CHAPTER 6

DIRECTION COSINES AND DIRECTION RATIOS

TOPICS:

1.DEFINITION OF D.CS., RELATION BETWEEN D.CS. OF A LINE, CO-ORDINATES OF A POINT WHEN D.CS. ARE GIVEN AND DIRECTION COSINES OF A LINE JOINING TWO POINTS.

2. ANGLE BETWEEN TWO LINES WHEN D.CS ARE GIVEN, FINDING THE ANGLE BETWEEN TWO LINES WHEN THEIR D.CS ARE CONNECTED BY EQUATIONS.

3.DEFINITION OF DIRECTION RATIOS, D.RS. OF A LINE JOINING TWO POINTS

4. RELATION BETWEEN D.CS AND D.RS

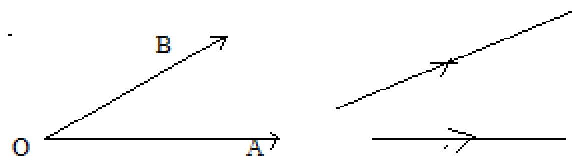
5. CONDITIONS FOR PARALLEL AND PERPENDICULA LINES WHEN D.CS/D.RS ARE GIVEN.

6. ANGLE BETWEEN TWO LINES WHEN D.RS ARE GIVEN, FINDING THE ANGLE BETWEEN TWO LINES WHEN THEIR D.RS ARE CONNECTED BY EQUATIONS.

DIRECTION COSINES & RATIOS (7 MARKS)

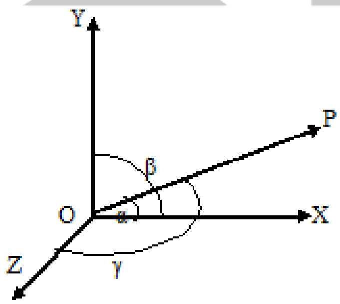
ANGLE BETWEEN TWO LINES:

The angle between two skew lines is the angle between two lines drawn parallel to them through any point in space.



DIRECTION COSINES

If α, β, γ are the angles made by a directed line segment with the positive directions of the coordinate axes respectively, then cos a, cos b, cos g are called the direction cosines of the given line and they are denoted by l, m, n respectively Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$



The direction cosines of $\overrightarrow{op}$ are

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

If l, m, n are the d.c's of a line L is one direction then the d.c's of the same line in the opposite direction are –l, –m, –n.

Note : The angles α, β, γ are known as the direction angles and satisfy the condition $0 \le \alpha, \beta, \gamma \le \pi$.

Note : The sum of the angles α, β, γ is not equal to 2p because they do not lie in the same plane.

Note: Direction cosines of coordinate axes.

The direction cosines of the x-axis are $\cos 0, \cos\frac{\pi}{2}, \cos\frac{\pi}{2}$ i.e., 1, 0, 0

Similarly the direction cosines of the y-axis are (0,1,0) and z-axis are (0,0,1)

THEOREM

If P(x,y,z) is any point in space such that OP =r and if l, m, n are direction cosines of $\overrightarrow{OP}$ then
x =lr,y =mr, z = nr.

Note: If P(x,y,z) is any point in space such that OP =r then the direction cosines of $\overrightarrow{OP}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$

Note: If P is any point in space such that OP =r and direction cosines of $\overrightarrow{OP}$ are l,m,n then the point P =(lr,mr,nr)

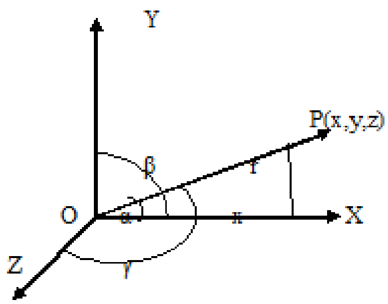
Note: If P(x,y,z) is any point in space then the direction cosines of $\overrightarrow{OP}$ are

$$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$$

THEOREM

If l, m, n are the direction cosines of a line L then $l^2 + m^2 + n^2 = 1$.

Proof :



From the figure

$$l = \cos\alpha = \frac{x}{r}, m = \cos\beta = \frac{y}{r}, n = \cos\gamma = \frac{z}{r} \quad \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1. \qquad \therefore l^2 + m^2 + n^2 = 1$$

THEOREM

The direction cosines of the line joining the points $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ are $\left(\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}\right)$ where $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

EXERCISE

1. A line makes angle $90^0, 60^0 \text{ and } 30^0$ with positive directions of x, y, z –axes respectively. Find the direction cosines.

Sol: Suppose l, m, n are the direction cosines of the line, then

$$l = \cos\alpha = \cos 90^0 = 0$$

$$m = \cos\beta = \cos 60^0 = \frac{1}{2} \qquad \text{And } n = \cos\gamma = \cos 30^0 = \frac{\sqrt{3}}{2}$$

Direction cosines of the line are $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

2. **If a line makes angles α, β, γ with the positive direction of X, Y, Z axes, what is the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$**

Solution: We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - 1 = 2$$

3. If $P\left(\sqrt{3}, 1, 2\sqrt{3}\right)$ is a point in space, find the direction cosines of $\overrightarrow{OP}$

Solution: Direction ratios of P are $\left(\sqrt{3}, 1, 2\sqrt{3}\right) = (a, b, c)$

$$\Rightarrow a^2 + b^2 + c^2 = 3 + 1 + 12 = 16 \qquad \Rightarrow \sqrt{a^2 + b^2 + c^2} = 4$$

Direction cosines of $\overrightarrow{OP}$ are

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}\right) = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{2\sqrt{3}}{4}\right) = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$$

4. Find the direction cosines of the line joining the points $(-4,1,7)\ and\ (2,-3,2)$

Solution: $A(-4,1,2)\ and\ B(2,-3,2)$ are given points

d.rs of PQ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

i.e. $(2+4, 1+3, 2-7)$ i.e, $(6, 4, -5) = (a, b, c)$

$\Rightarrow \sqrt{a^2 + b^2 + c^2} = \sqrt{36+16+25} = \sqrt{77}$

Direction cosines of $\overrightarrow{AB}$ are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) = \left(\frac{6}{\sqrt{77}}, \frac{4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}\right)$$

II.

1. Find the direction cosines of the sides of the triangle whose vertices are $(3,5,-4), (-1,1,2)$ **and** $(-5,-5,-2)$

Sol: $A(3,5,-4), B(-1,1,2)\ and\ C(-5,-5,-2)$ are the vertices of ΔABC

d.rs of AB are $(-1-3, 1-5, 2+4) = (-4, -4, 6)$

Dividing with $\sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$

d.cs of AB are $\frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}$ i.e., $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$

D.rs of BC are $(-5+1, -5-1, -2-2)$ i.e., $(-4, -6, -4)$

Dividing with $\sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$ d.cs of BC are $\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$

i.e., $\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-4}{2\sqrt{17}}$

d.rs of CA are $(3+5, 5+5, -4+2) = (8, 10, -2)$

Dividing with $\sqrt{64+100+4}=\sqrt{168}=2\sqrt{42}$

Then d.cs of CA are $\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$

2. **Show that the lines $\overrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel where P, Q, R, S are two points (2, 3, 4), (-1, -2, -1) and (1, 2, 5) respectively**

Sol : $P(2, 3, 4), Q(4, 7, 8)$, R (-1, -2, 1) and S(1, 2, 5) are the given points.

d.rs of PQ are (4 -2, 7-3, 8-4) i.e.,(2, 4, 4)

d.rs of RS are (1 + 1, 2 + 2, 5 -1) i.e., (2, 4, 4)

$\therefore$ d.rs of PQ and RS are proportional. Hence, PQ and RS are parallel

III.

1. **Find the direction cosines of two lines which are connected by the relation** $l-5m+3n=0$ *and* $7l^2+5m^2-3n^2=0$

Sol. Given $l-5m+3n=0$

$\Rightarrow l=5m-3n -----(1)$

and $7l^2+5m^2-3n^2=0---(2)$

Substituting the value of l in (2)

$7(5m-3n)^2+5m^2-3n^2=0$

$\Rightarrow 7(25^2+9n^2-30mn)+5m^2-3n^2=0$

$\Rightarrow 175m^2+63n^2-210mn+5m^2-3n^2=0$

$\Rightarrow 180m^2-210mn+60n^2=0$

$\Rightarrow 6m^2-7mn+2n^2=0$

$\Rightarrow (3m-2n)(2m-n)=0$

Case (i): $3m_1 = 2n_1 \Rightarrow \frac{m_1}{2} = \frac{n_1}{3}$

Then $m_1 = \frac{2}{3}n_1$

From $(1)\, l_1 = 5m_1 - 3n_1 = \frac{10}{3}n_1 - 3n_1$

$= \frac{10n_1 - 9n_1}{3} = \frac{n_1}{3}$

$\therefore \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3}$

d.rs of the first line are (1, 2, 3)

Dividing with $\sqrt{1+4+9} = \sqrt{14}$

d.cs of the first line are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Case (ii) $2m_2 = n_2$

From (1) $l_2 - 5m_2 + 3n_2 = 0$

$\Rightarrow l_2 - 5m_2 + 6m_2 = 0$

$\Rightarrow -l_2 = m_2$

$\therefore \frac{l_2}{-1} = \frac{m_2}{1} = \frac{n_2}{2}$

d.rs of the second line are -1, 1, 2

Dividing with $\sqrt{1+1+4} = \sqrt{6}$

d.cs of the second line are $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

DIRECTION RATIOS

A set of three numbers a,b,c which are proportional to the direction cosines l,m,n respectively are called DIRECTION RATIOS (d.r's) of a line.

Note : If (a, b, c) are the direction ratios of a line then for any non-zero real number λ , $(\lambda a, \lambda b, \lambda c)$ are also the direction ratios of the same line.

Direction cosines of a line in terms of its direction ratios

If (a, b, c) are direction ratios of a line then the direction cosines of the line are

$$\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

THEOREM

The direction ratios of the line joining the points are $(x_2-x_1, y_2-y_1, z_2-z_1)$

ANGLE BETWEEN TWO LINES

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosines of two lines θ and is the acute angle between them, then $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

Note.

If θ is the angle between two lines having d.c's (l_1, m_1, n_1) and (l_2, m_2, n_2) then

$$\sin\theta = \sqrt{\sum (l_1m_2 - l_2m_1)^2}$$

and $\tan\theta = \dfrac{\sqrt{\sum (l_1m_2 - l_2m_1)^2}}{|l_1l_2 + m_1m_2 + n_1n_2|}$ **when** $\theta \neq \dfrac{\pi}{2}$

Note 1 : The condition for the lines to be perpendicular is $l_1l_2 + m_1m_2 + n_1n_2 = 0$

Note 2 : The condition for the lines to be parallel is $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

THEOREM

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are direction ratios of two lines and θ is the angle between them then $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Note 1 : If the two lines are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Note 2 : If the two lines are parallel then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Note 3 : If one of the angle between the two lines is θ then other angle is $180^0 - \theta$

EXERCISE

I

1. Find the direction ratios of the line joining the points (3, 4, 0) are (4, 4, 4)

Sol. $A(3,4,0)$ and $B(4,4,4)$ are the given points

d.rs of AB are $(x_2-x_1, y_2-y_1, z_2-z_1) = (4-3, 4-4, 4-0)$ *i.e.*, $(1,0,4)$

2. **The direction ratios of a line are $(-6,2,3)$. Find the direction cosines.**

Sol: D.rs of the line are -6, 2, 3

Dividing with $\sqrt{36+4+9}=7$

Direction cosines of the line are $-\frac{6}{7},\frac{2}{7},\frac{3}{7}$

3. Find the cosine of the angle between the lines, whose direction cosines are

$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ *and* $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$

Sol: D.cs of the given lines are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ *and* $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$.

Let θ be the angle between the lines. Then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \qquad = \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}.0 = \frac{1}{\sqrt{6}}=\frac{2}{\sqrt{6}}=\sqrt{\frac{4}{6}} \qquad =\sqrt{\frac{2}{3}}$

4. Find the angle between the lines whose direction ratios are $(1,1,2)\left(\sqrt{3},-\sqrt{3},0\right)$

Sol: D.rs of the given lines are $(1,1,2)$ and $\left(\sqrt{3},-\sqrt{3},0\right)$

Let θ be the angle between the lines. Then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{1\sqrt{3}+1\left(-\sqrt{3}\right)+2.0}{\sqrt{1+1+4}\sqrt{3+3}} = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

5. Show that the lines with direction cosines $\left(\frac{12}{13},\frac{-3}{13},\frac{-4}{13}\right)$ *and* $\left(\frac{4}{13},\frac{12}{13},\frac{3}{13}\right)$ are perpendicular to each other.

Sol: Direction cosines of the lines are $\left(\frac{12}{13},\frac{-3}{13},\frac{-4}{13}\right)$ *and* $\left(\frac{4}{13},\frac{12}{13},\frac{3}{13}\right)$

Now $l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13}.\frac{4}{13} - \frac{3}{13}.\frac{12}{13} - \frac{4}{13}.\frac{3}{13} = \frac{48-36-12}{169} = 0$

$\therefore\ l_1l_2 + m_1m_2 + n_1n_2 = 0 \Rightarrow$ the two lines are perpendicular .

6. O is the origin, P(2, 3, 4) and Q(1, k, 1) are points such that $\overline{OP} \perp \overline{OQ}$ find K

Sol: O (0,0,0), P(2, 3, 4) and Q(1, k, 1)

d.rs of OP are 2, 3, 4

d.rs of OQ are 1, k, 1

OP and OQ are perpendicular $\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow \quad 2 + 3k + 4 = 0 \Rightarrow 3k = -6 \quad \Rightarrow \quad k = -2$

II.

1 If the direction ratios of a line are (3, 4, 0) find its direction cosines are also the angles made the co-ordinate axes.

Sol: Direction ratios of the line are (3, 4, 0)

Dividing with $\sqrt{9+16+0} = 5$

D.cs of the line are $\left(\frac{3}{5}, \frac{4}{5}, 0\right)$

If α, β, γ are the angles made by the line with the co-ordinate axes, then

$\cos\alpha = \frac{3}{5} \cos\beta = \frac{4}{5} \cos\gamma = 0$

$\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \cos^{-1}\left(\frac{4}{5}\right), \gamma = \frac{\pi}{2}$

Angles made with co-ordinate axes are $\cos^{-1}\left(\frac{3}{5}\right), \cos^{-1}\left(\frac{4}{5}\right), \frac{\pi}{2}$

2. Show that the line through the points (1, -1, 2) (3, 4, -2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6)

Sol:

Given points $A(1, -1, 2)\, B(3, 4, -2)\, C(0, 3, 2)$ and $D(3, 5, 6)$

d.rs of AB are (3 -1, 4 +1, -2-2) i.e., 2, 5, -4 and d.rs of CD are

(3 -0, 5 -3, 6 – 2) i.e., 3, 2, 4

now $a_1\, a_2 + b_1 b_2 + c_1 c_2 = 2.3 + 5.2 - 4.4 = 6 + 10 - 16 = 0$

Therefore, AB and CD are perpendicular

3. Find the angle between $\overline{DC}$ *and* $\overline{AB}$ where
$A = (3, 4, 5), B = (4, 6, 3)\, C = (-1, 2, 4)$ **are** $D(1, 0, 5)$

Sol: $A(3, 4, 5), B(4, 6, 3), C(-1, 2, 4), D(1, 0, 5)$ are the given points

d.rs of AB are(4, -3, 6 -4, 3 – 5) i.e., (1, 2, -2) and

d.rs of CD are (1 + 1, 0 – 2, 5 -4) i.e., (2, -2, 1)

let θ be the angle between the lines, then $\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$$= \frac{|1.2 + 2(-2) + (-2).1}{\sqrt{1+4+4}\,\sqrt{4+4}} = \frac{4}{9} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{9}\right)$$

4. Find the direction cosines of a line which is perpendicular to the lines, whose direction ratios are (1, -1, 2) and (2, 1, -1)

Sol: Let l, m, n be the d.rs of the required line. This line is perpendicular to the lines with d.rs (1, -1, 2) and (2, 1, -1)

$\therefore l - m + 2n = 0$ and $2l + m - n = 0$

Solving these two equations

	l		m		n
-1		2		1	-1
1		-1		2	1

$$\frac{l}{1-2} = \frac{m}{4+1} = \frac{n}{1+2} \Rightarrow \frac{l}{-1} = \frac{m}{5} = \frac{n}{3}$$

d.rs of the line are -1, 5, 3

Dividing with $\sqrt{1+25+9} = \sqrt{35}$

d.cs of the required line are $-\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$

5. Show that the points (2, 3, -4), (1, -2, 3) and (3, 8, -11) are collinear

6. Show that the points (4, 7, 8), (2, 3, 4), $(-1, -2, 1), (1, 2, 5)$ **are vertices of a parallelogram**

Sol: Given points are $A(4, 7, 8), B(2, 3, 4), C(-1, -2, 1)$ *and* $D(1, 2, 5)$ Now

$$AB = \sqrt{(4-2)^2 + (7-3)^2 + (8-4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

$$BC = \sqrt{(2+1)^2 + (3+2)^2 + (4-1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$CD = \sqrt{(-1, -1)^2 + (-2-2)^2 + (1-5)^2} \qquad = \sqrt{4+16+16} = 6$

and

$DA = \sqrt{(1-4)^2 + (2-7)^2 + (5-8)^2} \quad = \sqrt{9+25+9} = \sqrt{43}$

$\therefore AB = CD \text{ and } BC = DA$

$\therefore A, B, C, D$ are the vertices of parallelogram

III

1. Show that the lines whose direction cosines are given by $l + m + n = 0$ $2m + 3nl - 5lm = 0$ **are perpendicular to each other**

Sol: Given equations are $l + m + n = 0 ------(1)$

$2mn + 3nl - 5lm = 0 -----(2)$

From (1), $l = -(m+n)$ Substituting in (2)

$\Rightarrow 2mn - 3n(m+n) + 5m(m+n) = 0$

$\Rightarrow 2mn - 3mn - 3n^2 + 5m^2 + 5mn = 0$

$\Rightarrow 5m^2 + 4mn - 3n^2 = 0$

$\Rightarrow 5\left(\frac{m}{n}\right)^2 + 4\frac{m}{n} - 3 = 0$

$\Rightarrow \frac{m_1 m_2}{n_1 n_2} = \frac{-3}{5} \Rightarrow \frac{m_1 m_2}{-3} = \frac{n_1 n_2}{5} ------(3)$

From (1), n = $-(l+m)$

Substituting in (2) , $-2m(l+m) - 3l(l+m) - 5lm = 0$

$\Rightarrow -2lm - 2m^2 - 3l^2 - 3lm - 5lm = 0$

$\Rightarrow 3l^2 + 10lm + 2m^2 = 0$

$$\Rightarrow 3\left(\frac{l}{m}\right)^2 + 10\frac{l}{m} + 2 = 0$$

$$\Rightarrow \quad \frac{l_1 l_2}{m_1 m_2} = \frac{2}{3} \Rightarrow \frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} ------(4)$$

Form (3) and (4)

$$\frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5} = k\,(say) \quad \Rightarrow l_1 l_2 = 2k,\ m_1 m_2 = 3k,\ n_1 n_2 = -5k$$

$$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 2k + 3k - 5k = 0$$

The two lines are perpendicular

2.Find the angle between the lines whose direction cosines satisfy the equation $l + m + n = 0, l^2 + m^2 - n^2 = 0$

Sol: Given equations are

$l + m + n = 0$ ……….(1)

$l^2 + m^2 - n^2 = 0$ ……..(2)

From (1), $l = -(m + n)$

Substituting in (2)

$$(m + n)^2 + m^2 - n^2 = 0$$

$$\Rightarrow m^2 + n^2 + 2mn + m^2 - n^2 = 0$$

$$\Rightarrow 2m^2 + 2mn = 0$$

$$\Rightarrow 2m(m + n) = 0$$

$$\Rightarrow m = 0 \text{ and } m + n = 0$$

Case (i) m = 0, Substituting in (1) $l + n = 0$

$$l = -n \Rightarrow \frac{l}{1} = \frac{n}{-1}$$

D.rs of the first line *are* $(1, 0, -1)$

Case (ii) : $m+n=0 \Rightarrow m=-n \Rightarrow \frac{m}{1}=\frac{n}{-1}$

Substituting in (1) $l=0$

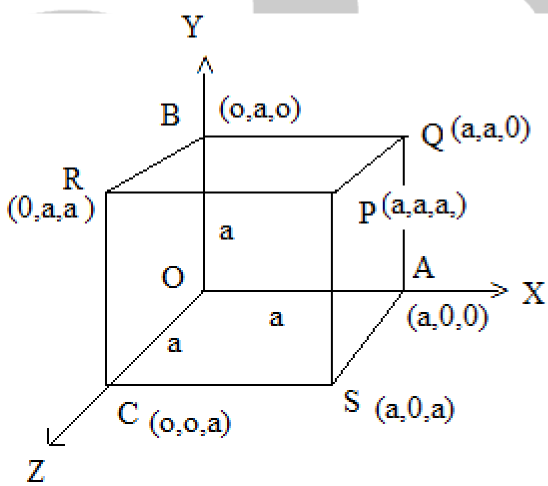
D.rs of the second line *are* $(0, 1, -1)$

let θ be the angle between the two lines , then

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$=\frac{|0+0+1|}{\sqrt{2}.\sqrt{2}}=\frac{1}{2} \qquad \therefore \theta=\frac{\pi}{3}$$

3. If a ray makes angle α, β, γ and δ with the four diagonals of a cube find $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$



Sol:

Let OABC;PQRS be the cube.

Let a be the side of the cube. Let one of the vertices of the cube be the origin O and the co-ordinate axes be along the three edges $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ passing through the origin.

The co-ordinate of the vertices of the cube with respect to the frame of reference OABC are as shown in figure are A (a,o,o), B(o,a,o), C(0,o,a) P(a,a,a) Q(a,a,o)

R(o,a,a) and S(a,o,a)

The diagonals of the cube are $\overline{OP}, \overline{CQ}, \overline{AR} and \overline{BS}$. and their d.rs are respectively

(a, a, a), (a, a, -a), (-a, a, a) and (a, -a, a).

Let the direction cosines of the given ray be (l, m, n).

Then $l^2 + m^2 + n^2 = 1$

If this ray is making the angles α, β, γ and δ with the four diagonals of the cube, then

$$\cos\alpha = \frac{|a\times l + a\times m + a\times n|}{\sqrt{a^2+a^2+a^2}.1} = \frac{|l+m+n|}{\sqrt{3}}$$

Similarly, $\cos\beta = \dfrac{|l+m-n|}{\sqrt{3}}$

$$\cos\gamma = \frac{|-l+m+n|}{\sqrt{3}} \text{ and } \cos\delta = \frac{|-l+m+n|}{\sqrt{3}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$$

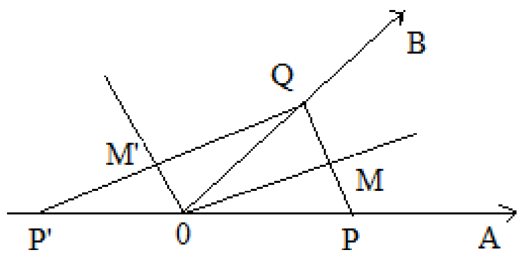
$$\frac{1}{3}\{|l+m+n|^2 + |l+m-n|^2 + |-l+m+n|^2 + |l-m+n|^2\}$$

$$\frac{1}{3}[(l+m+n)^2 + (l+m-n)^2 + (-l+m+n)^2 + (l-m+n)^2]$$

$$\frac{1}{3}[4(l^2+m^2+n^2)] = \frac{4}{3} \qquad (\text{since } l^2+m^2+n^2=1)$$

4. If $(l_1, m_1, n_1), (l_2, m_2, n_2)$ and d.cs of two intersecting lines show that d.c.s of two lines, bisecting the angles between them are proportional to $l_1 \pm l_2, m_1 \pm m_2\ n_1 \pm n_2$

Sol:



Let OA and OB be the given lines whose d.cs are given by $(l_1, m_1, n_1), (l_2, m_2, n_2)$.

Let OP = OQ = 1 unit. Also take a point P^1 on AO produced such that

OP^1 =OP =1.

Join PQ and P^1Q.

LET M,M^1 be the mid points of PQ and P^1Q.

Then OM & OM^1 are the required bisectors.

Now point P = (l_1, m_1, n_1) & Q = (l_2, m_2, n_2)

and $P^1 = (-l_1, -m_1, -n_1)$.

And mid points

M = $\left(\frac{l_1 + l_2}{2}, \frac{m_1 + m_2}{2}, \frac{n_1 + nl2}{2}\right)$,

$M^1 = \left(\frac{l_1 - l_2}{2}, \frac{m_1 - m_2}{2}, \frac{n_1 - n_2}{2}\right)$

Hence the d.cs of the bisector OM are proportional to

$\left(\frac{l_1 + l_2}{2} - 0, \frac{m_1 + m_2}{2} - 0, \frac{n_1 + n_2}{2} - 0\right)$ Or $l_1 + l_2, m_1 + m_2, n_1 + n_2$

Similarly, d.cs of the bisector OM^1 are proportional to

$$\left(\frac{l_1-l_2}{2}-0, \frac{m_1-m_2}{2}-0, \frac{n_1-n_2}{2}-0\right) \text{ or } l_1-l_2, m_1-m_2, n_1-n_2.$$

Hence d.cs of bisectors are proportional to $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$.

5. A (-1, 2, -3), B(5, 0, -6), C(0, 4, -1) are three points. Show that the direction cosines of the bisector of $\underline{|BAC}$ are proportional to (25, 8, 5) and (-11, 20, 23)

Sol: Given points are A(-1, 2, -3), B (5, 0, -6) and C(0, 4, -1)

D.rs of AB are (5 + 1, 0-2, -6 + 3)

i.e.,(6, -2, -3)=(a,b,c)

Now $\sqrt{a^2+b^2+c^2}=\sqrt{36+4+9}=7$ $\therefore$ D.rs of AB are $\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}$

D.rs of AC are (0 + 1, 4 -2, -1 +3) i.e., 1, 2, 2

$\sqrt{a^2+b^2+c^2}=\sqrt{1+4+4}=3$ $\Rightarrow$D.rs of AC are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

$\therefore$ D.rs of one of the bisector are proportional to $l_1+l_2, m_1+m_2, n_1+n_2$

$$=\left(\frac{6}{7}+\frac{1}{3}, \frac{-2}{7}+\frac{2}{3}, \frac{-3}{7}+\frac{2}{3}\right)=\left(\frac{18+7}{21}, \frac{-6+14}{21}, \frac{-9+14}{21}\right) = \left(\frac{25}{21}, \frac{8}{21}, \frac{5}{21}\right)$$

D.rs of one of the bisector are (25, 8, 5)

D.rs of the other bisectors are proportional to

$$l_1-l_2, m_1-m_2, n_1-n_2 = \left(\frac{6}{7}-\frac{1}{3}, \frac{-2}{7}-\frac{2}{3}, \frac{-3}{7}-\frac{2}{3}\right)=\left(\frac{18-7}{21}, \frac{-6-14}{21}, \frac{-23}{21}\right)$$

$$=\left(\frac{11}{21}, \frac{-20}{21}, \frac{-23}{21}\right)$$

D.rs of the second bisector are (-11, 20, 23)

6. If (6, 10, 10), (1, 0, -5), (6, -10, 0) are vertices of a triangle, find the direction ratios of its sides. Determine whether it is right angle or isosceles

Sol:

Given vertices are $A(6,10,10), B(1,0,-5), C(6,-10,0)$

D.rs of AB are 5, 10, 15 i.e., 1, 2, 3

D.rs of BC are -5, 10, -5 i.e., 1, -2, 1

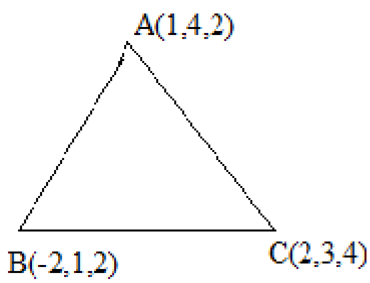
D.rs of AC are 0, 20, 10, i.e., 0, 2, 1

$$\cos \angle ABC = \frac{[1.1 + 2(-2) + 3.1]}{\sqrt{1+4+9}\,\sqrt{1+4+1}} = 0 \;\Rightarrow \angle B = \frac{\pi}{2}$$

Therefore, the triangle is a rt. triangle.

7. The vertices of a triangle are A(1, 4, 2), B(-2, 1, 2) C(2, 3, -4). Find $\angle A, \angle B, \angle C$

Sol:



Vertices of the triangle are $A(1,4,2), B(-2,1,2), C(2,3,-4)$

D.rs of AB are 3, 3,0 i.e., 1, 1, 0

D.rs of BC are -4, -2, 5 i.e., 2, 1, -3

D.rs of AC are -1, 1, 6

$$\cos \angle ABC = \frac{|1.2 + 1.0 + 0(-3)|}{\sqrt{1+1}\,\sqrt{4+1+9}} = \frac{3}{\sqrt{28}} = \frac{3}{2\sqrt{7}} \qquad \therefore \angle B = \cos^{-1}\left(\frac{3}{2\sqrt{7}}\right)$$

$$\cos\underline{|BCA} = \frac{1(-1)+1.1+(-3)6}{\sqrt{4+1+9}\,\sqrt{1+1+36}} = \frac{19}{\sqrt{19}\,\sqrt{28}} = \sqrt{\frac{19}{28}} \qquad \therefore\ \underline{|C} = \cos^{-1}\left(\sqrt{\frac{19}{28}}\right)$$

$$\cos\underline{|CAB} = \frac{|-1.1+1.1+6.0|}{\sqrt{1+1+36}\,\sqrt{1+1+0}} = 0 \qquad \Rightarrow \underline{|A} = \pi/2$$

8. Find the angle between the lines whose direction cosines are given by the equation $3l+m+5n=0$ **and** $6mn-2nl+5l=0$

Sol: Given $3l+m+5n=0$

$6mn \;-2nl+5lm=0$

From (1), $m=-(3l+5n)$

Substituting in (2)

$\Rightarrow -6n(3l+5n)-2nl-5l(3l+5n)=0$

$\Rightarrow -18\ln-30n^2-2nl-15l^2-25\ln=0$

$\Rightarrow -15l^2-45\ln-30n^2=0$

$\Rightarrow l^2+3\ln+2n^2=0$

$\Rightarrow (l+2n)(l+n)=0$

$\Rightarrow l+2n=0 \; or \; l+n=0$

Case (i) :

$l_1+n_1=0 \Rightarrow n_1=-l_1; \Rightarrow n_1=-l_1; \Rightarrow \frac{l_1}{1}=\frac{n_1}{-1}$

But $m_1=-(3l_1+5n_1)=-(-3n_1+5n_1)=-2n_1$

$\therefore \frac{m_1}{+2}=\frac{n_1}{-1}$

$\therefore \frac{l_1}{1}=\frac{m_1}{2}=\frac{n_1}{-1}$

D.rs of the first line l_1 *are* $(1, 2, -1)$

Case (ii) : $l_2 + 2n_2 = 0$

$\Rightarrow l_2 = -2n_2 \Rightarrow \frac{l_2}{-2} = \frac{n_2}{1}$

$\Rightarrow m_2 = -(3l_2 + 5n_2) = -(-6n_2 + 5n_2) = n_2$

$\frac{m_2}{1} = \frac{n_2}{1}$

$\therefore \frac{l_2}{-2} = \frac{m_2}{1} = \frac{n_2}{1}$

D.rs of the second line l_2 *are* $(-2, 1, 1)$

Suppose 'θ' is the angle between the lines l_1 *and* l_2

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\frac{|1(-2) + 2.1 + (-1).1|}{\sqrt{1+4+1}\sqrt{4+1+1}} = \frac{1}{6} \Rightarrow \theta = \cos^{-1}(1/6)$$

9. If variable line in two adjacent position has direction cosines (l, m, n) **and** $(l + \delta l, m + \delta m, n + \delta n)$**, show that the small angle** $\delta\theta$ **between two position is given by** $(\delta\theta)^2 = (\delta l)^2 = (\delta m)^2 + (\delta n)^2$

Sol: The d.cs of the line in the two positions are (l, m, n) *and* $(l + \delta l, m + \delta n, n + n)$.

Therefore, $l^2 + m^2 + n^2 = 1$ ------ (1)

and $(l + \delta l^2) + (m + \delta m)^2 + (n + \delta n)^2 = 1$ ----(2)

$(2)-(1) \Rightarrow (l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 - (l^2 + m^2 + n^2) = 0$

$2(l.\delta l + m.\delta m + n\delta n) = -\left((\delta l)^2 + (\delta m)^2 + (\delta n)^2\right)$ ….. (3)

And $\cos\delta\theta = l(l+\delta l) + m(m+\delta m) + n(n+\delta n)$

$$= (l^2 + m^2 + n^2) + (l.\delta l + m.\delta m + n.\delta n)$$

$$\cos\delta\theta = 1 - \frac{1}{2}\left[(\delta l)^2 + (\delta m)^2 + (\delta n)^2\right]$$

$$(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 2(1-\cos\delta\theta)$$

$$= 2.\,2\sin^2\frac{\delta\theta}{2}$$

$\delta\theta$ being small, $\sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}$

$$\therefore\ 4\sin^2\theta = 4\left(\frac{\delta\theta}{2}\right)^2 = (\delta\theta)^2$$

$$\therefore (\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

PROBLEMS FOR PRACTICE

1. If P(2, 3, –6) Q(3, –4, 5) are two points, find the d.c's of $\overrightarrow{OP}, \overrightarrow{QO}$ and $\overrightarrow{PQ}$ where is the origin
2. If the d.c.s of line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$ find c.?
3. Find the d.c's of line that makes equal angles with the axes.?
4. Find the angle between two diagonals of a cube.?
5. Show that the points A(1,2,3), B(4,0,4), C(–2,4,2) are collinear?

6. A(1,8,4), B(0,–11,4), C(2,–3,1) are three points and D is the foot of the perpendicular from A to BC. Find the coordinates of D.

Solution: -

suppose D divides in the ratio m : n

Then $D = \left(\frac{2m}{m+n}, \frac{-3m-11n}{m+n}, \frac{m+4n}{m+n}\right)$

Direction ratios of $\overline{AD} = \left(\frac{m-n}{m+n}, \frac{-11m-19n}{m+n}, \frac{-3m}{m+n}\right)$

Direction ratios of $\overline{BC} : (2,8,-3)$

$\overline{AD} \perp \overline{BC} \Rightarrow 2\left(\frac{m-n}{m+n}\right) + 8\left(\frac{-11m-19n}{m+n}\right) \; 3\left(\frac{-3m}{m+n}\right) = 0$

2m–2n–88m–152n+9m=0

m = –2n

substituting in (1), D=(4,5,–2)

7. Lines $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ are drawn from O with direction cosines proportional to (1,–2,–1); (3,–2,3). Find the direction cosines of th enormal to the plane AOB.

Sol : -

Let (a, b, c) be the direction ratios of a normal to the plane AOB. since $\overleftrightarrow{OA}, \overleftrightarrow{OB}$ lie on the plane, they are perpendicular to the normal to the plane.

Using the condition of perpendicularity

a.1+b(–2)+ c (–1) = 0. ...... (1)

a.3 + b(–2) +c(3) = 0. ...... (2)

Solving (1) and (2) $\frac{a}{-8} = \frac{b}{-6} = \frac{c}{4} \; or \; \frac{a}{4} = \frac{b}{3} = \frac{c}{-2}$

The d.c's of the normal are

$\left(\frac{4}{\sqrt{16+9+4}}, \frac{3}{\sqrt{16+9+4}}, \frac{-2}{\sqrt{16+9+4}}\right)$ i.e., $\left(\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\right)$

8. Show that the line whose d.c's are proportional to (2,1,1), $\left(4, \sqrt{3}-1, -\sqrt{3}-1\right)$ are inclined to one another at angle .

9. Find the d.r's and d.c's of the line joining the points (4, –7, 3), (6, –5, 2)

10. For what value of x the line joining A(4,1,2) B(5,x,0) is perpendicular to the line joining C(1,2,3) and D(3,5,7).

11. Find the direction cosines of two lines which are connected by the relations $l + m + n = 0$ and mn-2nl-2lm = 0