

THEOREMS ON STANDARD LIMITS

THEOREM

If n is a rational number and $a > 0$ then $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$

Note 1: If n is a positive integer, then for any $a \in R$, $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$

Note 2: If n is a real number and $a > 0$ then $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$

Note 3: If m and n are any real numbers and $a > 0$, then $\lim_{x \rightarrow a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$.

THEOREM 2

If $0 < x < \frac{\pi}{2}$ then $\sin x < x < \tan x$.

Corollary 1:

If $-\frac{\pi}{2} < x < 0$ then $\tan x < x < \sin x$

Corollary 2:

If $0 < |x| < \frac{\pi}{2}$ then $|\sin x| < |x| < |\tan x|$

STANDARD LIMITS

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a, \quad \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e,$$
$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad \text{and} \quad \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

EXERCISE

I. Compute the following limits

1. $\lim_{x \rightarrow 1} \left[\frac{2x+1}{3x^2-4x+5} \right]$

Sol: $\lim_{x \rightarrow 1} \frac{2x+1}{3x^2-4x+5} = \frac{2.1+1}{3.1^2-4.1+5} = \frac{2+1}{8-4} = \frac{3}{4}$

2. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\left[x - \frac{\pi}{2} \right]}$

Sol: Let $y = x - \frac{\pi}{2}$ so that as $x \rightarrow \frac{\pi}{2}$, $y \rightarrow 0$ and $x = y + \frac{\pi}{2}$

$\therefore \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = \lim_{y \rightarrow 0} \frac{\cos\left(\frac{\pi}{2} + y\right)}{y} = \lim_{y \rightarrow 0} \frac{-\sin y}{y} = -1$

3. $\lim_{x \rightarrow 0} \frac{\sin ax}{x \cos x}$

Sol: $\lim_{x \rightarrow 0} \frac{\sin ax}{x \cos x} = \frac{\lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot a}{\lim_{x \rightarrow 0} \cos x} = \frac{a}{1} = a$

4. $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x^2-1)}$

Sol. $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2-1} = \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)} \cdot \lim_{x \rightarrow 1} \frac{1}{x+1}$

Put $y = x - 1$ so that as $x \rightarrow 1$, $y \rightarrow 0$

$$= \lim_{y \rightarrow 0} \frac{\sin y}{y} \cdot \frac{1}{1+1} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

5. $\lim_{x \rightarrow 0} \frac{\sin(a+bx) - \sin(a-bx)}{x}$

Sol :

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin(a+bx) - \sin(a-bx)}{x} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos\left(\frac{a+bx+a-bx}{2}\right) \cdot \sin\left(\frac{a+bx-a-bx}{2}\right)}{x} \\ &= \lim_{x \rightarrow 0} 2 \cos a \cdot \lim_{x \rightarrow 0} \frac{\sin bx}{bx} \cdot b = 2 \cos a \cdot b = 2b \cos a. \end{aligned}$$

6. $\lim_{x \rightarrow a} \frac{\tan(x-a)}{x^2-a^2} (a \neq 0)$

Sol :

$$\lim_{x \rightarrow a} \frac{\tan(x-a)}{x^2-a^2} = \lim_{x \rightarrow a} \frac{\tan(x-a)}{(x-a)(x+a)} = \lim_{x \rightarrow a} \frac{\tan(x-a)}{(x-a)} \cdot \lim_{x \rightarrow a} \frac{1}{(x+a)}$$

In the first limit Put $x-a = h$ so that as $x \rightarrow a$, $h \rightarrow 0$

$$G.L = \lim_{h \rightarrow 0} \frac{\tan h}{h} \cdot \left(\frac{1}{a+a}\right) = 1 \cdot \frac{1}{2a} = \frac{1}{2a}$$

II.

1. $\lim_{x \rightarrow 1} \frac{(2x-1)(\sqrt{x}-1)}{(2x^2+x-3)}$

Sol : $\lim_{x \rightarrow 1} \frac{(2x-1)(\sqrt{x}-1)}{2x^2+x-3} = \lim_{x \rightarrow 1} \frac{(2x-1)(\sqrt{x}-1)}{(x-1)(2x+3)} = \lim_{x \rightarrow 1} \frac{(2x-1)(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)(2x+3)}$

$$= \frac{2.1 - 1}{(\sqrt{1} + 1)(2.1 + 3)} = \frac{1}{(1 - 1)5} = \frac{1}{10}$$

2. $\lim_{x \rightarrow a} \left[\frac{x \sin a - a \sin x}{x - a} \right]$

Sol : $\lim_{x \rightarrow a} \frac{x \sin a - a \sin x}{x - a} = \lim_{x \rightarrow a} \frac{(x \sin a - a \sin a) - (a \sin x - a \sin a)}{(x - a)}$

$$= \lim_{x \rightarrow a} \frac{(x - a) \sin a - a(\sin x - \sin a)}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{(x - a) \sin a}{x - a} - \lim_{x \rightarrow a} a \left(\frac{\sin x - \sin a}{x - a} \right)$$

$$= \sin a - a \cdot \lim_{x \rightarrow a} \frac{2 \cos \frac{x+a}{2} \cdot \sin \frac{x-a}{2}}{x - a}$$

$$= \sin a - a \cdot \lim_{x \rightarrow a} \frac{\cos(x+a)}{2} \cdot \lim_{x \rightarrow a} \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}$$

$$= \sin a - a \cos a - 1 = \sin a - a \cos a$$

3. $\lim_{x \rightarrow 0} \left[\frac{\cos ax - \cos bx}{x^2} \right]$

Sol : $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin \frac{(a+b)x}{2} \cdot \sin \frac{(b-a)x}{2}}{x^2}$

$$= 2 \cdot \lim_{x \rightarrow 0} \frac{\sin(b+a) \frac{x}{2}}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin(b-a) \frac{x}{2}}{x}$$

$$= 2 \cdot \lim_{x \rightarrow 0} \frac{\sin(b+a) \frac{x}{2}}{(b+a) \frac{x}{2}} \times \frac{(b+a)}{2} \cdot \lim_{x \rightarrow 0} \frac{\sin(b-a) \frac{x}{2}}{(b-a) \frac{x}{2}} \times \frac{(b-a)}{2}$$

$$= 2 \cdot \left(\frac{b+a}{2}\right) \left(\frac{b-a}{2}\right) = \frac{1}{2}(b^2 - a^2)$$

4. $\lim_{x \rightarrow 2} \frac{(2x^2 - 7x - 4)}{(2x - 1)(\sqrt{x} - 2)}$

Sol: $\lim_{x \rightarrow 2} \frac{2x^2 - 7x - 4}{(2x - 1)(\sqrt{x} - 2)} = \lim_{x \rightarrow 2} \frac{(x - 4)(2x + 1)}{(2x - 1)(\sqrt{x} - 2)} = \lim_{x \rightarrow 2} \frac{(\sqrt{x} + 2)(2x + 1)}{(2x - 1)}$

$$= \frac{(\sqrt{2} + 2)(4 + 1)}{(4 - 1)} = \frac{5(2 + \sqrt{2})}{3}$$

III.

1. $\lim_{x \rightarrow 0} \left[\frac{(1+x)^{\frac{1}{8}} - (1-x)^{\frac{1}{8}}}{x} \right]$

Sol: $\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{8}} - (1-x)^{\frac{1}{8}}}{x} = \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{8}} - 1 + 1 - (1-x)^{\frac{1}{8}}}{x}$

$$= \lim_{(1+x) \rightarrow 1} \frac{(1+x)^{\frac{1}{8}} - 1}{(1+x) - 1} + \lim_{(1-x) \rightarrow 1} \frac{(1-x)^{\frac{1}{8}} - 1}{(1-x) - 1}$$

$$= \frac{1}{8} 1^{-7/8} + \frac{1}{8} 1^{-7/8} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

2. $\lim_{x \rightarrow 0} \left[\frac{3^x - 1}{\sqrt{1+x} - 1} \right]$

Sol : $\lim_{x \rightarrow 0} \frac{3^x - 1}{\sqrt{1+x} - 1} = \lim_{x \rightarrow 0} \frac{3^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$ (rationalising Dr.)

$$= \lim_{x \rightarrow 0} \frac{(3^x - 1)(\sqrt{1+x} + 1)}{1 + x - 1} = \lim_{x \rightarrow 0} \frac{3^x - 1}{x} \cdot \lim_{x \rightarrow 0} (\sqrt{1+x} + 1)$$

$$= (\log 3)(\sqrt{1+0} + 1) = 2 \cdot \log 3$$

3. $\lim_{x \rightarrow a} \left[\frac{(\sqrt{a+2x} - \sqrt{3x})}{(\sqrt{3a+x} - 2\sqrt{x})} \right]$

Sol : rationalise both nr. And dr., then

$$\text{G.L.} = \frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{a+2x} + \sqrt{3x})}{(\sqrt{a+2x} + \sqrt{3x})} + \lim_{x \rightarrow a} \frac{(\sqrt{3a+x} + \sqrt{4x})}{(\sqrt{3a+x} - \sqrt{4x})(\sqrt{3a+x} + \sqrt{4x})}$$

$$= \lim_{x \rightarrow a} \frac{a+2x-3x}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + \sqrt{4x}}{3a+x-4x} = \lim_{x \rightarrow a} \frac{(a-x)(\sqrt{3a+x} + \sqrt{4x})}{(\sqrt{a+2x} + \sqrt{3x})3(a-x)}$$

$$= \frac{2(2a)}{2(\sqrt{3a})3} = \frac{2}{3\sqrt{3}}$$

4.
$$\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}$$

Sol:
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x} &= \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x} \\ &= \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{3}} - 1 + 1 - (1-x)^{\frac{1}{3}}}{x} \\ &= \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{3}} - 1}{(1+x) - 1} + \lim_{x \rightarrow 0} \frac{(1-x)^{\frac{1}{3}} - 1}{(1-x) - 1} = \frac{1}{3} \cdot 1^{-2/3} + \frac{1}{3} \cdot 1^{-2/3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

5.
$$\lim_{x \rightarrow a} \frac{\sin(x-a) \tan^2(x-a)}{(x^2 - a^2)^2}$$

Sol:
$$\begin{aligned} \lim_{x \rightarrow a} \frac{\sin(x-a) \tan^2(x-a)}{(x^2 - a^2)^2} &= \lim_{x \rightarrow a} \frac{\sin(x-a)}{x-a} \left(\lim_{x \rightarrow a} \frac{\tan(x-a)}{(x-a)} \right)^2 + \lim_{x \rightarrow a} \frac{(x-a)}{(x+a)^2} = 1 \cdot 1 \cdot \frac{0}{(2a)^2} = 0 \end{aligned}$$

6.
$$\lim_{x \rightarrow 0} \frac{1 - \cos 2mx}{\sin^2 nx} \quad (m, n \in \mathbb{R})$$

Sol:
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos 2mx}{\sin^2 nx} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 mx}{\sin^2 nx} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 mx}{\sin^2 nx} \times \frac{x^2}{x^2} = \lim_{x \rightarrow 0} \frac{2 \frac{\sin^2 mx}{x^2}}{\frac{\sin^2 nx}{x^2}} = 2 \frac{\left(\lim_{x \rightarrow 0} \frac{\sin mx}{mx} \right)^2}{\left(\lim_{x \rightarrow 0} \frac{\sin nx}{nx} \right)^2} \times \frac{m^2}{n^2} = \frac{2m^2}{n^2} \end{aligned}$$