PAIR OF LINES-SECOND DEGREE GENERAL EQUATION

THEOREM

If the equation $\boldsymbol{S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ represents a pair of straight lines then

i) $\boldsymbol{\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$ and (ii) $\boldsymbol{h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc}$

Proof:

Let the equation S = 0 represent the two lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$.

Then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$\equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0$$

Equating the co-efficients of like terms, we get

$l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, m_1m_2 = b, and $l_1n_2 + l_2n_1 = 2g$, $m_1n_2 + m_2n_1 = 2f$, $n_1n_2 = c$

(i) Consider the product $(2h)(2g)(2f)$

$$= (l_1m_2 + l_2m_1)(l_1n_2 + l_2n_1)(m_1n_2 + m_2n_1)$$
$$= l_1l_2\left(m_1^2n_2^2 + m_2^2n_1^2\right) + m_1m_2\left(l_1^2n_2^2 + l_2^2n_1^2\right) + n_1n_2\left(l_1^2m_2^2 + l_2^2m_1^2\right) + 2l_1l_2m_1m_2n_1n_2$$
$$= l_1l_2[(m_1n_2 + m_2n_1)^2 - 2m_1m_2n_1n_2] + m_1m_2[(l_1n_2 + l_2n_1)^2 - 2l_1l_2n_1n_2]$$
$$+ n_1n_2[(l_1m_2 + l_2m_1)^2 - 2l_1l_2m_1m_2] + 2l_1l_2m_1m_2n_1n_2$$
$$= a(4f^2 - 2bc) + b(4g^2 - 2ac) + c(4h^2 - 2ab)$$
$$8fgh = 4[af^2 + bg^2 + ch^2 - abc]$$
$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

ii) $h^2 - ab = \left(\dfrac{l_1m_2 + l_2m_1}{2}\right)^2 - l_1l_2m_1m_2 \quad = \dfrac{(l_1m_2 + l_2m_1)^2 - 4 - l_1l_2m_1m_2}{4}$

$$= \frac{(l_1m_2 - l_2m_1)^2}{4} \geq 0$$

Similarly we can prove $g^2 \geq ac$ and $f^2 \geq bc$

NOTE :

If $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, $\boldsymbol{h^2 \geq ab}$, $\boldsymbol{g^2 \geq ac}$ and $\boldsymbol{f^2 \geq bc}$, then the equation $\boldsymbol{S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ represents a pair of straight lines

CONDITIONS FOR PARALLEL LINES-DISTANCE BETWEEN THEM

THEOREM

If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines then $h^2 = ab$ and $bg^2 = af^2$. Also the distance between the two parallel lines is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}} \text{ (or) } 2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$

Proof :

Let the parallel lines represented by S = 0 be

$lx + my + n_1 = 0$ -- (1) $lx + my + n_2 = 0$ -- (2)

$\therefore ax^2 + 2hxy + 2gx + 2fy + c$

$\equiv (lx + my + n_1)(lx + my + n_2)$

Equating the like terms

$l^2 = a$ -- (3) $\quad 2lm = 2h$ -- (4)

$m^2 = b$ -- (5) $\quad l(n_1 + n_2) = 2g$ -- (6)

$m(n_1 + n_2) = 2f$ -- (7) $\quad n_1 n_2 = c$ -- (8)

From (3) and (5), $l^2 m^2 = ab$ and from (4) $h^2 = ab$.

Dividing (6) and (7) $\frac{l}{m} = \frac{g}{f} \Rightarrow \frac{l^2}{m^2} = \frac{g^2}{f^2}$,

$\therefore \frac{a}{b} = \frac{g^2}{f^2} \Rightarrow bg^2 = af^2$

Distance between the parallel lines (1) and (2) is

$$= \left| \frac{n_1 - n_2}{\sqrt{(l^2 + m^2)}} \right| = \frac{\sqrt{(n_1 + n_2)^2 - 4n_1 n_2}}{\sqrt{l^2 + m^2}}$$

$$= \frac{\sqrt{(4g^2 / l^2) - 4c}}{\sqrt{a+b}} \text{ or } \frac{\sqrt{(4f^2 / m^2) - 4c}}{\sqrt{a+b}}$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a+b)}} \text{ (or) } 2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$

POINT OF INTERSECTION OF PAIR OF LINES THEOREM

The point of intersection of the pair of lines represented by

$\boldsymbol{ax^2+2hxy+by^2+2gx+2fy+c=0}$ when $\boldsymbol{h^2>ab}$ is $\boldsymbol{\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)}$

Proof:

Let the point of intersection of the given pair of lines be (x_1, y_1). Transfer the origin to (x_1, y_1) without changing the direction of the axes.

Let (X,Y) represent the new coordinates of (x, y). Then $x = X + x_1$ and $y = Y + y_1$.

Now the given equation referred to new axes will be

$$a(X+x_1)^2+2h(X+x_1)(Y+y_1)+b(Y+y_1)^2+2g(X+x_1)+2f(Y+y_1)+c=0$$

$$\Rightarrow aX^2+2hXY+bY^2+2X(ax_1+hy_1+g)+2Y(hx_1+by_1+f)$$
$$+(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)=0$$

Since this equation represents a pair of lines passing through the origin it should be a homogeneous second degree equation in X and Y. Hence the first degree terms and the constant term must be zero. Therefore,

$$ax_1+hy_1+g=0 \quad \text{-- (1)}$$

$$hx_1+by_1+f=0 \quad \text{-- (2)}$$

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \text{-- (3)}$$

But (3) can be rearranged as

$$x_1(ax_1+hy_1+g)+y_1(hx_1+by_1+f)+(gx_1+fy_1+c)=0$$

$$\Rightarrow gx_1+fy_1+c=0 \text{ --(4)}$$

Solving (1) and (2) for x_1 and y_1

$$\frac{x_1}{hf-bg}=\frac{y}{gh-af}=\frac{1}{ab-h^2}$$

$$\therefore x_1=\frac{hf-bg}{ab-h^2} \text{ and } y_1=\frac{gh-af}{ab-h^2}$$

Hence the point of intersection of the given pair of lines is $\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$

THEOREM

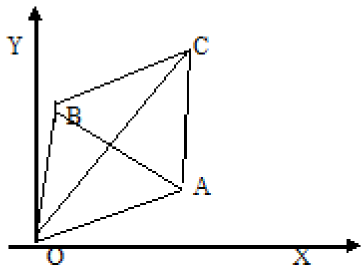
If the pair of lines $ax^2 + 2hxy + by^2 = 0$ and the pair of lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ form a rhombus then $(a-b)fg + h(f^2 - g^2) = 0$.

Proof:

The pair of lines $ax^2 + 2hxy + by^2 = 0$ -- (1) is parallel to the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ -- (2)



Now the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda(ax^2 + 2hxy + by^2) = 0$

Represents a curve passing through the points of intersection of (1) and (2).
Substituting $\lambda = -1$, in (3) we obtain $2gx + 2fy + c = 0$ ...(4) Equation (4) is a straight line passing through A and B and it is the diagonal $\overleftrightarrow{AB}$

The point of intersection of (2) is $C = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

$\Rightarrow$ Slope of $\overleftrightarrow{OC} = \frac{gh - af}{hf - bg}$

In a rhombus the diagonals are perpendicular $\Rightarrow \left(\text{Slope of } \overleftrightarrow{OC}\right)\left(\text{Slope of } \overleftrightarrow{AB}\right) = -1$

$\Rightarrow \left(\frac{gh - af}{hf - bg}\right)\left(-\frac{g}{f}\right) = -1$

$\Rightarrow g^2h - afg = hf^2 - bfg$

$\Rightarrow (a-b)fg + h(f^2 - g^2) = 0$

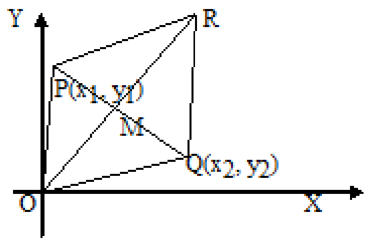
$\frac{g^2 - f^2}{a - b} = \frac{fg}{h}$

THEOREM

If $ax^2+2hxy+by^2=0$ be two sides of a parallelogram and $px+qy=1$ is one diagonal, then the other diagonal is $y(bp-hq)=x(aq-hp)$

proof:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points where the digonal



px + qy = 1 meets the pair of lines.

$\overline{OR}$ and $\overline{PQ}$ biset each other at $M(\alpha,\beta)$.

$\therefore\ \alpha=\frac{x_1+x_2}{2}$ and $\beta=\frac{y_1+y_2}{2}$

Eliminating y from $ax^2+2hxy+by^2=0$ -- (1)

and px + qy = 1 -- (2)

$$ax^2+2hx\left(\frac{1-px}{q}\right)+b\left(\frac{1-px}{q}\right)^2=0$$

$$\Rightarrow x^2(aq^2-2hpq+bp^2)+2x(hp-bp)+b=0$$

The roots of this quadratic equation are x_1 and x_2 where

$$x_1+x_2=-\frac{2(hq-bp)}{aq^2-2hpq-bp^2}$$

$$\Rightarrow \alpha=\frac{(bp-hq)}{(aq^2-2hpq+bp^2)}$$

Similarly by eliminating x from (1) and (2) a quadratic equation in y is obtained and y_1, y_2 are its roots where

$$y_1+y_2=-\frac{2(hp-aq)}{aq^2-2hpq-np^2}\Rightarrow \beta=\frac{(aq-hp)}{(aq^2-2hpq+bp^2)}$$

Now the equation to the join of O(0, 0) and $M(\alpha,\beta)$ is $(y-0)(0-\alpha)=(x-0)(0-\beta)$

$\Rightarrow \alpha y=\beta x$

Substituting the values of α and β, the equation of the diagonal OR

is $\boldsymbol{y(bp-hq)=x(aq-hp)}$.

EXERCISE

I

1. Find the angle between the lines represented by $2x^2+xy-6y^2+7y-2=0$.

Sol. Given equation is

$2x^2+xy-6y^2+7y-2=0$ Comparing with

$ax^2+2hxy+by^2+2gx+2fy+c=0$ then

a = 2, b = - 6, c = - 2, g = 0, $f=\frac{7}{2}, h=\frac{1}{2}$

Angle between the lines is given by

$$\cos\alpha=\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}=\frac{|2-6|}{\sqrt{(2+6)^2+1}}=\frac{4}{\sqrt{65}} \Rightarrow \alpha=\cos^{-1}\left(\frac{4}{\sqrt{65}}\right)$$

2. Prove that the equation $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ represents a pair of perpendicular lines.

Sol. From given equation a = 2, b = - 2 **and a + b = 2 + (-2)=0**

$\Rightarrow$ angle between the lines is 90^0. $\therefore$ The given lines are perpendicular.

II

1. Prove that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y+2 = 0$ represents a pair of straight lines and find the co-ordinates of the point of intersection.

Sol. The given equation is $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 3 \quad , \; b = 2, c = 2, \quad 2f = 3 \quad \Rightarrow f = \frac{5}{2}$

$2g = 5 \Rightarrow g = \frac{5}{2}, \qquad 2h = 7 \Rightarrow h = \frac{7}{2}$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 3(2)(2) + 2.\frac{5}{2}.\frac{5}{2}.\frac{7}{2} - 3.\frac{25}{4} - 2.\frac{25}{4} - 2.\frac{49}{4}$

$= \frac{1}{4}(48 + 175 - 75 - 50 - 98)$

$= \frac{1}{2}(223 - 223) = 0$

$h^2 - ab = \left(\frac{7}{2}\right)^2 - 3.2 = \frac{49}{4} - 6 = \frac{25}{4} > 0$

$f^2 - bc = \left(\frac{5}{2}\right)^2 - 2.2 = \frac{25}{4} - 4 = \frac{9}{4} > 0$

$g^2 - ac = \left(\frac{5}{2}\right)^2 - 3.2 = \frac{25}{4} - 6 = \frac{1}{4} > 0$

$\therefore$ The given equation represents a pair of lines.

The point of intersection of the lines is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

$$= \left(\frac{\frac{7}{2}.\frac{5}{2} - 2\frac{5}{2}}{6 - \frac{49}{4}}, \frac{\frac{5}{2}.\frac{7}{2} - 3.\frac{5}{2}}{6 - \frac{49}{4}} \right) = \left(\frac{35-20}{24-29}, \frac{35-30}{24-49} \right)$$

$$= \left(\frac{+15}{-25}, \frac{5}{-28} \right) = \left(\frac{-3}{5}, -\frac{1}{5} \right)$$

Point of intersection is $p\left(\frac{-3}{5}, \frac{-1}{5} \right)$

2. Find the value of k, if the equation $2x^2 + kxy - 6y^2 + 3x + y + 1 = 0$ **represents a pair of straight lines. Find the point of intersection of the lines and the angle between the straight lines for this value of k.**

Sol. The given equation is $2x^2 + kxy - 6y^2 + 3x + y + 1 = 0$

a = 2, b = - 6, c = 1, $f = \frac{1}{2}, 2g = 3\, g = \frac{3}{2}, h = \frac{k}{2}$

Since the given equation is representing a pair of straight lines, therefore

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow -12 + 2.\frac{1}{2}.\frac{3}{2}.\left(+\frac{k}{2} \right) - 2.\frac{1}{4} + 6.\frac{9}{4} - \frac{k^2}{4} = 0$$

$$\Rightarrow -48 + 3k - 2 + 54 - k^2 = 0$$

$$\Rightarrow -k^2 + 3k + 4 = 0 \Rightarrow k^2 - 3k - 4 = 0$$

$\Rightarrow$ (k – 4) (k + 1) = 0

$\Rightarrow$ k = 4 or – 1

Case (i) k = - 1

Point of intersection is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$

$$\left(\frac{+\frac{1}{2}.\frac{1}{2}+6.\frac{3}{2}}{-12-\frac{1}{4}}, \frac{\frac{3}{2}\left(-\frac{1}{2}\right)-2.\frac{1}{2}}{-12-\frac{1}{4}}\right)=\left(\frac{-1+36}{-49}, \frac{-3-4}{-49}\right)$$

$$=\left(\frac{35}{-49}, \frac{-7}{-49}\right)=\left(\frac{-5}{7}, \frac{1}{7}\right)$$

Point of intersection is $\left(\frac{-5}{7}, \frac{1}{7}\right)$

Angle between the lines $=\cos\alpha=\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}} = \frac{|2-6|}{\sqrt{(2+6)^2+4}}=\left(\frac{4}{\sqrt{65}}\right)$

Case (ii) k = 4

$$\left(\frac{2.\frac{1}{2}+6.\frac{3}{2}}{-12-4}, \frac{\frac{3}{2}.2-2.\frac{1}{2}}{-12-4}\right)=\left(-\frac{5}{8}, -\frac{1}{8}\right)$$

Point of intersection is $P\left(-\frac{5}{8}, -\frac{1}{8}\right)$ and angle between the lines is

$$\cos\alpha=\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$$

$$=\frac{|2-6|}{\sqrt{(2+6)^2+16}}=\frac{4}{4\sqrt{5}}=\frac{1}{\sqrt{5}}$$

$$\alpha=\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

3. Show that the equation $x^2 - y^2 - x + 3y - 2 = 0$ represents a pair of perpendicular lines and find their equations.

Sol. Given equation is $x^2 - y^2 - x + 3y - 2 = 0$ $\Rightarrow$ a = 1, b = 1, c = - 2 $f = \frac{3}{2}$, $g = -\frac{1}{2}$,

h = 0

Now $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1(-1)(-2) + 0 - 1.\frac{9}{4} + 1.\frac{1}{4} + 0 \quad = +2 - \frac{9}{4} + \frac{1}{4} = 0$

$h^2 - ab = 0 - 1(-1) = 1 > 0$

$f^2 - bc = \frac{9}{4} - 2 = \frac{1}{4} > 0$

$g^2 - ac = \frac{1}{4} + 2 = \frac{9}{4} > 0$

And a +b = 1 – 1 = 0

The given equation represent a pair of perpendicular lines.

Let $x^2 - y^2 - x + 3y - 2 = (x + y + c_1)(x - y + c_2)$

Equating the coefficients of x $\Rightarrow c_1 + c_2 = -1$

Equating the co-efficient of y $\Rightarrow -c_1 + c_2 = 3$

Adding $2c_1 = 2 \Rightarrow c_2 = 1$

$c_1 + c_2 = -1 \Rightarrow c_1 + 1 = -1, \; c_1 = -2$

Equations of the lines are x + y – 2 = 0 and x – y + 1 = 0

4. **Show that the lines** $x^2+2xy-35y^2-4x+44y-12=0$ **are** $5x+2y-8=0$ **are concurrent.**

Sol. Equation of the given lines are $x^2+2xy-35y^2-4x+44y-12=0$

a = 1, b = - 35, c = - 12, f = 22, g = - 2, h = 1

Point of intersection is $\left(\frac{hf-bg}{ab-h^2},\frac{gh=af}{ab-h^2}\right)$

$=\left(\frac{22-70}{-35-1},\frac{-2-22}{-35-1}\right)=\left(\frac{-48}{-36},\frac{-24}{-36}\right)=\left(\frac{4}{3},\frac{2}{3}\right)$

Point of intersection of the given lines is $P\left(\frac{4}{3},\frac{2}{3}\right)$. Given line is $5x+2y-8=0$.

Substituting P in above line,

$5x+2y-8=5.\frac{4}{3}+2.\frac{2}{3}-8=\frac{20+4-24}{3}=0$

P lies on the third line $5x+2y-8=0$

$\therefore$ The given lines are concurrent.

5. **Find the distances between the following pairs of parallels straight lines :**

i). $9x^2-6xy+y^2+18x-6y+8=0$

Sol. **Given equation is**

$9x^2-6xy+y^2+18x-6y+8=0$.

From above equation a =9,b=1,c=8,h =-3,g=9,f=-3.

Distance between parallel lines $=2\sqrt{\frac{g^2-ac}{a(a+b)}}$

$$= 2\sqrt{\frac{9^2 - 9.8}{9(9+1)}} = 2\sqrt{\frac{9}{9.10}} = \sqrt{\frac{4}{10}} = \sqrt{\frac{2}{5}}$$

ii. $x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0$

ans. $\frac{5}{2}$

6. Show that the pairs of lines $3x^2 + 8xy - 3y^2 = 0$ **and** $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ **form a squares.**

Sol. Equation of the first pair of lines is $3x^2 + 8xy - 3y^2 = 0$

$\Rightarrow (x + 3y)(3x - y) = 0 \qquad \Rightarrow 3x - y = 0, x + 3y = 0$

Equations of the lines are $3x - y = 0$ ……..(1)and $x + 3y = 0$ ……..(2)

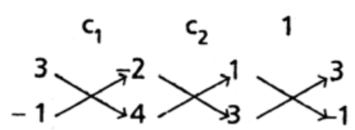
Equation of the second pair of lines is $3x^2 + 8xy - 3y^2 + 2x - 4y + 1 = 0$

Since $3x^2 + 8xy - 3y^2 = (x + 3y)(3x - y)$

Let $3x^2 + 8xy - 3y^2 + 2x - 4y + 1 = \quad (3x - y + c_1)(x + 3y + c_2)$

Equating the co-efficient of x, we get $c_1 + 3c_2 = 2$

Equation the co-efficient of y, we get $3c_1 + c_2 = -4$



$$\frac{c_1}{12 - 2} = \frac{c_2}{-6 - 4} = \frac{1}{-1 - 9}$$

$$c_1 = \frac{10}{-10} = -1, c_2 = \frac{-10}{-10} = 1$$

Equations of the lines represented by $3x^2 + 8xy - 3y^2 + 2x - 4y + 1 = 0$ are

$3x - y - 1 = 0$ ….(3)and $x + 3y + 1 = 0$…..(4)

From above equations, lines (1) and (3) are parallel and lines (2) and(4) are parallel.

Therefore given lines form a parallelogram.

But the adjacent sides are perpendicular, it is a rectangle.(since,(1),(2) are perpendicular and (3),(4) and perpendicular.)

The point of intersection of the pair of lines $3x^2 + 8xy - 3y^2 = 0$ is O(0,0).

Length of the perpendicular from O to (3) = $\frac{|0+0+1|}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}$

Length of the perpendicular from O to (4) = $\frac{|0+0+1|}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}$

Therefore, O is equidistant from lines (3),(4).

Therefore, the distance between the parallel lines is same. Hence the rectangle is a square.

III

1. Find the product of the length of the perpendiculars drawn from (2,1) upon the lines $12x^2 + 25xy + 12y^2 + 10x + 11y + 2 = 0$

Sol. Given pair of lines is $12x^2 + 25xy + 12y^2 + 10x + 11y + 2 = 0$

Now

$$12x^2 + 25xy + 12y^2 = 12x^2 + 16xy + 9xy + 12y^2$$
$$= 4x(3x+4y) + 3y(3x+4y) = (3x+4y)(4x+3y)$$

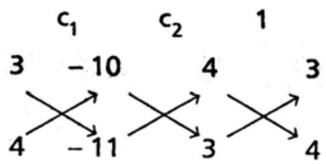
Let $12x^2 + 25xy + 12y^2 + 10x + 11y + 2 = (3x+4y+c_1)(4x+3y+c_2)$

Equating the co-efficient of x, y we get

$4c_1 + 3c_2 = 10 \Rightarrow 4c_1 + 3c_2 - 10 = 0$ ….(1)

$3c_1 + 4c_2 = 11 \Rightarrow 3c_1 + 4c_2 - 11 = 0$ ....(2)

Solving,



$$\frac{c_1}{-33+40} = \frac{c_2}{-30+44} = \frac{1}{16-9}$$

$$c_1 = \frac{7}{7} = 1, c_2 = \frac{14}{7} = 2$$

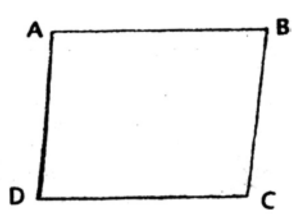
Therefore given lines are $3x + 4y + 1 = 0$ -----(3) and $4x + 3y + 2 = 0$ ----(4)

Length of the perpendicular form P(2,1) on $(1) = \frac{6+4+1}{\sqrt{9+16}} = \frac{11}{5}$

Length of the perpendicular from P(2,1) on $(2) = \frac{|8+3+2|}{\sqrt{16+9}} = \frac{13}{5}$

Product of the length of the perpendicular $= \frac{11}{5} \times \frac{13}{5} = \frac{143}{25}$

2. Show that the straight lines $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 + 5x + 10y + 4 = 0$ from a parallelogram and find the lengths of its sides.



Sol. Equation of the first pair of lines is

$$y^2 - 4y + 3 = 0, \Rightarrow (y-1)(y-3) = 0$$

$\Rightarrow y-1=0 \text{ or } y-3=0$

$\Rightarrow$ Equations of the lines are $y - 1 = 0$ ……..(1)

and $y - 3 = 0$ ……..(2)

Equations of (1) and (2) are parallel.

Equation of the second pair of lines is $x^2+4xy+4y^2+5x+10y+4=0$

$\Rightarrow (x+2y)^2+5(x+2y)+4=0$

$\Rightarrow (x+2y)^2+4(x+2y)+(x+2y)+4=0$

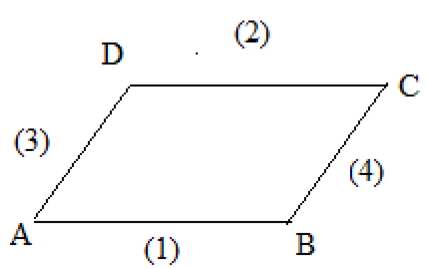
$\Rightarrow (x+2y)(x+2y+4)+1(x+2y+4)=0$

$\Rightarrow (x+2y+1)(x+2y+4)=0$

$\Rightarrow x+2y+1=0, x+2y+4=0$

Equations of the lines are $x+2y+1=0$ ……..(3)and $x+2y+4=0$ ……(4)

Equations of (3) and (4) are parallel .



Solving (1), (3) x + 2 + 1= 0, x = - 3

Co-ordinates of A are (-3, 1)

Solving (2), (3) x + 6 + 1 = 0, x = - 7

Co-ordinates of D are (-7,3)

Solving (1), (4) x + 2 + 4 = 0, x = - 6

Co-ordinates of B are (-6, 1)

$$AB = \sqrt{(-3+6)^2 + (1-1)^2} = \sqrt{9+0} = 3$$

$$AD = \sqrt{(-3+7)^2 + (1-3)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

Length of the sides of the parallelogram are $3, 2\sqrt{5}$

3. Show that the product of the perpendicular distances from the origin to the pair of straight lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\frac{|c|}{\sqrt{(a-b)^2 + 4h^2}}$$

Sol. Let $l_1x + m_1y + n_1 = 0$ ....(1)

$l_2x + m_2y + n_2 = 0$ ....(2)be the lines represented by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\Rightarrow ax^2 + 2hxy + by^2 + 2gx + 2fy + c \quad = (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

$\Rightarrow l_1l_2 = a, m_1m_2 = b, l_1m_2 + l_2m_1 = 2h$

$l_1n_2 + l_2n_1 = 2g, m_1n_2 + m_2n_1 = 2f, n_1n_2 = c$

Perpendicular from origin to (1) $= \dfrac{|n_1|}{\sqrt{l_1^2 + m_1^2}}$

Perpendicular from origin to (2) $= \dfrac{|n_2|}{\sqrt{l_2^2 + m_2^2}}$

Product of perpendiculars

$$=\frac{|n_1|}{\sqrt{l_1^2+m_1^2}}\cdot\frac{|n_2|}{\sqrt{l_2^2+m_2^2}}$$

$$=\frac{|n_1n_2|}{\sqrt{l_1^2l_2^2+m_1^2m_2^2+l_1^2m_2^2+l_2^2m_1^2}}$$

$$=\frac{|n_1n_2|}{\sqrt{(l_1l_2-m_1m_2)^2+(l_1m_2+l_2m_1)^2}}$$

$$=\frac{|c|}{\sqrt{(a-b)^2+(2h)^2}}=\frac{|c|}{\sqrt{(a-b)^2+4h^2}}$$

4. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of intersecting lines, then show that the square of the distance of their point of intersection from the origin is $\frac{c(a+b)-f^2-g^2}{ab-h^2}$. Also show that the square of this distance from origin is $\frac{f^2+g^2}{h^2+b^2}$ if the given lines are perpendicular.

Sol. Let $l_1x+m_1y+n_1=0$ …….(1)

$l_2x+m_2y+n_2=0$ ……..(2)

be the lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$

$\Rightarrow ax^2+2hxy+by^2+2gx+2fy+c$

$=(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)$

$l_1l_2=a, m_1m_2=b, l_1m_2+l_2m_1=2h$

$l_1n_2+l_2n_1=2g, m_1n_2+m_2n_1=2f, n_1n_2=c$ **Solving (1) and (2)**

$$\frac{x}{m_1n_2-m_2n_1}=\frac{y}{l_2n_1-l_1n_2}=\frac{1}{l_1m_2-l_2m_2}$$

The point of intersection is P= $\left[\dfrac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1}, \dfrac{l_2n_1 - l_1n_2}{l_1m_2 - l_2m_1}\right]$

$$OP^2 = \frac{(m_1n_2 - m_2n_1)^2 + (l_2n_1 - l_1n_2)^2}{(l_1m_2 - l_2m_1)^2}$$

$$= \frac{(m_1n_2 + m_2n_1)^2 - 4m_1m_2n_1n_2 + (l_1n_2 + l_2n_1)^2 - 4l_1l_2n_1n_2}{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}$$

$$= \frac{4f^2 - 4abc + 4g^2 - 4ac}{4h^2 - 4ab}$$

$$= \frac{c(a+b) - f^2 - g^2}{ab - h^2}.$$

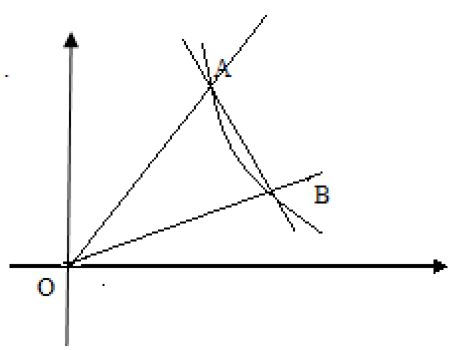
If the given pair of lines are perpendicular, then $a + b = 0 \Rightarrow a = -b$

$$\Rightarrow OP^2 = \frac{0 - f^2 - g^2}{(-b)b - h^2} = \frac{f^2 + g^2}{h^2 + b^2}$$

HOMOGENISATION

THEOREM

The equation to the pair of lines joining the origin to the points of intersection of the curve $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the line $L \equiv lx + my + n = 0$ is $ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\dfrac{lx+my}{-n}\right) + c\left(\dfrac{lx+my}{-n}\right)^2 = 0$ ---(1)



Eq (1) represents the combined equation of the pair of lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

EXERCISE

I

1. **Find the equation of the lines joining the origin to the point of intersection of $x^2+y^2=1$ and $x+y=1$**

Sol. The given curves are $x^2+y^2=1$ ……..(1)

$$x+y=1 \qquad ……..(2)$$

Homogenising (1) with the help of (2) then $x^2+y^2=1^2$

$\Rightarrow x^2+y^2=(x+y)^2=x^2+y^2+2xy$ i.e. $2xy=0 \Rightarrow xy=0$

2. **Find the angle between the lines joining the origin to points of intersection of $y^2=x$ and $x+y=1$.**

Sol. Equation of the curve is $y^2=x$ …..(1) and Equation of line is $x+y=1$…….(2)

Harmogonsing (1) with the help of (2)

Y^2 –x.1 =0 $\Rightarrow y^2=x(x+y)=x^2+xy$

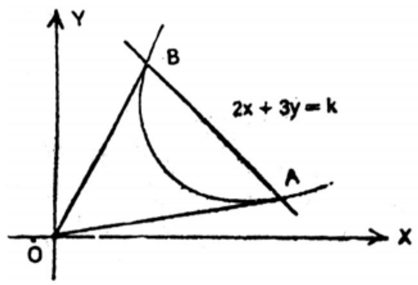
$\Rightarrow x^2+xy-y^2=0$ which represents a pair of lines. From this equation

$a+b=1-1=0$

The angle between the lines is 90^0.

II

1. **Show that the lines joining the origin to the points of intersection of the curve $x^2-xy+y^2+3x+3y-2=0$ and the straight line $x-y-\sqrt{2}=0$ are mutually perpendicular.**

Sol. 

Le t A,B the the points of intersection of the line and the curve.

Equation of the curve is $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ …….(1)

Equation of the line AB is $x - y - \sqrt{2} = 0$

$$\Rightarrow \quad x - y = \sqrt{2} \Rightarrow \frac{x-y}{\sqrt{2}} = 1 \qquad \text{…….(2)}$$

Homogenising, (1) with the help of (2) combined equation of OA, OB is

$$x^2 - xy + y^2 + 3x.1 + 3y.1 - 2.1^2 = 0$$

$$\Rightarrow x^2 - xy + y^2 + 3(x+y)\frac{x-y}{\sqrt{2}} - 2\frac{(x-y)^2}{2} = 0$$

$$\Rightarrow x^2 - xy + y^2 + \frac{3}{\sqrt{2}}(x^2 - y^2) - (x^2 - 2xy + y^2) = 0$$

$$\Rightarrow x^2 - xy + y^2 + \frac{3}{\sqrt{2}}x^2 - \frac{3}{\sqrt{2}}y^2 - x^2 + 2xy - y^2 = 0$$

$$\Rightarrow \frac{3}{\sqrt{2}}x^2 + xy - \frac{3}{\sqrt{2}}y^2 = 0$$

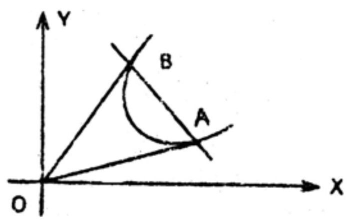
$$\Rightarrow a + b = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}} = 0$$

$\therefore$ OA, OB are perpendicular.

2. Find the values of k, if the lines joining the origin to the points of intersection of the curve $2x^2-2xy+3y^2+2x-y-1=0$ and the line x + 2y = k are mutually perpendicular.

Sol. Given equation of the curve is $S \equiv 2x^2-2xy+3y^2+2x-y-1=0$ ……(1)

Equation of AB is x + 2y =k



$\frac{x+2y}{k}=1$ ……..(2)

Le t A,B the the points of intersection of the line and the curve.

Homogenising, (1) with the help of (2), the combined equation of OA,OB is

$$2x^2-2xy+3y^2+2x.1-y.1-1^2=0$$

$$2x^2-2xy+3y^2+2x\frac{(x+2y)}{k}-y\frac{(x+2y)}{k}=\frac{(x+2y)^2}{k^2}=0$$

$$\Rightarrow 2k^2x^2-2k^2xy+3k^2y^2+2kx(x+2y)-ky(x+2y)-(x+2y)^2=0$$

$$\Rightarrow 2k^2x^2-2k^2xy+3k^2y^2+2kx^2+4kxy-kxy-2ky^2-x^2-4xy-4y^2=0$$

$$\Rightarrow (2k^2+2k-1)x^2+(-2k^2+3k-4)xy+(3k^2-2k-4)y^2=0$$

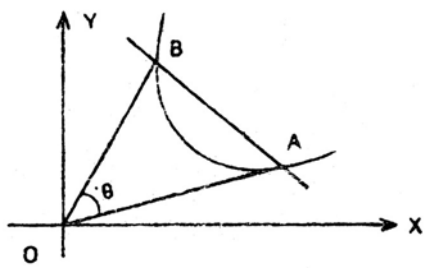
Given that above lines are perpendicular, Co-efficient x^2 + co-efficient of y^2 =0

$$\Rightarrow 2k^2+2k-1+3k^2-2k-4=0$$

$$\Rightarrow 5k^2=5 \Rightarrow k^2=1 \ \therefore k=\pm 1$$

3. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2+2xy+y^2+2x+2y-5=0$ and the line $3x-y+1=0$

Sol.



Equation of the curve is $x^2+2xy+y^2+2x+2y-5=0$ ……(1)

Equation of AB is $3x-y+1=0 \Rightarrow y-3x=1$ ………(2)

Le t A,B the the points of intersection of the line and the curve.

Homogenising (1) with the help of (2), combined equation of OA, OB is

$$x^2+2xy+y^2+2x.1+2y.1-5.1^2=0$$

$$\Rightarrow x^2+2xy+y^2+2x(y-3x)+2y(y-3x)-5(y-3x)^2=0$$

$$\Rightarrow x^2+2xy+y^2+2xy-6x^2+2y^2-6xy-5(y^2+9x^2-6xy)=0$$

$$\Rightarrow -5x^2-2xy+3y^2-5y^2-45x^2+30xy=0$$

$$\Rightarrow -50x^2+28xy-2y^2=0 \Rightarrow 25x^2-14xy+y^2=0$$

let θ be the angle between OA and OB ,then

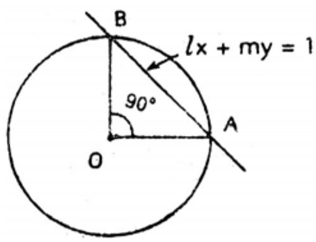
$$\cos\theta=\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}=\frac{|25+1|}{\sqrt{(25-1)^2+196}}=\frac{26}{\sqrt{576+196}}=\frac{26}{\sqrt{772}}$$

$$=\frac{26}{2\sqrt{193}}=\frac{13}{\sqrt{193}} \quad \therefore \theta=\cos^{-1}\left(\frac{13}{\sqrt{193}}\right)$$

III

1. Find the condition for the chord $lx + my = 1$ of the circle $x^2 + y^2 = a^2$ (whose centre is the origin) to subtend a right angle at the origin.

Sol.



Equation of the circle $x^2 + y^2 = a^2$ …….(1)

Equation of AB is $lx + my = 1$ ………(2)

Let A,B the the points of intersection of the line and the curve

Homogenising (1) with the help of (2) ,the combined equation of OA, OB is

$$x^2 + y^2 = a^2.1^2 \Rightarrow x^2 + y^2 = a^2 (lx + my)^2$$

$$= a^2 \left(l^2x^2 + m^2y^2 + 2lmxy\right) = a^2l^2x^2 + a^2m^2y^2 + 2a^2lmxy$$

$$\Rightarrow a^2l^2x^2 + 2a^2lmxy + a^2m^2y^2 - x^2 - y^2 = 0$$

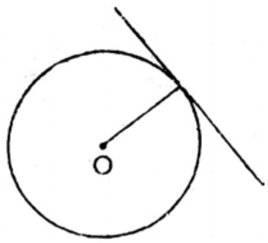
$$\Rightarrow \left(a^2l^2 - 1\right)x^2 + 2a^2lmxy + \left(a^2m^2 - 1\right)y^2 = 0$$

Since OA, OB are perpendicular, Coefficient of x^2 + co-efficient of y^2 =0

$\Rightarrow a^2l^2 - 1 + a^2m^2 - 1 = 0 \Rightarrow a^2\left(l^2 + m^2\right) = 2$ which is the required condition

2. Find the condition for the lines joining the origin to the points of intersection of the circle $x^2+y^2=a^2$ and the line $lx+my=1$ to coincide.

Sol.



Equation of the circle is $x^2+y^2=a^2$ ……(1)

Equation of AB is $lx+my=1$…….(2) .

Le t A,B the the points of intersection of the line and the curve.

Homogenising (1) with the help of (2) ,

Then the combined equation of OA, OB is $x^2+y^2=a^2.1^2$

$x^2+y^2=a^2(lx+my)^2 \quad =a^2\left(l^2x^2+m^2y^2+2lmxy\right)$

$\Rightarrow x^2+y^2=a^2l^2x^2+a^2m^2y^2+2a^2lmxy$

$\Rightarrow \left(a^2l^2-1\right)x^2+2a^2lmxy+\left(a^2m^2-1\right)y^2=0$

Since OA, OB are coincide $\Rightarrow h^2=ab$

$\Rightarrow a^4l^2m^2=\left(a^2l^2-1\right)\left(a^2m^2-1\right) \Rightarrow a^4l^2m^2=a^4l^2m^2-a^2l^2-a^2m^2+1$

$\therefore a^2l^2-a^2m^2+1=0 \Rightarrow a^2\left(l^2+m^2\right)=1$

This is the required condition.

3. Write down the equation of the pair of straight lines joining the origin to the points of intersection of the lines 6x – y + 8 = 0 with the pair of straight lines $3x^2+4xy-4y^2-11x+2y+6=0$. **Show that the lines so obtained make equal angles with the coordinate axes.**

Sol. Given pair of line is $3x^2+4xy-4y^2-11x+2y+6=0$ …(1)

Given line is $6x-y+8=0 \Rightarrow \frac{6x-y}{-8}=1 \quad \Rightarrow \frac{y-6x}{8}=1$-----(2)

Homogenising (1) w.r.t (2)

$$3x^2+4xy-4y^2-(11x-2y)\left(\frac{y-6x}{8}\right)+6\left(\frac{y-6x}{8}\right)^2=0$$

$$64\left[3x^2+4xy-4y^2\right]-8\left[11xy-66x^2-2y^2+12xy\right]+6\left[y^2+36x^2-12xy\right]=0$$

$$\Rightarrow 936x^2+256xy-256xy-234y^2=0$$

$$\Rightarrow 468x^2-117y^2=0$$

$$\Rightarrow 4x^2-y^2=0 \text{ ---- (3)}$$

Is eq. of pair of lines joining the origin to the point of intersection of (1) and (2).

The eq. pair of angle bisectors of (3) is $h\left(x^2-y^2\right)-(a-b)xy=0$

$$\Rightarrow 0\left(x^2-y^2\right)-(4-1)xy=0 \quad \Rightarrow xy=0$$

x = 0 or y = 0 which are the eqs. is of co-ordinates axes

$\therefore$ The pair of lines are equally inclined to the co-ordinate axes

4. If the straight lines given by $ax^2+2hxy+by^2+2gx+2fy+c=0$ intersect on Y-axis, show that $2fgh-bg^2-ch^2=0$

Sol. Given pair of lines is $ax^2+2hxy+by^2+2gx+2fy+c=0$

Equation of Y-axis is x = 0 then equation becomes $by^2+2fy+c=0$ ……(1)

Since the given pair of lines intersect on Y – axis, the roots or equation (1) are equal.

$\therefore$ Discriminate = 0

$\Rightarrow (2f)^2-4.b.c=0 \Rightarrow 4f^2-4bc=0$

$\Rightarrow f^2-bc=0 \Rightarrow f^2=bc$

Since the given equation represents a pair of lines

$abc+2fgh+af^2-bg^2-ch^2=0$

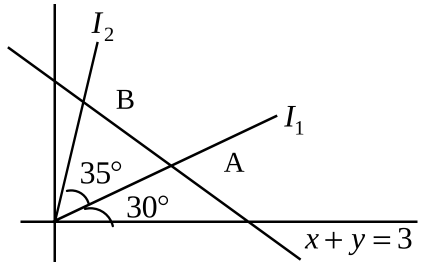
$\Rightarrow a\left(f^2\right)+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow 2fgh-bg^2-ch^2=0$

5. Prove that the lines represented by the equations $x^2-4xy+y^2=0$ and $x+y=3$ form an equilateral triangle.

Sol. Since the straight line $L: x+y=3$ makes $45°$ with the negative direction of the X –axis, none of the lines which makes $60°$ with the line L is vertical. If 'm' is the slope of one such straight line, then $\sqrt{3}=\tan 60°=\left|\frac{m+1}{1-m}\right|$ and so, satisfies the equation $(m+1)^2=3(m-1)^2$

Or $m^2-4m+1=0$ ....(1)



But the straight line having slope ‘m’ and passing through the origin is

$y = mx$ .... (2)

So the equation of the pair of lines passing through the origin and inclined at $60°$ with the line L is obtained by eliminating ‘m’ from the equations (1) and (2).

Therefore the combined equation of this pair of lines is $\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) + 1 = 0$ (i.e,)

$x^2 - 4xy + y^2 = 0$

Which is the same as the given pair of lines. Hence, the given traid of lines form an equilateral triangle.

6. Show that the product of the perpendicular distances from a point (α, β) to the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{(a-b)^2 + 4h^2}}$

Sol. Let $ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y)(l_2x + m_2y)$

Then the separate equations of the lines represented by the equation

$ax^2 + 2hxy + by^2 = 0$ are $L_1 : l_1x + m_1y = 0$ and $L_2 : l_2x + m_2y = 0$

Also, we have $l_1l_2 = a; m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$

d_1 =length of the perpendicular from (α, β) to $L_1 = \frac{|l_1\alpha + m_1\beta|}{\sqrt{l_1^2 + m_1^2}}$

d_2 =length of the perpendicular from (α, β) to L_2 $L_2 = \frac{|l_2\alpha + m_2\beta|}{\sqrt{l_2^2 + m_2^2}}$

Then, the product of the lengths of the perpendiculars from (α,β) to the given pair of lines $= d_1 d_2$

$$=\frac{\left|(l_1\alpha+m_1\beta)(l_2\alpha+m_2\beta)\right|}{\sqrt{(l_1^2+m_1^2)(l_2^2+m_2^2)}}=\frac{\left|a\alpha^2+2h\alpha\beta+b\beta^2\right|}{\sqrt{(a-b)^2+4h^2}}$$

PROBLEMS FOR PRACTICE.

1. **If the lines xy+x+y+1 = 0 and x +ay- 3= 0 are concurrent, find a.**

2. **The equation** $ax^2+3xy-2y^2-5x+5y+c=0$ **represents two straight lines perpendicular to each other. Find a and c.**

3. **Find** λ **so that** $x^2+5xy+4y^2+3x+2y+\lambda=0$ **may represent a pair of straight lines. Find also the angle between them for this value of** λ.

4. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents the straight lines equidistant from the origin, show that $f^4-g^4=c(bf^2-ag^2)$

5. **Find the centroid of the triangle formed by the lines** $12x^2-20xy+7y^2=0$ **and** $2x-3y+4=0$

ANS. $=\left(\frac{8}{3},\frac{8}{3}\right)$

6. **Let** $aX^2+2hXY+bY^2=0$ **represent a pair of straight lines. Then show that the equation of the pair of straight lines.**

i)Passing through (x_o,y_o) **and parallel to the given pair of lines is**

$a(x-x_o)^2+2h(x-x_o)(y-y_o)+b(y-y_o)^2=0$ **ii) Passing through** (x_o,y_o) **and perpendicular to the given pair of lines is** $b(x-x_o)^2-2h(x-x_o)(y-y_o)+a(y-y_o)^2=0$

7. **Find the angle between the straight lines represented by** $2x^2+3xy-2y^2-5x+5y-3=0$

8. **Find the equation of the pair of lines passing through the origin and perpendicular to the pair of lines** $ax^2+2hxy+by^2+2gx+2fy+c=0$

9. **If** $x^2+xy+2y^2+4x-y+k=0$ **represents a pair of straight lines find k.**

10. **Prove that equation** $2x^2+xy-6y^2+7y-2=0$ **represents a pair of straight line.**

11. **Prove that the equation** $2x^2+3xy-2y^2-x+3y-1=0$ **represents a pair of perpendicular straight lines.**

12. **Show that the equation** $2x^2-13xy-7y^2+x+23y-6=0$ **represents a pair of straight lines. Also find the angle between the co-ordinates of the point of intersection of the lines.**

13. **Find that value of** λ **for which the equation** $\lambda x^2-10xy+12y^2+5x-16y-3=0$ **represents a pair of straight lines.**

14. **Show that the pair of straight lines** $6x^2-5xy-6y^2=0$ **and** $6x^2-5xy-6y^2+x+5y-1=0$ **form a square.**

15. **Show that the equation** $8x^2-24xy+18y^2-6x+9y-5=0$ **represents pair of parallel straight lines are find the distance between them.**