

LIMITS AT INFINITY

Definition:

Let $f(x)$ be a function defined on $A = (K, \infty)$.

- (i) A real number l is said to be the limit of $f(x)$ at ∞ if to each $\epsilon > 0$, an $M > 0$ (however large M may be) such that $x \in A$ and $x > M \Rightarrow |f(x) - l| < \epsilon$. In this case we write $f(x) \rightarrow l$ as $x \rightarrow +\infty$ or $\lim_{x \rightarrow +\infty} f(x) = l$.
- (ii) A real number l is said to be the limit of $f(x)$ at $-\infty$ if to each $\epsilon > 0$, an $M > 0$ (however large it may be) such that $x \in A$ and $x < -M \Rightarrow |f(x) - l| < \epsilon$. In this case, we write $f(x) \rightarrow l$ as $x \rightarrow -\infty$ or $\lim_{x \rightarrow -\infty} f(x) = l$.

INFINITE LIMITS

Definition:

- (i) Let f be a function defined in a deleted neighbourhood of D of a . (i) The limit of f at a is said to be ∞ if to each $M > 0$ (however large it may be) a $\delta > 0$ such that $x \in D, 0 < |x-a| < \delta \Rightarrow f(x) > M$. In this case we write $f(x) \rightarrow \infty$ as $x \rightarrow a$ or $\lim_{x \rightarrow a} f(x) = \infty$.
- (ii) The limit of $f(x)$ at a is said to be $-\infty$ if to each $M > 0$ (however large it may be) a $\delta > 0$ such that $x \in D, 0 < |x-a| < \delta \Rightarrow f(x) < -M$. In this case we write $f(x) \rightarrow -\infty$ as $x \rightarrow a$ or $\lim_{x \rightarrow a} f(x) = -\infty$.

INDETERMINATE FORMS

While evaluating limits of functions, we often get forms of the type

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0 , 1^∞ , ∞^0 which are termed as indeterminate forms.

EXERCISE

I. Compute the following limits.

1. $\lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{x^2 - 6x + 9}$

Sol : $\lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{x^2 - 6x + 9} = \lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{(x - 3)^2} = \frac{9 + 9 + 2}{0} = \infty$

2. $\lim_{x \rightarrow 1^-} \frac{1 + 5x^3}{1 - x^2}$

Sol : $\lim_{x \rightarrow 1^-} \frac{1 + 5x^3}{1 - x^2} = \frac{1 + 5(1)^3}{1 - 1^2} = \frac{1 + 5}{1 - 1} = \frac{6}{0} = \infty$

3. $\lim_{x \rightarrow \infty} \frac{3x^2 + 4x + 5}{2x^3 + 3x - 7}$

Sol : $\lim_{x \rightarrow \infty} \frac{3x^2 + 4x + 5}{2x^3 + 3x - 7}$

$$= \lim_{x \rightarrow \infty} \frac{\left(3 + \frac{4}{x} + \frac{5}{x^2}\right)x^2}{\left(2 + \frac{3}{x^2} - \frac{7}{x^3}\right)x^3} = \lim_{x \rightarrow \infty} \frac{\left(3 + \frac{4}{x} + \frac{5}{x^2}\right)}{2 + \frac{3}{x^2} - \frac{7}{x^3}} \cdot \frac{1}{x}$$

As $x \rightarrow \infty, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3} \rightarrow 0$

$$= \lim_{x \rightarrow \infty} \frac{3x^2 + 4x + 5}{2x^3 + 3x - 7} = \frac{(3 + 0 + 0)}{2 + 0 - 0} \cdot 0 = 0$$

4. $\lim_{x \rightarrow \infty} \frac{6x^2 - x + 7}{x + 3}$

Sol : $\lim_{x \rightarrow \infty} \frac{6x^2 - x + 7}{x + 3} = \lim_{x \rightarrow \infty} \frac{x^2 \left(6 - \frac{1}{x} + \frac{7}{x^2}\right)}{x \left(1 + \frac{3}{x}\right)}$

$$\lim_{x \rightarrow \infty} \frac{6 - \frac{1}{x} + \frac{7}{x^2}}{1 + \frac{3}{x}} \cdot \lim_{x \rightarrow \infty} x = \frac{6 - 0 + 0}{1 + 0} = \infty$$

5. $\lim_{x \rightarrow \infty} e^{-x^2}$

Sol : $\lim_{x \rightarrow \infty} e^{-x^2} = \lim_{x \rightarrow \infty} \frac{1}{e^{x^2}} = \frac{1}{\infty} = 0$ (since $e > 1$)

6. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 6}}{2x^2 - 1}$

Sol : $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 6}}{2x^2 - 1} = \lim_{x \rightarrow \infty} \frac{x \sqrt{1 + \frac{6}{x^2}}}{x^2 \left(2 - \frac{1}{x^2}\right)}$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} \frac{\sqrt{1 + \frac{6}{x^2}}}{2 - \frac{1}{x^2}}$$

As $x \rightarrow \infty, \frac{1}{x} \rightarrow 0 \therefore \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 6}}{2x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1}{x}$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \frac{1}{x^2}}}{2 - \frac{1}{x^2}} = \frac{\sqrt{1 + 0}}{2 - 0} 0 = 0$$

II.

$$1. \quad \underset{x \rightarrow 0}{Lt} \frac{8|x| + 3x}{3|x| - 2x}$$

Sol : as $x \rightarrow \infty \Rightarrow |x| = x$ (\because here x is positive)

$$\underset{x \rightarrow \infty}{Lt} \frac{8|x| + 3x}{3|x| - 2x} = \underset{x \rightarrow \infty}{Lt} \frac{8x + 3x}{3x - 2x} = \underset{x \rightarrow \infty}{Lt} \frac{11x}{x} = 11$$

$$2. \quad \underset{x \rightarrow \infty}{Lt} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1}$$

$$\text{Sol : } = \underset{x \rightarrow \infty}{Lt} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1} = \underset{x \rightarrow \infty}{Lt} \frac{x^2 \left[1 + \frac{5}{x} + \frac{2}{x^2} \right]}{x^2 \left(2 - \frac{5}{x} + \frac{1}{x^2} \right)}$$

As $x \rightarrow \infty, \frac{1}{x}$ and $\frac{1}{x^2} \rightarrow 0$

$$\underset{x \rightarrow \infty}{Lt} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1} = \underset{x \rightarrow \infty}{Lt} \frac{1 + \frac{5}{x} + \frac{2}{x^2}}{2 - \frac{5}{x} + \frac{1}{x^2}} = \frac{1 + 0 + 0}{2 - 0 + 0} = \frac{1}{2}$$

$$3. \quad \underset{x \rightarrow -\infty}{Lt} \frac{2x^2 - x + 3}{x^2 - 2x + 5}$$

$$\text{Sol ; } \underset{x \rightarrow -\infty}{Lt} \frac{2x^2 - x + 3}{x^2 - 2x + 5} = \underset{x \rightarrow -\infty}{Lt} \frac{x^2 \left(2 - \frac{1}{x} + \frac{3}{x^2} \right)}{x^2 \left(1 + \frac{2}{x} + \frac{5}{x^2} \right)}$$

$$\underset{x \rightarrow -\infty}{Lt} \frac{2x^2 - x + 3}{x^2 - 2x + 5} = \underset{x \rightarrow 0-\infty}{Lt} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{2}{x} + \frac{5}{x^2}} \quad \text{As } x \rightarrow -\infty, \frac{1}{x} \text{ and } \frac{1}{x^2} \rightarrow 0$$

$$= \frac{2 - 0 + 0}{1 - 0 + 0} = \frac{2}{1} = 2$$

4. $\lim_{x \rightarrow -\infty} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

Sol : $\lim_{x \rightarrow -\infty} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

$$= \lim_{x \rightarrow \infty} \frac{x^3 \left(11 - \frac{3}{x^2} + \frac{4}{x^3} \right)}{x^3 \left(13 - \frac{5}{x} - \frac{7}{x^3} \right)}$$

As $x \rightarrow \infty$, $\frac{1}{x}, \frac{1}{x^2}$ and $\frac{1}{x^3} \rightarrow 0$

$$\lim_{x \rightarrow \infty} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$$

$$= \lim_{x \rightarrow \infty} \frac{11 - \frac{3}{x^2} + \frac{4}{x^3}}{13 - \frac{5}{x} - \frac{7}{x^3}} = \frac{11 - 0 + 0}{13 - 0 - 0} = \frac{11}{13}$$

5. $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$

Sol : $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) = \lim_{x \rightarrow 2} \frac{x+2-4}{x^2-4}$

$$= \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x+2-4}{x^2-4} = \frac{1}{4}$$

6. $\lim_{x \rightarrow -\infty} \frac{5x^3 + 4}{\sqrt{2x^4 + 1}}$

Sol : $\lim_{x \rightarrow -\infty} \frac{5x^3 + 4}{\sqrt{2x^4 + 1}} = \lim_{x \rightarrow -\infty} \frac{x^3 \left(5 + \frac{4}{x^3}\right)}{x^2 \sqrt{2 + \frac{1}{x^4}}}$

$$= \lim_{x \rightarrow -\infty} x \cdot \frac{5 + \frac{4}{x^3}}{\sqrt{2 + \frac{1}{x^4}}}$$

As $x \rightarrow -\infty, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4} \rightarrow 0$

$$\lim_{x \rightarrow -\infty} \frac{5x^3 + 4}{\sqrt{2x^4 + 1}}(-\infty) = (-\infty) \cdot \frac{5}{\sqrt{1}} = -\infty.$$

7. $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$

Sol : $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x+1-x}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x} \left(\sqrt{1} + \frac{1}{x} + 1 \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x}}}{\sqrt{1} + \frac{1}{x} + 1} = \frac{0}{1+1} = 0$$

$$8. \quad = \underset{x \rightarrow \infty}{Lt} \left(\sqrt{x^2 + x} - x \right)$$

$$\text{Sol : } = \underset{x \rightarrow \infty}{Lt} \sqrt{x^2 + x} - x$$

$$\begin{aligned} &= \underset{x \rightarrow \infty}{Lt} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + \sqrt{x})}{\sqrt{x^2 + x} + x} = \underset{x \rightarrow \infty}{Lt} \frac{x^2 + x - x^2}{x \left(\sqrt{1 + \frac{1}{x}} + 1 \right)} = \underset{x \rightarrow \infty}{Lt} \frac{x}{x \left(\sqrt{1 + \frac{1}{x}} + 1 \right)} \\ &= \frac{1}{\sqrt{1 + 0 + 1}} = \frac{1}{1 + 1} = \frac{1}{2} \end{aligned}$$

III.

$$1. \quad \underset{x \rightarrow -\infty}{Lt} \left(\frac{2x + 3}{\sqrt{x^2 - 1}} \right)$$

$$\underset{x \rightarrow -\infty}{Lt} \frac{2x + 3}{\sqrt{x^2 - 1}}$$

$$\text{Sol : } = \underset{x \rightarrow -\infty}{Lt} \frac{x \left(2 + \frac{3}{x} \right)}{-x \sqrt{1 - \frac{1}{x^2}}} \left(\because \text{here } x \rightarrow -\infty \text{ i.e., } x \text{ is negative} \Rightarrow \sqrt{x^2} = -x \right)$$

$$\text{As } x \rightarrow -\infty, \frac{1}{x}, \frac{1}{x^2} \rightarrow 0$$

$$\underset{x \rightarrow -\infty}{Lt} \frac{2x + 3}{\sqrt{x^2 - 1}} = -\frac{2 + 0}{\sqrt{1 - 0}} = -\frac{2}{1} = -2$$

$$2. \quad \underset{x \rightarrow \infty}{Lt} \frac{2 + \sin x}{x^2 + 3}$$

Sol : $\lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 + 3} = \lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 \left(1 + \frac{3}{x^2}\right)}$

$$= \lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 \left(1 + \frac{3}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} + \frac{\sin x}{x^2}}{\left(1 + \frac{3}{x^2}\right)}$$

as $x \rightarrow \infty$, $\frac{1}{x^2}$ and $\frac{\sin x}{x^2} \rightarrow 0$. ($\because -1 \leq \sin x \leq 1$) $= \frac{0+0}{1+0} = 0$

3. $\lim_{x \rightarrow \infty} \frac{2 + \cos^2 x}{x + 2007}$ (ans 0)

4. $\lim_{x \rightarrow \infty} \frac{6x^2 - \cos 3x}{x^2 + 5}$ ans. 6

5. $\lim_{x \rightarrow \infty} \frac{\cos x + \sin^2 x}{x + 1}$ try yourself. ans $= \lim_{x \rightarrow \infty} \frac{\cos x + \sin^2 x}{x + 1} = 0$

PROBLEMS FOR PRACTICE

1. Evaluate $\lim_{x \rightarrow -3} \frac{1}{x + 2}$.

2. Compute $\lim_{x \rightarrow 2} \frac{x - 2}{x^3 - 8}$.

3. Find $\lim_{x \rightarrow 1} (x + 2)(2x + 1)$

4. Compute $\lim_{x \rightarrow 0} x^2 \cdot \sin \frac{1}{x}$.

5. Show that $\lim_{x \rightarrow 0+} \frac{|x|}{x} = 1$ and $\lim_{x \rightarrow 0-} \frac{|x|}{x} = -1$ ($x \neq 0$)

6. Let $f : R \rightarrow R$ be defined by $f(x) = \begin{cases} 2x - 1 & \text{if } x < 3 \\ 5 & \text{if } x \geq 3 \end{cases}$ show that $\lim_{x \rightarrow 3^-} f(x) = 5$.

7. Show that $\lim_{x \rightarrow -2} \sqrt{x^2 - 4} = 0, = \lim_{x \rightarrow 2} \sqrt{x^2 - 4}$

8. If $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1 & x > 1 \end{cases}$, then find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.

9. Show that $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$.

10. Find $\lim_{x \rightarrow 0} \left\{ \frac{\sqrt{1+x-1}}{x} \right\}$

11. Compute $\lim_{x \rightarrow 0} \left[\frac{e^x - 1}{\sqrt{1+x} - 1} \right]$.

Sol : For $0 < |x| < 1$,

$$\begin{aligned} \frac{e^x - 1}{\sqrt{1+x} - 1} &= \frac{e^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} \\ &= \frac{e^x - 1(\sqrt{1+x} + 1)}{14x - y} = \frac{e^x - 1}{x} (\sqrt{1+x} + 1) \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1+x} - 1} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

$$\lim_{x \rightarrow 0} \sqrt{1+x} + 1 = 1.(\sqrt{1+0} + 1) = (1+1) = 2$$

12. Show that $\lim_{x \rightarrow 0} \frac{x-3}{\sqrt{|x^2-9|}} = 0$

Sol : For $x^2 \neq 9$, $\left| \frac{x-3}{\sqrt{x^2-9}} \right| = \sqrt{\left| \frac{x-3}{x+3} \right|}$ (1)

$$\lim_{x \rightarrow 3} \sqrt{|x - 3|} = 0, \lim_{x \rightarrow 3} \sqrt{|x + 3|} = \sqrt{6}$$

$$\therefore \lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x^2 - 9}} = 0$$

13. Compute $\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1}$ ($a > 0, b > 0, b \neq 1$).

Sol : For $x \neq 0$, $\frac{a^x - 1}{b^x - 1} = \frac{\left[\frac{a^x - 1}{x} \right]}{\left[\frac{b^x - 1}{x} \right]}$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1} = \lim_{x \rightarrow 0} \frac{\frac{a^x - 1}{x}}{\frac{b^x - 1}{x}} = \frac{\log_e^a}{\log_e^b}$$

14. Show that $\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$.

15. Show that $\lim_{x \rightarrow \infty} e^x = \infty$.

16. Compute $\lim_{x \rightarrow 2} \frac{x^2 + 2x - 1}{x^2 - 4x + 4}$

17. Evaluate $\lim_{x \rightarrow \infty} \frac{3x^5 - 1}{4x^2 + 1}$

18. $\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx}, N \neq 0$

$$\text{Sol. } \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{mx}{2}}{2 \sin^2 \frac{nx}{2}} = \lim_{x \rightarrow 0} \frac{\frac{\sin^2 \frac{mx}{2}}{\left(\frac{mx}{2}\right)^2} \left(\frac{mx}{2}\right)^2}{\frac{\sin^2 \frac{nx}{2}}{\left(\frac{nx}{2}\right)^2} \left(\frac{nx}{2}\right)^2}$$

As $x \rightarrow 0$

Then $\frac{mx}{2} \rightarrow 0, \frac{nx}{2} \rightarrow 0$

$$= \frac{\lim_{\frac{mx}{2} \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right)^2 \times \frac{m^2}{n^2}}{\lim_{\frac{nx}{2} \rightarrow 0} \left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right)^2} = (1) = \frac{m^2}{n^2}$$

19. $\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$

$$\text{Sol. } \lim_{x \rightarrow 0} \frac{x^2(e^x - 1)}{x \cdot 2 \sin^2 \frac{x}{2}}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{2 \sin^2 \frac{x}{2}} \times \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

= As $x \rightarrow 0$

$$\frac{x}{2} \rightarrow 0$$

$$\begin{aligned}
 & \text{Lt}_{\frac{x}{2} \rightarrow 0} \frac{\frac{x^2}{4} \times 4}{2 \sin^2 \frac{x}{4}} \left(\text{Lt}_{x \rightarrow 0} \frac{e^x - 1}{x} \right) \\
 &= \frac{4}{2} \text{Lt}_{\frac{x}{2} \rightarrow 0} \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right) \cdot \text{Lt}_{x \rightarrow 0} \frac{e^x - 1}{x} \\
 &= 2(1)(1) = 2
 \end{aligned}$$

20. $\text{Lt}_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x}$

Sol. $\text{Lt}_{x \rightarrow 0} \frac{\log \frac{(1+x^3)}{x^3}}{\frac{\sin^3 x}{x^3}}$

As $x \rightarrow 0$

Then $x^3 \rightarrow 0$

$$\begin{aligned}
 & \text{Lt}_{x^3 \rightarrow 0} \frac{\log \frac{(1+x^3)}{x^3}}{\frac{\sin^3 x}{x^3}} = \frac{1}{1} = 1 \\
 & \text{Lt}_{x^3 \rightarrow 0} \left(\frac{\sin x}{x} \right)^3
 \end{aligned}$$

21. $\text{Lt}_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1-\cos 2x)^2}$

Sol. $\text{Lt}_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1-\cos 2x)^2}$

$$\begin{aligned}
 & \text{Lt}_{x \rightarrow 0} \frac{x \frac{\tan x}{1-\tan^2 x} - 2x \tan x}{(2 \sin^2 x)^2} \\
 &= \text{Lt}_{x \rightarrow 0} \frac{2x \tan \left[\frac{1}{1-\tan^2 x} - 1 \right]}{4 \sin^2 x} \\
 &= \text{Lt}_{x \rightarrow 0} \frac{2x \tan x \left[\frac{1}{1-\tan^2 x} - 1 \right]}{4 \sin^2 x \left[\frac{1-1+\tan^2 x}{1-\tan^4 x} \right]}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \frac{2x \tan^3 x}{4 \sin^4 x} \\
 &= \lim_{x \rightarrow 0} \frac{2x^4 \tan^3 x}{x^3 4 \sin^4 x} dx \\
 &= \frac{2}{4} \lim_{x \rightarrow 0} \frac{x^4}{\sin^4 x} \cdot \lim_{x \rightarrow 0} \frac{\tan^3 x}{x^3} \\
 &= \frac{1}{2}(1)(1) = \frac{1}{2}
 \end{aligned}$$

22. $\lim_{x \rightarrow \infty} \frac{2 + \cos^2 x}{x + 2007}$

Sol. $\lim_{x \rightarrow \infty} \frac{2 + \cos^2 x}{x + 2007}$

We know that

$$0 \leq \cos^2 x \leq 1, 2 \leq 2 + \cos^2 x \leq 3$$

$$\frac{2}{x + 2007} \leq \frac{2 + \cos^2 x}{x + 2007} \leq \frac{3}{x + 2007}$$

$$\text{Let } g(x) = \frac{2}{x + 2007}, h(x) = \frac{3}{x + 2007}$$

$$f(x) = \frac{2 + \cos^2 x}{x + 2007} \quad \dots(1)$$

$$\therefore g(x) \leq f(x) \leq h(x)$$

$$\lim_{x \rightarrow \infty} g(x), \lim_{x \rightarrow \infty} f(x) \leq \lim_{x \rightarrow \infty} h(x) = 0$$

$$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} \frac{2}{x + 2007}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x}}{1 + \frac{2007}{x}} = \frac{0}{1+0} = 0$$

$$\lim_{x \rightarrow \infty} h(x) = \lim_{x \rightarrow \infty} \frac{3}{x + 2007}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{3}{x}}{1 + \frac{2007}{x}} = \frac{0}{1+0} = 0$$

$$\therefore \lim_{x \rightarrow \infty} g(x) = 0 = \lim_{x \rightarrow \infty} h(x)$$

$$\Rightarrow \lim_{x \rightarrow \infty} f(x) = 0$$

23. Compute $\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x}$

Sol. We have $\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x}$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \left[\frac{e^x - 1}{x} - \frac{\sin x}{x} \right] \\ &= \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) - \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 1 - 1 = 0 \end{aligned}$$

24. If $f(x) = \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$ when $a_n > 0, b_m > 0$, then show that $\lim_{x \rightarrow \infty} f(x) = \infty$ if $n > m$.

Sol. $f(x) = \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$

$$\begin{aligned} &= \frac{x^n \left(a_n + \frac{a_n - 1}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)}{x^m \left(b_m + \frac{b_m - 1}{x} + \dots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m} \right)} \\ &= x^{n-m} \left(\frac{a_n + \frac{a_n - 1}{x} + \dots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}{b_m + \frac{b_m - 1}{x} + \dots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} \right) \end{aligned}$$

As $x \rightarrow \infty$, $\frac{a_{n-i}}{x^i}, \frac{b_{m-i}}{x^i}$ approach zero

\therefore The quantity in the brackets above approach $\frac{a_n}{b_m} (> 0)$

But $\lim_{x \rightarrow \infty} x^{n-m} = \infty$

$\therefore \lim_{x \rightarrow \infty} f(x) = \infty$