CHAPTER 8

LIMITS

TOPICS:

LIMITS

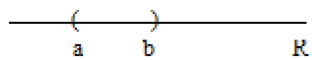
INTERVALS

Definition:

Let a, b∈ R and a < b. Then the set {x∈ R: a≤ x≤ b} is called a closed interval. It is denoted by [a, b]. Thus

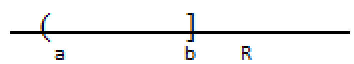
Closed interval [a, b] = {x∈ R: a ≤ x ≤ b}. It is geometrically represented by

Open interval (a,b) = $\{x \in R : a < x < b\}$ It is geometrically represented by



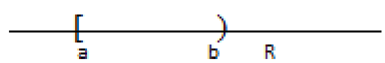
Left open interval

(a, b] = $\{x \in R : a < x \leq b\}$. It is geometrically represented by

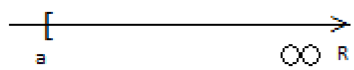


Right open interval

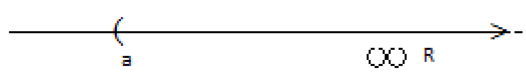
[a, b) = $\{x \in R : a \leq x < b\}$. It is geometrically represented by



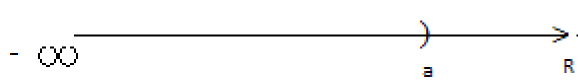
$[a, \infty) = \{x \in R : x \geq a\} = \{x \in R : a \leq x < \infty\}$ It is geometrically represented by



$(a, \infty) = \{x \in R : x > a\} = \{x \in R : a < x < \infty\}$



$(-\infty, a] = \{x \in R : x \leq a\} = \{x \in R : -\infty < x < a\}$



NEIGHBOURHOOD OF A POINT

Definition: Let $a \in R$. If $\delta > 0$ then the open interval $(a-\delta, a+\delta)$ is called the neighbourhood (δ - nbd) of the point a. It is denoted by $\boldsymbol{N_\delta(a)}$. a is called the centre and δ is called the radius of the neighbourhood .

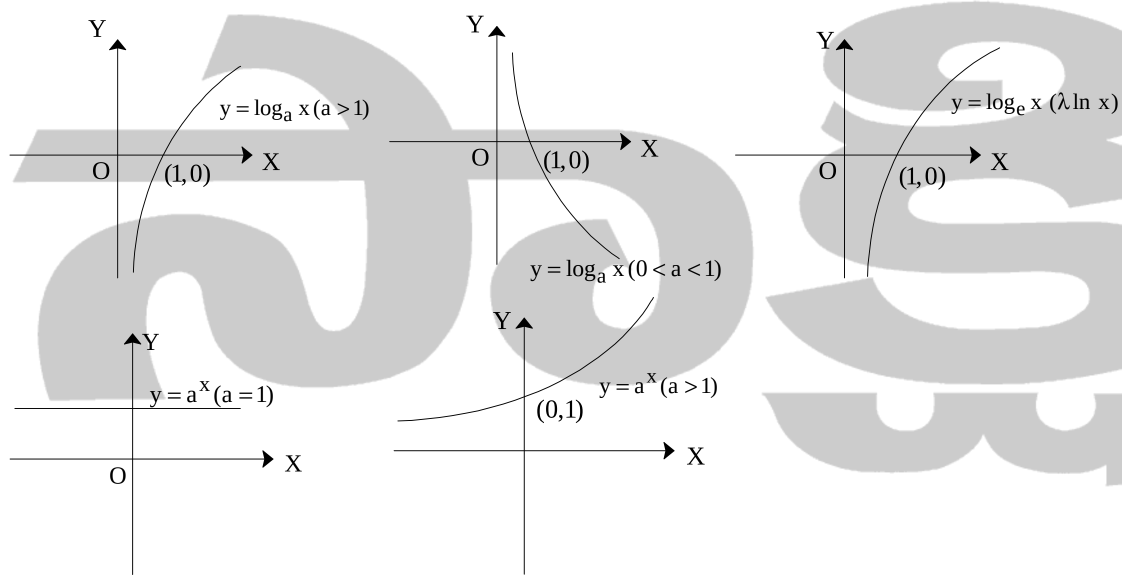
$\therefore N_\delta(a) = (a-\delta,\ a+\delta) = \{x \in R : a-\delta < x < a+\delta\} = \left\{x \in R : |x-a| < \delta\right\}$

The set $N_\delta(a) - \{a\}$ is called a deleted
δ - neighbourhood of the point a.

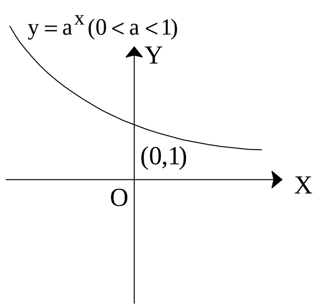
$\therefore N_\delta(a) - \{a\} = (a-\delta,\ a) \cup (a,\ a+\delta) = \{x \in R : 0 < |x-a| < \delta\}$

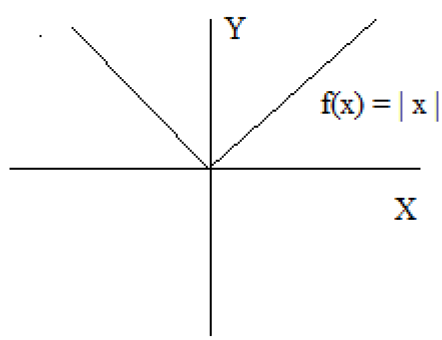
Note: $(a-\delta,\ a)$ is called left δ -neighbourhood, $(a,\ a+\delta)$ is called right δ - neighbourhood of a

GRAPH OF A FUNCTION:



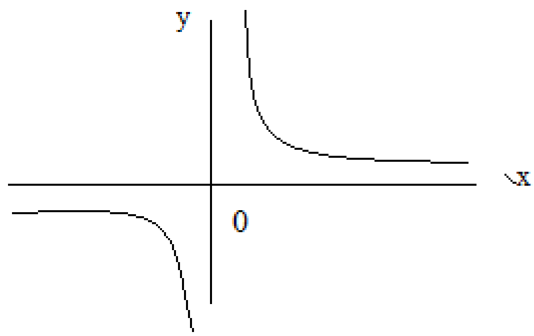
Mod function:
The function f : R-R defined by f(x) = | x | is called the mod function or modulus function or absolute value function.
Dom f = R, Range f = [0,)



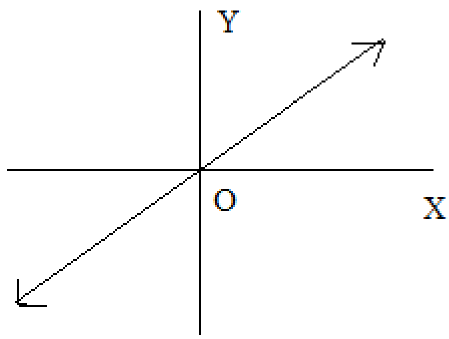


Reciprocal function :
The function $f : R - \{0\}$- R defined by
$f(x) = \frac{1}{x}$ is called the reciprocal function.

Dom $f = R - \{0\}$, Range $f = R$



Identity funciton:
The function f: R-R defined by $f(x) = x$ is called the identity function on R. It is denoted by I(x) .



LIMIT OF A FUNCTION

Concept of limit:
Before giving the formal definition of limit consider the following example.

Let f be a function defined by $f(x) = \frac{x^2 - 4}{x - 2}$. clearly, f is not defined at x= 2.

When $x \neq 2, x - 2 \neq 0 and f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$

Now consider the values of f(x) when x≠2, but very very close to 2 and <2.

x	1.9	1.99	1.999	1.9999	1.99999
F(x)	3.9	3.99	3.999	3.9999	3.99999

It is clear from the above table that as x approaches 2 i.e.,x$\to$2 through the values less than 2, the value of f(x) approaches 4 i.e., f(x)$\to$4. We will express this fact by saying that left hand limit of f(x) as x$\to$2 exists and is equal to 4 and in symbols we shall write $\lim\limits_{x\to 2^-} f(x)=4$

Again we consider the values of f(x) when x$\neq$2, but is very-very close to 2 and x>2.

x	2.1	2.01	2.001	2.0001	2.00001
F(x)	4.1	4.01	4.001	4.0001	4.00001

It is clear from the above table that as x approaches 2 i.e.,x$\to$2 through the values greater than 2, the value of f(x) approaches 4 i.e., f(x)$\to$4. We will express this fact by saying that right hand limit of f(x) as x$\to$2 exists and is equal to 4 and in symbols we shall write $\lim\limits_{x\to 2^+} f(x)=4$.

Thus we see that f(x) is not defined at x=2 but its left hand and right hand limits as x$\to$2 exist and are equal.

When $\lim\limits_{x\to a^+} f(x), \lim\limits_{x\to a^-} f(x)$ are equal to the same number l, we say that $\lim\limits_{x\to a} f(x)$ exist and equal to l.

Thus , in above example, $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^-} f(x)=4$. $\therefore \lim\limits_{x\to 2} f(x)=4$

ONE SIDED LIMITS

DEFINITION OF LEFT HAND LIMIT

Let f be a function defined on (a – h, a), h > 0. A number ℓ_1 is said to be the left hand limit (LHL) or left limit (LL) of f at a if to each $\varepsilon>0, \exists$ a $\delta>0$ such that, $a-\delta < x < a \Rightarrow |f(x)-\ell_1|<\varepsilon$.

In this case we write $\underset{x\to a-}{Lt} f(x)=\ell_1$ (or) $\underset{x\to a-0}{Lt} f(x)=\ell_1$

DEFINITION OF RIGHT LIMIT:

Let f be a function defined on (a, a + h), h > 0. A number ℓ_2 is said to the right hand limit (RHL) or right limit (RL) of f at a if to each $\varepsilon>0, \exists$ a $\delta>0$ such that $a<x<a+\delta \Rightarrow |f(x)-\ell_2|<\varepsilon$

In this case we write $\underset{x\to a+}{Lt} f(x)=\ell_2$ (or) $\underset{x\to a+0}{Lt} f(x)=\ell_2$.

DEFINITION OF LIMIT.

Let $A\subseteq R$, a be a limit point of A and f : A$\to$R. A real number l is said to be the limit of f at a if to each $\varepsilon>0, \exists$ a $\delta>0$ such that x$\in$ A, 0 < |x – a| < $\delta \Rightarrow$ | f(x) – l | < $\in$.

In this case we write f(x) $\to$ l as x$\to$ a or $\underset{x\to a}{Lt} f(x)=\ell$

NOTE:

1.If a function f is defined on (a–h, a) for some $h > 0$ and is not defined on (a, a+h) and if $\underset{x\to a-}{Lt} f(x)$ exists then $\underset{x\to a}{Lt} f(x)= \underset{x\to a-}{Lt} f(x)$.

2. If a function f is defined on (a, $a + h$) for some $h > 0$ and is not defined on ($a - h$, a) and if $\underset{x\to a+}{Lt} f(x)$ exists then $\underset{x\to a}{Lt} f(x)= \underset{x\to a+}{Lt} f(x)$.

THEOREM

If $\underset{x\to a}{Lt} f(x)$ exists then $\underset{x\to a}{Lt} f(x)= \underset{x\to 0}{Lt} f(x+a)= \underset{x\to 0}{Lt} f(a-x)$

THEOREMS ON LIMITS WITH OUT PROOFS

1. If f : R$\to$R defined by f(x) = c, a constant then $\underset{x\to a}{Lt} f(x)=c$ for any a $\in$ R.

2. If f: R$\to$R defined by f(x) = x, then $\underset{x\to a}{Lt} f(x)=a$ i.e., $\underset{x\to a}{Lt}\, x=a \quad (a\in R)$

3. Algebra of limits

Let $\underset{x\to a}{Lt} f(x)=\ell,\ \underset{x\to a}{Lt} g(x)=m.$ then

i) $\underset{x\to a}{Lt} (f+g)(x)= \underset{x\to a}{Lt} (f(x)+g(x))=\ell+m$

ii) $\underset{x\to a}{Lt} (f-g)(x)= \underset{x\to a}{Lt} (f(x)-g(x))=\ell-m$

iii) $\underset{x\to a}{Lt} (cf)(x)= \underset{x\to a}{Lt} c.f(x)=c\underset{x\to a}{Lt} f(x)=c\ell$

iv) $\underset{x\to a}{Lt} (fg)(x)= \underset{x\to a}{Lt} (f(x).g(x))=\ell.m$

v) $\underset{x\to a}{Lt}\left(\frac{f}{g}\right)(x)= \underset{x\to a}{Lt}\left(\frac{f(x)}{g(x)}\right)=\frac{\ell}{m}$ **(m≠0)**

vi) $\underset{x\to a}{Lt} (f(x)-\ell)=0$ **and vii)** $\underset{x\to a}{Lt} |f(x)|=|\ell|$

vii) If $f(x) \le g(x)$ in some deleted neighbourhood of a, then $\underset{x\to a}{Lt} f(x)\le \underset{x\to a}{Lt} g(x)$

viii) If $f(x) \le h(x) \le g(x)$ in a deleted nbd of a and $\underset{x\to a}{Lt} f(x)=\ell= \underset{x\to a}{Lt} g(x)$ **then** $\underset{x\to a}{Lt} h(x)=\ell$

ix) If $\underset{x\to a}{Lt} f(x)=0$ and $g(x)$ is a bounded function in a deleted nbd of a then $\underset{x\to a}{Lt} f(x)g(x)=0$.

THEOREM

If n is a positive integer then $\underset{x \to a}{Lt}\, x^n = a^n, a \in R$

THEOREM

If f(x) is a polynomial function, then $\underset{x \to a}{Lt}\, f(x) = f(a)$

EVALUATION OF LIMITS

A) Evaluation of limits involving algebraic functions.

To evaluate the limits involving algebraic functions we use the following methods:

1) **Direct substitution method**
2) **Factorisation method**
3) **Rationalisation method**
4) **Application of the standard limits.**

1) **Direct substitution method:**

This method can be used in the following cases:

(i) If $f(x)$ is a polynomial function, then $\underset{x \to a}{Lt}\, f(x) = f(a)$.

(ii) If $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions then $\underset{x \to a}{Lt}\, f(x) = \dfrac{P(a)}{Q(a)}$, provided $Q(a) \neq 0$.

2) **Factiorisation Method:**

This method is used when $\underset{x \to a}{Lt}\, \dfrac{f(x)}{g(x)}$ is taking the indeterminate form of the type $\dfrac{0}{0}$ by the substitution of $x = a$.

In such a case the numerator (Nr.) and the denominator (Dr.) are factorized and the common factor $(x - a)$ is cancelled. After eliminating the common factor the substitution $x = a$ gives the limit, if it exists.

3) **Rationalisation Method :** This method is used when $\underset{x \to a}{Lt}\, \dfrac{f(x)}{g(x)}$ is a $\frac{0}{0}$ form and either the Nr. or Dr. consists of expressions involving radical signs.

4) **Application of the standard limits.**

In order to evaluate the given limits , we reduce the given limits into standard limits form and then we apply the standard limits.

EXERCISE

I. Compute the following limits.

1. $\underset{x \to a}{Lt} \dfrac{x^2 - a^2}{x - a}$

Sol : Given limit $= \underset{x \to a}{Lt} \dfrac{x^2 - a^2}{x - a} = \underset{x \to a}{Lt} \dfrac{(x+a)(x-a)}{x-a} = \underset{x \to a}{Lt}(x+a) \quad = a + a$

$= 2a$

2. $\underset{x \to a}{Lt}\left(x^2 + 2x + 3\right)$

Sol : given function $f(x) = x^2 + 2x + 3$ is a polynomial.

$\therefore \underset{x \to 1}{Lt}\left(x^2 + 2x + 3\right) = 1^2 + 2.1 + 3 \quad = 1 + 2 + 3 = 6$

3. $\underset{x \to 0}{Lt} \dfrac{1}{x^2 - 3x + 2}$

Sol : $\underset{x \to 0}{Lt} \dfrac{1}{x^2 - 3x + 2} = \dfrac{1}{0 - 0 + 2} = \dfrac{1}{2}$

4. $\underset{x \to 3}{Lt} \dfrac{1}{x + 1}$

Sol : $\underset{x \to 3}{Lt} \dfrac{1}{x + 1} = \dfrac{1}{3 + 1} = \dfrac{1}{4}$

5. $\underset{x \to 1}{Lt} \dfrac{2x + 1}{3x^2 - 4x + 5}$

Sol : $\underset{x \to 1}{Lt} \dfrac{2x + 1}{3x^2 - 4x + 5} = \dfrac{2.1 + 1}{3.1^2 - 4.1 + 5} = \dfrac{3}{4}$

6. $\underset{x\to 1}{Lt}\dfrac{x^2+2}{x^2-2}$

Sol : $\underset{x\to 1}{Lt}\dfrac{x^2+2}{x^2-2}=\dfrac{1^2+2}{1^2-2}=\dfrac{1+2}{1-2}=\dfrac{3}{-1}=-3$

7. $\underset{x\to 1}{Lt}\left(\dfrac{2}{x+1}-\dfrac{3}{x}\right)$

Sol : G.L.= $\underset{x\to 2}{Lt}\left(\dfrac{2}{x+1}-\dfrac{3}{x}\right)=\dfrac{2}{2+1}-\dfrac{3}{2}=\dfrac{2}{3}-\dfrac{3}{2}=\dfrac{4-9}{6}=\dfrac{-5}{6}$

8. $\underset{x\to 0}{Lt}\left[\dfrac{x-1}{x^2+4}\right]$

Sol : $\underset{x\to 0}{Lt}\left[\dfrac{x-1}{x^2+4}\right]=\dfrac{0-1}{0+4}=-\dfrac{1}{4}$

9. $\underset{x\to 0}{Lt}\ x^{3/2}\,(x>0)$

Sol : $\underset{x\to 0}{Lt}\ x^{3/2}\,(x>0)=0^{3/2}=0$

10. $\underset{x\to 0}{Lt}\left(\sqrt{x}+x^{5/2}\right)(x>0)$

Sol : $\underset{x\to 0}{Lt}\left(\sqrt{x}+x^{5/2}\right)=\sqrt{0}+0^{5/2}=0+0=0$

11. $\underset{x\to 0}{Lt}\ x^2\cos\dfrac{2}{x}$

Sol : $\underset{x\to 0}{Lt}\ x^2.\underset{x\to 0}{Lt}\cos\dfrac{2}{x}=0.k$ Where $|k|\le 1=0$

EXERCISE

I. Find the right and left hand limits of the functions in 1,2,3 of I and 1,2,3 of II at the point a mentioned against them. Hence, check whether the functions have limits at those a' s.

1. $f(x) = \begin{cases} 1-x & \text{if} \quad x \le 1 \\ 1+x & \text{if} \quad x > 1 \end{cases}$; $a = 1.$

Sol : $\textit{Left}$ limit at x=1 is $\underset{x \to 1-}{Lt}(1-x) = 1-1 = 0$

$\textit{Right}$ limit at x=1 is $\underset{x \to 1+}{Lt}(1+x) = 1+1 = 2$

$\underset{x \to 1-}{Lt} f(x) \ne \underset{x \to 1+}{Lt} f(x)$

$\therefore \underset{x \to 1}{Lt} f(x)$ does not exist.

2. $f(x) = \begin{cases} x+2 & \text{if} \quad -1 < x \le 3 \\ x^2 & \text{if} \quad 3 < x < 5 \end{cases}$; $a = 3.$

Sol : $L.L = \underset{x \to 3-}{Lt}(x+2) = 3+2 = 5$

$R.L = \underset{x \to 3+}{Lt} x^2 = 3^2 = 9$

$\underset{x \to 3-}{Lt} f(x) \ne \underset{x \to 3+}{Lt} f(x)$

$\underset{x \to 3}{Lt} f(x)$ does not exists.

3. $f(x) = \begin{cases} \dfrac{x}{2} & if \quad x < 2 \\ \dfrac{x^2}{3} & if \quad x \geq 2 \end{cases}$; $x = 2.$

Sol: At x =21

$$LL = \underset{x \to 2-}{Lt} f(x) = \underset{x \to 2-}{Lt} \frac{x}{2} = \frac{2}{2} = 1$$

$$R.L = \underset{x \to 2+}{Lt} f(x) = \underset{x \to 2+}{Lt} \frac{x^2}{3} = \frac{4}{3}$$

$$\underset{x \to 2-}{Lt} f(x) \neq \underset{x \to 2+}{Lt} f(x)$$

$\underset{x \to 2}{Lt} f(x)$ does not exist.

II.

1. $f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 2x+1 & \text{if} \quad 0 \leq x < 1 \\ 3x & \text{if} \quad x > 1 \end{cases}$;a = 1.

Sol : At x =1

$$L.L = \underset{x \to 1-}{Lt} f(x) = \underset{x \to 1-}{Lt} 2x + 1 = 2(1) + 1 = 3$$

$$R.L = \underset{x \to 1+}{Lt} f(x) = \underset{x \to 1+}{Lt} 3x = 3(1) = 3$$

$$\underset{x \to 1-}{Lt} f(x) = \underset{x \to 1+}{Lt} f(x) = 3 \quad \therefore \underset{x \to 1}{Lt} f(x) = 3$$

2. $$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 1 \\ x & \text{if} \quad 1 < x \le 2 \\ x-3 & \text{if} \quad x > 2 \end{cases} ; a = 2.$$

At x =2

$$L.L = \underset{x\to 2-}{Lt} f(x) = \underset{x\to 2-}{Lt} x = 20$$

$$R.L = \underset{x\to 2+}{Lt} f(x) = \underset{x\to 2+}{Lt} (x-3) \quad = 2-3 = -1$$

$\underset{x\to 2-}{Lt} f(x) \ne \underset{x\to 2+}{Lt} f(x)$. hence $\underset{x\to 2}{Lt} f(x)$ does not exist.

3. **Show that** $\underset{x\to 2-}{Lt} \dfrac{|x-2|}{x-2} = -1$

Sol : $x \to 2- \Rightarrow x < 2 \Rightarrow \quad x-2 < 0$

Then, $|x-2| = -(x-2)$

$$\underset{x\to 2-}{Lt} \frac{|x-2|}{x-2} = \underset{x\to 2-}{Lt} \frac{-(x-2)}{(x-2)} = -1$$

4. **Show that** $\underset{x\to 0+}{Lt} \left(\dfrac{2|x|}{x} + x + 1\right) = 3.$

Sol : $x \to 0+ \Rightarrow x > 0 \Rightarrow \quad |x| = x$

$$\therefore \underset{x\to 0+}{Lt} \left(\frac{2|x|}{x} + x + 1\right)$$

$$= \underset{x\to 0+}{Lt} \left(\frac{2x}{x} + x + 1\right) = \underset{x\to 0+}{Lt} (2 + x + 1) = \underset{x\to 0+}{Lt} (2 + 0 + 1) = 3$$

5. **Compute** $\underset{x\to 2+}{Lt}([x]+x)$ **and** $\underset{x\to 2-}{Lt}([x]+x)$.

Sol : $\underset{x\to 2^+}{Lt}\{[x]+x\}.=\underset{h\to 0+}{Lt}\{[2+h]+(2+h)\} = [2+0]+2+0 \quad \left(\because \left[2^+\right]=2\right)$

$=2+2=4$

$\underset{x\to 2-}{Lt}\{[x]+x\}=\left[2^-\right]+2 \quad =1+2=3$

6. **Show that** $\underset{x\to 0-}{Lt}\, x^3\cos\frac{3}{x}=0$

Sol : For any $x, -1\le \cos\frac{3}{x}\le 1$

$\underset{x\to 0-}{Lt} -x^3.\cos\frac{3}{x}=-\underset{x\to 0-}{Lt}\, x^3.\ \underset{x\to 0-}{Lt}\ \cos\frac{3}{x}=0.k$ =0. where $-1\le k\le 1$

III. **1.** **Compute** $\underset{x\to 2-}{Lt}\sqrt{2-x}\ (x<2)$. **What is** $\underset{x\to 2}{Lt}\sqrt{2-x}$?

Sol : $\underset{x\to 2-}{Lt}\sqrt{2-x}=\underset{h\to 0^+}{Lt}\sqrt{2-(2-h)}\ \left(\because \underset{x\to a-}{Lt} f(x)=\underset{h\to o^+}{Lt} f(x)\right)$

$=\underset{h\to 0+}{Lt}\sqrt{h}=0$ The function is not defined when x>2. Therefore we consider only the left limit.. Hence we will not consider the right limit of the function.

So we consider $\underset{x\to 2}{Lt}\sqrt{2-x}=\underset{x\to 2-}{Lt}\sqrt{2-x}$ $\quad\therefore \underset{x\to 2}{Lt}\sqrt{2-x}=0$

2. **Compute** $\underset{x \to \left(-\frac{1}{2}\right)+}{Lt} \sqrt{1+2x}$. **Hence find** $\underset{x \to -\frac{1}{2}}{Lt} \sqrt{1+2x}$.

Sol : $\underset{x \to -\frac{1}{2}+}{Lt} \sqrt{1+2x} = \underset{h \to 0+}{Lt} \sqrt{1+2\left\{\left(-\frac{1}{2}\right)+h\right\}} = \underset{h \to 0+}{Lt} \sqrt{1-1+2h} = 0$

The function is not defined When $x < \frac{-1}{2}$ is not defined.

Hence $\underset{x \to -\frac{1}{2}}{Lt} \sqrt{1+2x} = \underset{x \to -\frac{1}{2}+}{Lt} \sqrt{1+2x} = 0$