0)$

$$\frac{dy}{dx} = \frac{(-1)}{\left(ax^2+bx+c\right)^2}\frac{d}{dx}\left(ax^2+bx+c\right)$$

$$= \frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

xvi) $y = \log_7(\log x)(x > 0)$

sol : $y = \log_7(\log x) = \frac{\log(\log x)}{\log 7}$

$$\frac{dy}{dx} = \frac{1}{\log_7}.\frac{1}{\log x}.\frac{1}{x}$$

2. **If** $f(x) = 1 + x + x^2 + ... + x^{100}$, **then find** $f'(1)$.

Sol : $f(x) = 1 + x + x^2 + ... + x^{100}$,

$\Rightarrow f'(x) = 1 + 2x + 3x^2 .... + 100x^{99}$

$\Rightarrow f'(1) = 1 + 2 + 3 .... + 100$

$= \frac{100 \times 101}{2} = 5050 \left(\Sigma x = \frac{x(x+1)}{2} \right)$

3. **if** $f(x) = 2x^2 + 3x - 5,$ **then prove that** $f'(0) + 3f'(-1) = 0$**.**

Sol : $f(x) = 2x^2 + 3x - 5,$

$\Rightarrow f'(x) = 4x + 3$

$\Rightarrow f'(0) = 0 + 3 = 3$

$\Rightarrow f'(-1) = -4 + 3 = -1$

$\Rightarrow f'(0) + 3f'(-1)3 + 3(-1) = 3 - 3 = 0$

2. **Find the derivatives of the following functions** $f(x)$ **from the first principles.**

i) $f(x) = x^3$

Sol : $f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{Lt} \frac{(x+h)^3 - x^3}{h}$

$= \underset{h \to 0}{Lt} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \underset{h \to 0}{Lt} \frac{h\left(3x^2 + 3xh + h^2\right)}{h}$

$= \underset{h \to 0}{Lt} \left(3x^2 + 3xh + h^2\right) = 3x^2 + 0 + 0 = 3x^2$

ii) $f(x) = x^4 + 4$

sol : $f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$$= \underset{h\to 0}{Lt} \frac{\left((x+h)^4+4\right)-\left(x^4+4\right)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{(x+h)^4+4-x^4-4}{h}$$

$$= \underset{h\to 0}{Lt} \frac{h\left[4x^3+6x^2h+4xh^2+h^3\right]}{h} = \underset{h\to 0}{Lt} 4x^3+6x^2h+4xh^2+h^3$$

$$= 4x^3+0+0+4x^3$$

iii) $f(x) = ax^2+bx+c$

Ans: $2ax+b$

iv) $f(x) = \sqrt{x+1}$

sol : $f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$

$$= \underset{h\to 0}{Lt} \frac{\left(\sqrt{x+h+1}-\sqrt{x+1}\right)\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{\sqrt{x+h+1}+\sqrt{x+1}} = \underset{h\to 0}{Lt} \frac{x+h+1-x-1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \underset{h\to 0}{Lt} \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \underset{h\to 0}{Lt} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

v) $f(x) = \sin 2x$

Ans: $= 2\cos 2x$

vi) $f(x) = \cos ax$

sol : $f^1(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\dfrac{\cos a(x+h)-\cos ax}{h}$

$$= \underset{h\to 0}{Lt}\frac{-2\sin\dfrac{ax+ah+ax}{2}.\sin\dfrac{ax+ah-ax}{2}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{-2\sin\left(ax+\dfrac{ah}{2}\right).\sin\left(\dfrac{ah}{2}\right)}{h}$$

$$\underset{h\to 0}{Lt}-2\sin\left(ax+\frac{ah}{2}\right)\underset{h\to 0}{Lt}\frac{\sin\left(\dfrac{ah}{2}\right)}{h}$$

$$= -2\sin ax.\frac{a}{2} \quad = -a.\sin ax$$

vii) $f(x) = \tan 2x$

sol : $f'(x) = \underset{h\to 0}{Lt}\dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\dfrac{\tan 2(x+h)-\tan 2x}{h}$

$$= \underset{h\to 0}{Lt}\frac{\dfrac{\sin(2x+2h)}{\cos(2x+2h)}-\dfrac{\sin 2x}{\cos 2x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\dfrac{\sin(2x+2h).\cos 2x-\cos(2x+2h).\sin 2x}{\cos(2x+2h).\cos 2x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{\sin(2x+2h-2x)}{\cos(2x+2h)\cos 2x}$$

$$= \underset{h\to 0}{Lt}\frac{\sin 2h}{h}.\underset{h\to 0}{Lt}\frac{1}{\cos(2x+2h)\cos 2x}$$

$$= 2.\frac{1}{\cos 2x.\cos 2x} = 2\sec^2 2x.$$

viii) $f(x) = \cot x$

sol : $f^1(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\frac{\cot(x+h)-\cot x}{h}$

$$= \underset{h\to 0}{Lt}\frac{\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\frac{\cos(x+h).\sin x-\sin(x+h).\cos x}{\sin(x+h).\sin x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{-\sin(x+h-x)}{\sin(x+h).\sin x}\frac{1}{h}$$

$$\underset{h\to 0}{Lt}\frac{-\sin h}{h}\ \underset{h\to 0}{Lt}\frac{1}{\sin(x+h).\sin x}$$

$$= -\frac{1}{\sin x.\sin x} = -\operatorname{cosec}^2 x$$

ix) $f(x) = \sec 3x$

sol : $f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{1}{\cos(3x+3h)}-\frac{1}{\cos 3x}\right)$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{\cos 3x-\cos(3x+3h)}{\cos(3x+3h).\cos 3x}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{2.\sin\frac{3x+3h+3x}{2}.\sin\frac{3x+3h-3x}{2}}{\cos(3x+3h).\cos 3x}$$

$$f'(x) = \frac{\underset{h\to 0}{Lt}\, 2\sin\left(3x + \frac{3h}{2}\right).\underset{h\to 0}{Lt}\frac{\sin\frac{3h}{2}}{h}}{\underset{h\to 0}{Lt}\cos(3x+3h).\cos 3x}$$

$$= \frac{2\sin 3x.\frac{3}{2}}{(\cos 3x)(\cos 3x)} = 3.\frac{\sin 3x}{\cos 3x}.\frac{1}{\cos 3x}$$

$$= 3.\tan 3x.\sec 3x$$

x) $f(x) = x\sin x$

sol : $$f^1(x) = \underset{h\to 0}{Lt}\frac{f(x+h) - f(x)}{h} = \underset{h\to 0}{Lt}\frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \underset{h\to 0}{Lt}\frac{x(\sin(x+h) - \sin x) + h.\sin(x+h)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{x\left(2\cos\frac{x+h+x}{2}.\sin\frac{x+h-x}{2}\right) + h.\sin(x+h)}{h}$$

$$= 2x.\underset{h\to 0}{Lt}\cos\left(x + \frac{h}{2}\right).\underset{h\to 0}{Lt}\frac{\sin\frac{h}{2}}{h} + \underset{h\to 0}{Lt} h.\frac{\sin(x+h)}{h}$$

$$= 2x.\cos x.\frac{1}{2} + \sin x$$

$$= x.\cos x. + \sin x$$

xi) $f(x)=\cos^2 x$

sol : $$f^1(x) = \underset{h\to 0}{Lt}\frac{f(x+h) - f(x)}{h}$$

$$f^1(x) = \underset{h\to 0}{Lt}\frac{\cos^2(x+h) - \cos^2 x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-\left(\cos^2 x - \cos^2 (x+h)\right)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-\sin(x+h+x)\sin(x+h-x)}{h}$$

$$f'(x) = \underset{h\to 0}{Lt} - \sin(2x+h) \underset{h\to 0}{Lt} . \frac{\sin h}{h}$$

$$= -\sin 2x.1 = -\sin 2x$$

3. Show that the function $f(x) = |x| + |x+1|$ is differentiable for real numbers except for 0 and 1.

Sol : $f(x) = |x| + |x+1| \forall x \in R$

Case (i) : at $x = 0$

$$Rf'(0) = \underset{x\to 0+}{Lt} \frac{f(x) - f(0)}{x}$$

$$= \underset{x\to 0+}{Lt} \frac{2x+1-1}{x} = 2$$

$$Lf'(0) = \underset{x\to -}{Lt} \frac{f(x) - f(0)}{x}$$

$$= \underset{x\to 0-}{Lt} \frac{1-1}{x} = 0$$

$$R f'(0) \neq L f'(0)$$

$\therefore f'(0)$ does not exist.

$f(x)$ is not differentiable at $x = 0$.

Case (ii) : at $x = 1$

$$R\, f'(1) = \underset{x \to 1+}{Lt} \frac{f(x) - f(1)}{x - 1}$$

$$= \underset{x \to 1+}{Lt} \frac{2x + 1 - 3}{x - 1} = \underset{x \to 1+}{Lt} \frac{2x - 2}{x - 1} = 2$$

$$L\, f'(1) = \underset{x \to 1-}{Lt} \frac{f(x) - f(1)}{x - 1} = \underset{x \to 1-}{Lt} \frac{1 - 1}{x - 1} = 0$$

$R\, f'(1) \neq Lf'(1)$

$f'(1)$ Does not exist.

$f(x)$ is not differentiable at $x = 1$

$\therefore$ $f(x)$ is differentiable on $R - \{0,1\}$.

4. Verify whether the following function is differentiable at 1 and 3

$$f(x) = \begin{cases} x & \text{if} \quad x < 1 \\ 3 - x & \text{if} \quad 1 \leq x \leq 3 \\ x^2 - 4x + 3 & \text{if} \quad x > 3 \end{cases}$$

Sol : Case (i) : at $x = 1$

$$f'\left(1^-\right) = \underset{x \to 1-}{Lt} \frac{f(x) - f(1)}{x - 1}$$

$$= \underset{x \to 1-}{Lt} \frac{x - (3 - 1)}{x - 1} = \underset{x \to 1}{Lt} \frac{x - 2}{x - 1} = \infty$$

$$f'\left(1^+\right) = \underset{x \to 1+}{Lt} \frac{f(x) - f(1)}{x - 1}$$

$$\underset{x \to 1+}{Lt} \frac{(3 - x) - 2}{x - 1} = \underset{x \to 1+}{Lt} \frac{1 - x}{x - 1} = -1$$

$R\,f\,'(1) \neq L\,f\,'(1)$

$f(x)$ is not differentiable at $x = 1$

Case (ii) : at $x = 3$

$$f\,'\left(3^{-}\right) = \underset{x \to 3-}{Lt} \frac{f(x) - f(3)}{x - 3}$$

$$\underset{x \to 3-}{Lt} \frac{(3 - x) - 0}{x - 3} = \underset{x \to 3-}{Lt} \frac{3 - x}{x - 3} = -1$$

$$f\,'\left(3^{+}\right) = \underset{x \to 3+}{Lt} \frac{f(x) - f(3)}{x - 3}$$

$$= \underset{x \to 3+}{Lt} \frac{\left(x^2 - 4x + 3\right) - 0}{x - 3} = \underset{x \to 3+}{Lt} \frac{(x - 3)(x - 1)}{x - 3}$$

$$= \underset{x \to 3+}{Lt} x - 1 = 3 - 1 = 2$$

5. Is the following function f derivable at 2? Justify $f(x) = \begin{cases} x, & if \quad 0 \leq x \leq 2 \\ 2, & if \quad x \geq 2 \end{cases}$

Sol : $f\,'\left(2^{-}\right) = \underset{x \to 2^{-}}{Lt} \frac{f(x) - f(2)}{x - 2} = \underset{x \to 2-}{Lt} \frac{x - 2}{x - 2} = 1$

$$f\,'\left(2^{+}\right) = \underset{x \to 2^{+}}{Lt} \frac{f(x) - f(2)}{x - 2} = \underset{x \to 2+}{Lt} \frac{2 - 2}{x - 2} = 0$$

$f\,'(2-) \neq f\,'\left(2^{+}\right)$; $f(x)$ is not derivable at $x = 2$.