CONTINUITY

CONTINUITY AT A POINT

Let f be a function defined in a neighbourhood of a point a. Then f is said to be continuous at the point a if and only if $\underset{x \to a}{Lt} f(x) = f(a)$.

In other words, f is continuous at a iff the limit of f at a is equal to the value of f at a.

NOTE:

1.If f is not continuous at a it is said to be discontinuous at a, and a is called a point of discontinuity of f.

2. Let f be a function defined in a nbd of a point a. Then f is said to be

(i) Left continuous at a iff $\underset{x \to a-}{Lt} f(x) = f(a)$.

(ii) Right continuous at a iff $\underset{x \to a+}{Lt} f(x) = f(a)$.

3. f is continuous at a iff f is both left continuous and right continuous at a

i.e, $\underset{x \to a}{Lt} f(x) = f(a) \Leftrightarrow \underset{x \to a-}{Lt} f(x) = f(a) = \underset{x \to a+}{Lt} f(x)$

CONTINUITY OF A FUNCTION OVER AN INTERVAL

I) A function f defined on (a, b) is said to be continuous (a,b) if

it is continuous at everypoint of (a, b) i.e., if $\underset{x \to c}{Lt} f(x) = f(c) \forall c \in (a, b)$

II) A function f defined on [a, b] is said to be continuous on [a, b] if

(i) f is continuous on (a, b) i.e., $\underset{x \to c}{Lt} f(x) = f(c) \forall c \in (a, b)$

(ii) f is right continuous at a i.e., $\underset{x \to a+}{Lt} f(x) = f(a)$

(iii) f is left continuous at b i.e., $\underset{x \to b-}{Lt} f(x) = f(b)$.

NOTE :

1. Let the functions f and g be continuous at a and k€R. Then f + g, f – g, kf , kf + lg, f.g are continuous at a and $\frac{f}{g}$ is continuous at a provided g(a)≠ 0.

2. All trigonometric functions, Inverse trigonometric functions, hyperbolic functions and inverse hyperbolic functions are continuous in their domains of definition.

3. A constant function is continuous on R

4. The identity function is continuous on R.

5. Every polynomial function is continuous on R.

EXERCISE

I.

1. Is the function f defined by $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x > 1 \end{cases}$ **Continuous on R ?**

Sol : $\underset{x \to 1-}{Lt} f(x) = \underset{x \to 1-}{Lt} x^2 = 1^2 = 1$

$\underset{x \to 1+}{Lt} f(x) = \underset{x \to 1+}{Lt} x = 1$

$\therefore \underset{x \to 1-}{Lt} f(x) = \underset{x \to 1+}{Lt} f(x) = 1$ and $f(1) = 1^2 = 1$

$\underset{x \to 1}{Lt} f(x) = f(1)$

f is continuous $x = 1$

Hence f is continuous on R.

2. Is f defined by $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$ **Continuous at 0 ?**

Sol : $\underset{x \to 0}{Lt} f(x) = \underset{x \to 0}{Lt} \dfrac{\sin 2x}{x} = 2$ But $f(0) = 1$

$\therefore \underset{x \to 0}{Lt} f(x) \ne f(0)$

Hence f is not continuous at 0

3. Show that the function $f(x) = \left[\cos\left(x^{10} + 1\right)\right]^{1/3}, x \in R$ **is a continuous functions.**

Sol : We know that $\cos x$ continuous for every $x \in R$

$\therefore$ The given function $f(x)$ is continuous for every $x \in R$

II.

1. Check the continuity of the following function at 2 for the function

$$f(x) = \begin{cases} \frac{1}{2}\left(x^2 - 4\right) & \text{if} \quad 0 < x < 2 \\ 2 - 8x^{-3} & \text{if} \quad x > 2 \end{cases}$$

Sol : $l.l = \underset{x \to 2-}{Lt} f(x) = \underset{x \to 2-}{Lt} \frac{1}{2}\left(x^2 - 4\right) = \frac{1}{2}(4 - 4) = 0$

$$R.L = \underset{x \to 2+}{Lt} f(x) = \underset{x \to 2+}{Lt}\left(2 - \frac{8}{x^3}\right) = 2\frac{-8}{8} = 2 - 1 = 1$$

$$\underset{x \to 2-}{Lt} f(x) \neq \underset{x \to 2+}{Lt} f(x)$$

$\therefore \underset{x \to 2}{Lt} f(x)$ does not exist

Hence $f(x)$ is not continuous at 2.

2. Check the continuity of f given by $f(x) = \begin{cases} \frac{x^2 - 9}{x^2 - 2x - 3} & \text{if} \quad 0 < x < 5 \text{ and } x \neq 3 \\ 1.5 & \text{if} \quad x = 3 \end{cases}$ **At the point 3.**

Sol : Given f(3) =1.5.

$$\underset{x \to 3}{Lt} f(x) = \underset{x \to 3}{Lt} \frac{x^2 - 9}{x^2 - 2x - 3}$$

$$= \underset{x \to 3}{Lt} \frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{3+3}{3+1} = \frac{6}{4} = 1.5 = f(3)$$

$\therefore f(x)$ is continuous at $x = 3$.

3. Show that f, given by $f(x) = \frac{x - |x|}{x} (x \neq 0)$ is continuous on $R - \{0\}$.

Sol :

Left limit at x= 0 is $\underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt} 2 = 2$

Right limit at x=0 is $\underset{x \to 0^+}{Lt} f(x) = \underset{x \to 0^+}{Lt} 0 = 0$

$\underset{x \to 0-}{Lt} f(x) \neq \underset{x \to 0^+}{Lt} f(x)$ $\therefore \underset{x \to 0}{Lt} f(x)$ *does* not exist.

Hence the function is not continuous at x=0.

When x<0, f(x) = 2, a constant. And it is continuous for all x<0.

When x>0 , f(x) = 0, which is continuos for all x>0.

Hence the function is continuous on R-{0}.

4. If f is a function defined by $f(x) = \begin{cases} \frac{x-1}{\sqrt{x}-1} & \text{if} \quad x > 1 \\ 5 - 3x & \text{if} \quad -2 \le x \le 1 \\ \frac{6}{x-10} & \text{if} \quad x < -2 \end{cases}$ **then discuss the continuity of f**

Sol: Case (i) continuity at $x = 1$

$$\underset{x \to 1+}{Lt} f(x) = \underset{x \to 1+}{Lt} \frac{x-1}{\sqrt{x}-1} = \underset{x \to 1+}{Lt} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \underset{x \to 1+}{Lt} \sqrt{x-1} = \sqrt{1-1} = 0$$

$$\underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^-}{Lt} (5 - 3x) = 5 - 3 = 2$$

$\therefore \underset{x \to 1^-}{Lt} f(x) \neq \underset{x \to 1^+}{Lt} f(x)$. Hence f is not continuous at x=1

Case (ii) continuity at $x = -2$

$$\underset{x\to -2-}{Lt} f(x) = \underset{x\to -2-}{Lt} \frac{6}{x-10} = \frac{6}{-2-10} = \frac{-6}{12} = \frac{-1}{2}$$

$$\underset{x\to -2+}{Lt} f(x) = \underset{x\to -2+}{Lt} (5-3x) = 5-3(-2) = 5+6 = 11$$

$$\therefore \underset{x\to -2-}{Lt} f(x) \neq \underset{x\to -2+}{Lt} f(x)$$

Hence $f(x)$ is not continuous at $x = -2$.

5. If f is given by $f(x) = \begin{cases} k^2x - k & \text{if} \quad x \geq 1 \\ 2 & \text{if} \quad x < 1 \end{cases}$ **is a continuous function on R, then find the values of k.**

Sol : $\underset{x\to 1-}{Lt} f(x) = \underset{x\to 1-}{Lt} 2 = 2$

$\underset{x\to 1+}{Lt} f(x) = \underset{x\to 1+}{Lt} \left(k x^2 - k\right) = k^2 - k$ Given $f(x)$ is continuous at $x = 0$

$\underset{x\to 1-}{Lt} f(x) = \underset{x\to 1+}{Lt} f(x) \quad 2 = k^2 - k$

GIVEN f is continuous on R, hence it is continuous at x=1.

Therefore L.L =R.L

⇨ $k^2 - k - 2 = 0$

=> $(k-2)(k+1) = 0$ => $k = 2 \text{ or } -1$

6. Prove that the functions $'\sin x'$ **and** $'\cos x'$ **are continuous on R.**

Sol : i) Let $a \in R$

$$\underset{x\to a}{Lt} f(x) = \underset{x\to a}{Lt} \sin x = \sin a = f(a)$$

$\therefore\ f$ is continuous at a.

ii) Let $a \in R$

$$\underset{x\to a}{Lt}\ f(x)\ \underset{x\to a}{Lt}\ \cos x - \cos a = f(a)$$

$\therefore$ f is continuous at a.

III.

1 **Check the continuity of 'f' given by** $f(x) = \begin{cases} 4 - x^2 & \text{if} \quad x \le 0 \\ x - 5 & \text{if} \quad 0 < x \le 1 \\ 4x^2 - 9 & \text{if} \quad 1 < x < 2 \\ 3x + 4 & \text{if} \quad x \ge 2 \end{cases}$ **at the points 0, 1 and 2.**

Ans: f is continuous at $x = 0,1,2$

2. **Find real constant a, b so that the function f given by**

$$f(x) = \begin{cases} \sin x & \text{if} \quad x \le 0 \\ x^2 + a & \text{if} \quad 0 < x < 1 \\ bx + 3 & \text{if} \quad 1 \le x \le 3 \\ -3 & \text{if} \quad x > 3 \end{cases}$$ **is continuous on R.**

Sol : Given f(x) is continuous on R, hence it is continuous at 0,1,3.

At x=0.

$$\underset{x\to 0+}{Lt}\ f(x) = \underset{x\to 0}{Lt}\left(x^2 + a\right) = 0 + a = a$$

$$\underset{x\to 0-}{Lt}\ f(x) = \underset{x\to 0}{Lt}\ \sin x = 0$$

Since $f(x)$ is continuous at x=0,

. L.L = R.L→ a=0.

At x=3

R.L= $\underset{x\to 3+}{Lt} f(x) = \underset{x\to 3}{Lt} -3 = -3$

L.L= $\underset{x\to 3-}{Lt} f(x) = \underset{x\to 3}{Lt} (bx+3) = 3b+3$

Since $f(x)$ is continuous AT X=3,L.L=R.L

=> $3b+3=-3$ => $3b=-6 \Rightarrow b=-2$

3. **Show that** $f(x)\begin{cases} \dfrac{\cos ax - \cos bx}{x^2} & \text{if} \quad x \neq 0 \\ \dfrac{1}{2}\left(b^2-a^2\right) & \text{if} \quad x = 0 \end{cases}$

Where a and b are real constant, is continuous at 0.

Sol : $\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt} \dfrac{\cos ax - \cos bx}{x^2}$

$= \underset{x\to 0}{Lt} \dfrac{2\sin\dfrac{(a+b)x}{x}\sin\dfrac{(b-a)x}{2}}{x^2}$

$= 2 \underset{x\to 0}{Lt} \dfrac{\sin(a+b)\frac{x}{2}}{x} \underset{x\to 0}{Lt} \dfrac{\sin(b-a)\frac{x}{2}}{x}$

$= \dfrac{2(b+a)}{2}\dfrac{(b-a)}{2} = \dfrac{b^2-a^2}{2}$

Given $f(0) = \dfrac{b^2-a^2}{2}$. $\quad \therefore \underset{x\to 0}{Lt} f(x) = f(0)$

$\therefore \quad f(x)$ is continuous at $x=0$

PROBLEMS FOR PRACTICE

1. **If f is given by** $f(x) = \begin{cases} ax - b & \text{if} \quad x \le -1 \\ 3x^2 - 4ax + 2b & \text{if} \quad -1 < x < 1 \\ 10 & \text{if} \quad x \ge 1 \end{cases}$ **a continuous function is on R, then find the values of a, b.**

2. **Check the continuity of the function f given below at 1 and at 2.**

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x \le 1 \\ 2x & \text{if} \quad 1 < x > 2 \\ 1 + x^2 & \text{if} \quad x \ge 2 \end{cases}$$

3. **Show that** $f(x) = (x)\ (x \in R$ **is continuous at only those real numbers that are not integers.**

Sol : Case i) if $a \in z, f(a) = (a) = a$

$$\underset{x \to a-}{Lt} f(x) = \underset{h \to 0}{Lt} (a - h) = a +$$

$$\underset{x \to a+}{Lt} f(x) = \underset{h \to 0}{Lt} (a - h) = a$$

$$\therefore \underset{x \to a-}{Lt} f(x) \ne \underset{x \to a+}{Lt} f(x)$$

$\underset{x \to a}{Lt} f(x)$ does not exist

$\therefore$ f is not continuous at $x = a \in z.$

Case ii) : if $a \notin z,$ then $\exists\, n \in z$ such that $n < a < n + 1$ then $f(a) = (a) = n.$

$$\underset{x \to a-}{Lt} f(x) = \underset{h \to 0}{Lt} (a - h) = n,$$

$$\underset{x \to a+}{Lt} f(x) = \underset{h \to 0}{Lt} (a + h) = n,$$

$\therefore \ \underset{x \to a}{Lt} f(x) = n = f(a)$

$\therefore \ f(x)$ is continuous at $x = a \notin z.$

4. If $f : R \to R$ is such that $f(x+y) = f(x) + f(y)$ for all $x, y \in R$ then f is continuous on R if it is continuous at a single point in R.

Sol : Let f be continuous at $x_0 \in R$

$\underset{t \to x_0}{Lt} f(t) = f(x_0) \underset{h \to 0}{Lt} f(x_0 + h) = f(x_0)\ x \in R,\ f(x_0)$

$\Rightarrow f(x+h) - f(x) = f(h) = f(x_0 + h) - f(x_0) \underset{h \to 0}{Lt} \{f(x+h) - f(x)\}$

$= \underset{h \to 0}{Lt} \{f(x_0 + h) - f(x_0)\} = 0$

$\therefore$ f is continuous at x.

Since $x \in R$ is arbitrary, f is continuous on R.

5. Show that f(x) = [x] (x $\in$ R is continuous at only those real numbers that are not integers.

Sol.

Case (i) :

If a $\in$ z, f(a) = (a) = a

$\underset{x \to a^-}{Lt} f(x) = \underset{h \to 0}{Lt} (a - h) = a^+$

$\underset{x \to a^+}{Lt} f(x) = \underset{h \to 0}{Lt} (a + h) = a$

$\therefore \underset{x \to a^-}{Lt} f(x) \neq \underset{x \to a^+}{Lt} f(x)$

$\underset{x \to a}{Lt} f(x)$ does not exist.

$\therefore$ f is not continuous at x = a $\in$ z.

Case (ii) :

If $a \notin z$, then $\exists n \in z$ such that $n < a < n+1$ then f(a) = (a) = n.

$\underset{x \to a^-}{Lt} f(x) = \underset{h \to 0}{Lt} (a-h) = n$

$\underset{x \to a^+}{Lt} f(x) = \underset{h \to 0}{Lt} (a+h) = n$

$\therefore \underset{x \to a}{Lt} f(x) = n = f(a)$

$\therefore f(x)$ is continuous at $x = a \notin z$.

6. Show that the function f defined on R by f(x) = |1 + 2x + |x||, x $\in$ R is a continuous function.

Sol. We define $g : R \to R$ by

g(x) = 1 + 2x + |x|, x $\in$ R,

and $h : R \to R$ by h(x) = |x|, x $\in$ R. Then

(hog)(x) = h(g(x)) = h(1 + 2x + |x|)

= |1 + 2x + |x|| = f(x).