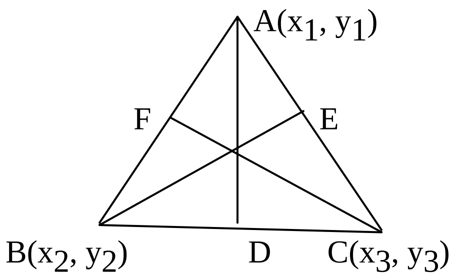
CONCURRENT LINES- PROPERTIES RELATED TO A TRIANGLE

THEOREM

The medians of a triangle are concurrent.

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle



Let D,E,F be the mid points of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively

$$\therefore D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right), \quad E = \left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$$

$$F = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Slope of $\overline{AD}$ is $\dfrac{\frac{y_2+y_3}{2} - y_1}{\frac{x_2+x_3}{2} - x_1} = \dfrac{y_2+y_3-2y_1}{x_2+x_3-2x_1}$

Equation of $\overline{AD}$ is

$$y - y_1 = \frac{y_2+y_3-2y_1}{x_2+x_3-2x_1}(x - x_1)$$

$\Rightarrow (y - y_1)(x_2 + x_3 - 2x_1) = (x - x_1)(y_2 + y_3 - 2y_1)$

$\Rightarrow L_1 \equiv (x - x_1)(y_2 + y_3 - 2y_1)$

$\quad - (y - y_1)(x_2 + x_3 - 2x_1) = 0.$

Similarly, the equations to $\overline{BE}$ and $\overline{CF}$ respectively are $L_2 \equiv (x - x_2)(y_3 + y_1 - 2y_2)$

$\quad - (y - y_2)(x_3 + x_1 - 2x_2) = 0.$

$L_3 \equiv (x - x_3)(y_1 + y_2 - 2y_3)$

$\quad - (y - y_3)(x_1 + x_2 - 2x_3) = 0.$

Now $1.\ L_1 + 1.L_2 + 1.\ L_3 = 0$

The medians $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are concurrent.

THEOREM

The altitudes of a triangle are concurrent.

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC.

Let AD, BE,CF be the altitudes.

Slope of $\overleftrightarrow{BC}$ is $\dfrac{y_3 - y_2}{x_3 - x_2}$ and AD $\perp$ BC

Slope of the altitude through A is $-\dfrac{x_3 - x_2}{y_3 - y_2}$

Equation of the altitude through A is $y - y_1 = \dfrac{x_3 - x_2}{y_3 - y_2}\ (x - x_1)$

$(y - y_1)\ (y_3 - y_2) = -(x - x_1)\ (x_3 - x_2)$

$L_1 = (x - x_1)(x_2 - x_3) + (y - y_1)(y_2 - y_3) = 0.$

Similarly equations of the altitudes through B,C are

$L_2 = (x - x_2)\ (x_3 - x_1) + (y - y_2)\ (y_2 - y_3) = 0,$

$L_3 = (x - x_3)\ (x_1 - x_2) + (y - y_3)\ (y_1 - y_2) = 0.$

Now $1.L_1 + 1.L_2 + 1.L_3 = 0$

The altitudes $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are concurrent.

THEOREM

The internal bisectors of the angles of a triangle are concurrent.

THEOREM

The perpendicular bisectors of the sides of a triangle are concurrent

EXERCISE – 3 (e)

I.

1. Find the in center of the triangle whose vertices are $(1,\sqrt{3})(2,0)$ **and (0, 0)**

Sol. let A(0, 0), B $(1,\sqrt{3})$, C(2, 0) be the vertices of Δ ABC

$$a = BC = \sqrt{(1-2)^2+(\sqrt{3}-0)^2} = \sqrt{1+3} = 2$$

$$b = CA = \sqrt{(2-0)^2-(0-0)^2} = \sqrt{4} = 2$$

$$C = AB = \sqrt{(0-1)^2+(0-\sqrt{3})^2} = \sqrt{4} = 2$$

$\therefore$ ABC is an equilateral triangle

co-ordinates of the in centre are

$$= \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right) = \left(\frac{2.0+2,1+2.2}{2+2+2}, \frac{2.0+2.\sqrt{3}+2.0}{2+2+2}\right)$$

$$= \left(\frac{6}{6}, \frac{2\sqrt{3}}{6}\right) = \left(1, \frac{1}{\sqrt{3}}\right)$$

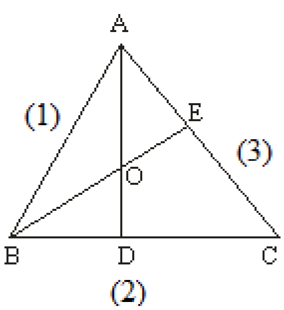
2. Find the orthocenter of the triangle are given by $x+y+10=0$, **x – y – 2 = 0 and 2x + y – 7 = 0**

Sol. Let equation of

AB be x + y + 10 = 0 ---(1)

BC be x – y – 2 = 0 ---(2)

and AC be 2x + y – 7 = 0 ---(3)



Solving (1) and (2) B = (- 4, - 6)

Solving (1) and (3) A =(17, -27)

Equation of BC is x – y – 2 = 0

Altitude AD is perpendicular to BC, therefore Equation of AD is x + y + k = 0

AD is passing through A (17, -27)

$\Rightarrow$ 17 – 27 + k = 0 $\Rightarrow$ k = 10

$\therefore$ Equation if AD is x + y + 10 = 0 ----(4)

Altitude BE is perpendicular to AC.

$\Rightarrow$Let the equation of DE be $x - 2y = k$

BE is passing through D (-4, -6)

$\Rightarrow -4 + 12 = k \Rightarrow k = 8$

Equation of BE is $x - 2y = 8$-----(5)

Solving (4) and (5), the point of intersection is (-4, -6).

Therefore the orthocenter of the triangle is (-4, -6).

3. Find the orthocentre of the triangle whose sides are given by $4x - 7y + 10 = 0$, $x + y = 5$ and $7x + 4y = 15$

Sol. Ans: O (1, 2)

4. Find the circumcentre of the triangle whose sides are $x = 1$, $y = 1$ and $x + y = 1$

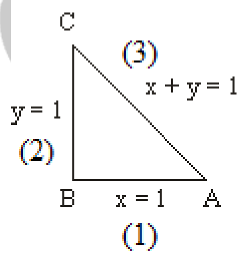
Sol. Let equation of AB be $x = 1$----(1)

BC be $y = 1$ ----(2)

and AC be $x + y = 1$ ----(3)

lines (1) and (2) are perpendicular to each other. Therefore, given triangle is a right triangle and $\angle B = 90^o$.

Therefore, circumcentre is the mid point of hypotenuse AC.



Solving (1) and (3), vertex A =(1, 0)

Solving (2) and (3), vertex c =(0, 1)

Circumcentre = mid point of AC= $\left(\frac{1}{2}, \frac{1}{2}\right)$

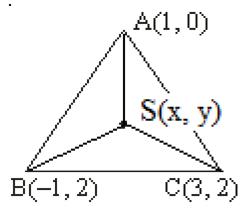
5. Find the incentre of the triangle formed by the lines $x = 1$, $y = 1$ and $x + y = 1$

Sol. ANS: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

6. Find the circumcentre of the triangle whose vertices are (1, 0), (-1, 2) and (3, 2)

Sol. vertices of the triangle are

A (1, 0), B (-1, 2), (3, 2)



Let S (x, y) be the circumcentre of Δ ABC.

Then SA = SB = SC

Let SA = SB $\Rightarrow SA^2 = SB^2$

$(x-1)^2 + y^2 = (x+1)^2 + (y-2)^2$

$\Rightarrow x^2 - 2x + 1 + y^2 = x^2 + 2x + 1 + y^2 - 4y + 4$

$\Rightarrow 4x - 4y = -4 \Rightarrow x - y = -1$ ---(1)

SB = SC $\Rightarrow SB^2 = SC^2$

$(x+1)^2 + (y-2)^2 = (x-3)^2 + (y-2)^2$

$\Rightarrow x^2 + 2x + 1 = x^2 - 6x + 9$

$\Rightarrow 8x = 8 \Rightarrow x = 1$

From (1), $1 - y = -1 \Rightarrow y = 2$

$\therefore$ Circum centre is (1, 2)

7. Find the value of k, if the angle between the straight lines $kx + y + 9 = 0$ and $3x - y + 4 = 0$ is $\pi/4$

Sol. Given lines are

$kx + y + 9 = 0$

$3x - y + 4 = 0$ and angle between the lines is $\pi/4$.

$\therefore \cos\frac{\pi}{4} = \frac{|3k-1|}{\sqrt{k^2+1}\sqrt{9+1}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{|3k-1|}{\sqrt{10}\sqrt{k^2+1}}$

Squaring

$\Rightarrow 5k^2 + 5 = (3k-1)^2 = 9k^2 - 6k + 1 \Rightarrow 4k^2 - 6k - 4 = 0 \Rightarrow 2k^2 - 3k - 2 = 0$

$\Rightarrow (k - 2)(2k + 1) = 0 \Rightarrow k = 2$ or $-1/2$

8. Find the equation of the straight line passing through the origin and also the point of intersection of the lines. 2x –y + 5 = 0 and x + y + 1 = 0

Sol. Given lines are $L_1 = 2x - y + 5 = 0$

$L_2 = x + y + 1 = 0$

Equation of any line passing through the point of intersection of the lines $L_1=0$ and $L_2=0$ is $L_1+KL_2=0$

$\Rightarrow$ (2x – y + 5) + k (x + y + 1) = 0 -----(1)

This line is passing through O (0, 0) $\Rightarrow$ 5 + k = 0$\Rightarrow$k = - 5

Substituting in (1), equation of OA is (x – y + 5) – 5 (x + y + 1) = 0

$\Rightarrow$2x – y + 5 – 5y – 5 = 0

$\Rightarrow$ -3x – 6y = 0 $\Rightarrow$ x + 2y = 0

9. Find the equation of the straight line parallel to the lines 3x + 4y = 7 and passing through the point of intersection of the lines x – 2y – 3 = 0 and x + 3y – 6 = 0

Sol. Given lines are $L_1 = x-2y-3=0$ and

$L_2 = x \_ 3y-6=0$

Equation of any line passing through the point of intersection of the lines $L_1=0$ and $L_2=0$ is $L_1+KL_2=0$

$\Rightarrow$ (x – 2y – 3) + k(x + 3y – 6) = 0

$\Rightarrow$ (1 + k)x + (-2 + 3k)y + (-3 -6k) = 0----(1)

This line is parallel to 3x + 4y = 7

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}\Rightarrow\frac{3}{(1+k)}=\frac{4}{(-2+3k)}$$

$\Rightarrow 3(-2+3k)=(1+k)4$

$\Rightarrow -6+9k=4+4k\Rightarrow 5k=10\Rightarrow k=2$

Equation of the required line is

3x + 4y –15 = 0

10. Find the equation of the straight line perpendicular to the line 2x + 3y = 0 and passing through the point of intersection of the lines x + 3y – 1 = 0 and x – 2y + 4 = 0

Sol. L_1=x + 3y – 1 = 0

L_2=x – 2y + 4 = 0

Equation of any line passing through the point of intersection of the lines $L_1=0$ and $L_2=0$ is $L_1+KL_2=0$

$\Rightarrow$ (x + 3y – 1) + k (x – 2y + 4) = 0

$\Rightarrow$ (1 + k)x + (3 – 2k)y + (4k – 1) = 0---(1)

This line is perpendicular to 2x + 3y = 0,

$a_1a_2+b_1b_2=0\Rightarrow 2(1+k)+3(3-2k)=0$

$$2+2k+9-6k=0\Rightarrow 4k=11\Rightarrow k=\frac{11}{4}$$

Substituting in (1), equation of the required line is

$$\left(1+\frac{11}{4}\right)x+\left(3-\frac{11}{2}\right)y+(11-1)=0$$

$$\frac{15}{4}x-\frac{5}{2}y+10=0$$

$$\Rightarrow 15x-10y=40=0$$

$$\Rightarrow 3x-2y+8=0$$

11. Find the equation of the straight line making non – zero equal intercepts on the axes and passing through the point of intersection of the lines 2x – 5y + 1 = 0 and x – 3y - 4 = 0

Sol. Let $L_1=2x+5y+1=0, L_2=x-3y-4=0$

Equation of any line passing through the point of intersection of the lines L_1=0 and L_2=0 is $L_1+KL_2=0$

$\Rightarrow$ (2x – 5y + 1) + k(x – 3y – 4) = 0

$\Rightarrow$ (2 + k)x – (5 + 3k)y + (1 – 4k) = 0 – (1)

Intercepts on co-ordinates axes are equal, coefficient of x = coefficient of y

$\Rightarrow$2 + k = -5 – 3k

$\Rightarrow$4k = - 7 $\Rightarrow$ k = - 7/4

Substituting in (1)

Equation of the required line is

$$\Rightarrow\left(-2\frac{7}{4}\right)x-\left(5-\frac{21}{4}\right)y+(1+7)=0$$

$$\Rightarrow\frac{1}{4}x+\frac{1}{4}y+8=0 \Rightarrow x + y + 32 = 0$$

12. Find the length of the perpendicular drawn from the point of intersection of the lines 3x + 2y + 4 = 0 and 2x+5y-1= to the straight line 7x + 24y – 15 = 0

Sol. Given lines are

3x + 2y + 4 = 0 -----(1)

2x + 5y – 1 = 0----(2)

Solving (1) and (2), point of intersection is P (-2, 1).

Length of the perpendicular from P (-2, 1) to the line 7x + 24y – 15 = 0 is

$$=\left|\frac{-14+24-15}{\sqrt{49+576}}\right|=\frac{5}{25}=\frac{1}{5}.$$

13. Find the value of 'a' if the distance of the points (2, 3) and (-4, a) from the straight line 3x + 4y – 8 = 0 are equal.

Sol. Equation of the line is 3x + 4y – 8 = 0 ---(1)

Given pointsP (2, 3), (-4, a)

Perpendicular from P(2,3) to (1) = perpendicular from Q(-4,a) to (1)

$\Rightarrow \frac{|3.2+4.3-8|}{\sqrt{9+16}} = \frac{|3.(-4)+4a-8|}{\sqrt{9+16}}$

$\Rightarrow 10 = |4a-20|$

$\Rightarrow 4a-20 = \pm 10 \Rightarrow 4a = 20 \pm 10 = 30 \text{or} 10$

$\Rightarrow a = \frac{30}{4} \text{or} \frac{10}{4}$

$\therefore\ a = \frac{15}{2} \text{or} 5/2$

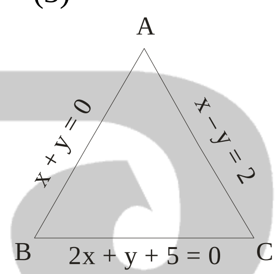
14. Fund the circumcentre of the triangle formed by the straight lines x + y = 0, 2x + y + 5 = 0 and x – y = 2

Sol. let the equation of

AB be $x + y = 0$ ---(1)

BC be $2x + y + 5 = 0$ ---(2)

And AC be $x - y = 2$ ---(3)



Solving (1) and (2) , vertex B = (-5, 5)

Solving (2) and (3) ,vertex C= (-1, -3)

Solving (1) and (3), vertex A = (1, -1)

Let S (x, y) be the circumcentre of Δ ABC.

Then SA = SB = SC

SA = SB $\Rightarrow SA^2 = SB^2$

$(x+5)^2 + (y-5)^2 = (x+1)^2 + (y+3)^2$

$x^2 + 10x + 25 + y^2 - 10y + 25 = x^2 + 2x + 1 + y^2 + 6y + 9$

$\Rightarrow 8x - 16y = -40$

$\Rightarrow x - 2y = -5$ ---(4)

SB = SC $\Rightarrow SB^2 = SC^2$

$\Rightarrow (x+1)^2 + (y+3)^2 = (x-1)^2 + (y+1)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 + 6y + 9 = x^2 - 2x + 1 + y^2 + 2y + 1$

$\Rightarrow 4x + 4y = -8$

$\Rightarrow x + y = -2$ ---(5)

Solving (4) & (5), point of intersection is (-3, 1)
circumcentre is S(-3, 1)

15. If θ is the angle between the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{x}{a}=1$, **find the value of** $\sin\theta$, **when a > b.**

Sol. Given equations are $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow bx+ay=ab$

And $\frac{x}{b}+\frac{y}{a}=1 \Rightarrow ax+by=ab$

Let θ be angle between the lines, then

$$\cos\theta=\frac{|a_1a_2+b_1b_2|}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$$

$$=\frac{|ab+ab|}{\sqrt{b^2+a^2}\sqrt{b^2+a^2}}=\frac{2ab}{a^2+b^2}$$

$$\sin^2\theta=1-\cos^2\theta=1-\frac{4a^2b^2}{\left(a^2+b^2\right)^2} \Rightarrow \sin\theta=\frac{a^2-b^2}{a^2+b^2}$$

II.

1. Find the equation of the straight lines passing through the point (–10, 4) and making an angle θ with the line x – 2y = 10 such that tan θ = 2.

Sol: Given line is $x-2y=10$ ---- (1) and point $(-10, 4)$.

$$\tan\theta=2 \Rightarrow \cos\theta=\frac{1}{\sqrt{5}}$$

Let m be the slope of the require line. This line is passing through (-10, 4), therefore equation of the line is

$y-4=m(x+10) \quad = mx+10m$

$\Rightarrow mx-y+(10m+4)=0$ ------(2)

Given θ is the angle between (1) and (2), therefore, $\cos\theta=\frac{|a_1a_2+b_1b_2|}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$

$$\frac{1}{\sqrt{5}}=\frac{|m+2|}{\sqrt{1+4}\sqrt{m^2+1}}$$

Squaring

$$m^2+1=(m+2)^2=m^2+4m+4$$

$$\Rightarrow 4m+3=0 \Rightarrow m=-\frac{3}{4}$$

Case (i): Co-efficient of $m^2 = 0$

$\Rightarrow$ One of the root is ∞

Hence the line is vertical.

$\therefore$ Equation of the vertical line passing through (–10, 4) is x + 10 = 0

Case (ii): $m = -\frac{3}{4}$

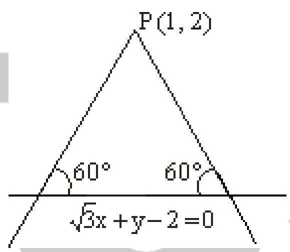
Substituting in (1)

Equation of the line is $-\frac{3}{4}x - y + \left(-\frac{30}{4} + 4\right) = 0$

$\frac{-3x-4y-14}{4} = 0 \Rightarrow 3x+4y+14 = 0$

2. Find the equation of the straight lines passing through the point (1, 2) and making an angle of 60º with the line $\sqrt{3}x + y - 2 = 0$.

Sol: equation of the given line is $\sqrt{3}x + y - 2 = 0$.-----(1)



Let P(1, 2) . let m be the slope of the required line.

Equation of the line passing through P(1, 2) and having slope m is

y – 2 = m(x – 1)= mx – m

$mx - y + (2-m) = 0$ ---(2)

This line is making an angle of 60º with (1), therefore,

$$\cos\theta = \frac{|a_1a_2 + b_1b_2|}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}} \Rightarrow \cos 60° = \frac{|\sqrt{3}m - 1|}{\sqrt{3+1}\sqrt{m^2+1}}$$

$$\Rightarrow \frac{1}{2} = \frac{|\sqrt{3}m-1|}{2\sqrt{m^2+1}}$$

Squaring on both sides, $\Rightarrow m^2 + 1 = \left(\sqrt{3}m - 1\right)^2 = 3m^2 + 1 - 2\sqrt{3}m$

$\Rightarrow 2m^2 - 2\sqrt{3}m = 0 \Rightarrow 2m\left(m - \sqrt{3}\right) = 0$

$\Rightarrow m = 0$ or $\sqrt{3}$

Case (i): m = 0, P(1, 2)

Equation of the line is $-y+2=0$ or $y-2=0$

Case (ii): $m=\sqrt{3}$, P(1, 2)

Equation is $\sqrt{3}x-y+\left(2-\sqrt{3}\right)=0$

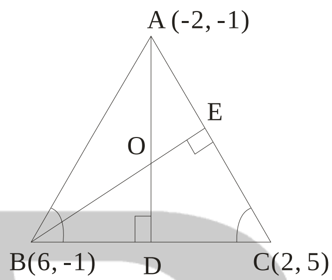
3. The base of an equilateral triangle is $x+y-2=0$ and the opposite vertex is $(2,-1)$. Find the equation of the remaining sides.

ANS: $y+1=\left(2+\sqrt{3}\right)(x-2)$, $y+1=\left(2-\sqrt{3}\right)(x-2)$

4. Find the orthocentre of the triangle whose sides are given below.

i) $(-2,-1),(6,-1)$ and $(2,5)$ **ii)** $(5,-2),(-1,2)$ and $(1,4)$

Sol: **i)** $A(-2,-1),B(6,-1),C(2,5)$ are the vertices of ΔABC.



Slope of $BC=\frac{5+1}{2-6}=\frac{6}{-4}=-\frac{3}{2}$

AD is perpendicular to BC $\Rightarrow$ Slope of $AD=\frac{2}{3}$

Equation of AD is $y+1=\frac{2}{3}(x+2)$

$\Rightarrow 2x-3y+1=0$ ---(1)

Slope of $AC=\frac{5+1}{2+2}=\frac{6}{4}=\frac{3}{2}$

BE is $\perp^{lr}$ to AC

Equation of BE is $y+1=-\frac{2}{3}(x-6)$

$2x-3y-9=0$ ---(2)

solving (1), (2)

$$\begin{array}{cccccc} & x & & y & & 1 \\ 3 & & -9 & & 2 & & 3 \\ -3 & & 1 & & 2 & & -3 \end{array}$$

$$\frac{x}{3-27}=\frac{y}{-18-2}=\frac{1}{-6-6}$$

$$\frac{x}{-24}=\frac{y}{-20}=\frac{1}{-12}$$

$$x=\frac{-24}{-12}=2,\ y=\frac{-20}{-12}=\frac{5}{3}$$

$\therefore$ Co-ordinates of the orthocenter O are $=\left(2,\frac{5}{3}\right)$

ii) $A(5,-2), B(-1,2), C(1,4)$ are the vertices of ΔABC.

ANS: $\left(\frac{1}{5},\frac{14}{5}\right)$

5. Find the circumcentre of the triangle whose vertices are given below.

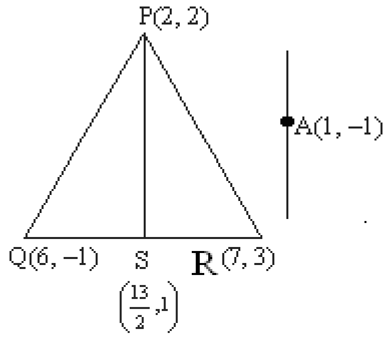
i) $(-2,3)(2,-1)$ and $(4,0)$ **ii)** $(1,3)$, $(0,-2)$ and $(-3,1)$

Sol: i) Ans $\left(\frac{3}{2},\frac{5}{2}\right)$

ii) $(1,3)$, $(0,-2)$ and $(-3,1)$

ANS: $\left(-\frac{1}{3},\frac{2}{3}\right)$

6. **Let $\overline{PS}$ be the median of the triangle with vertices $P(2,2)$ $Q(6,-1)$ and $R(7,3)$. Find the equation of the straight line passing through (1, –1) and parallel to the median $\overline{PS}$.**



Sol: $P(2,2)$, $Q(6,-1)$, $R(7,3)$ are the vertices of ΔABC. Let A(1, –1)

Given S is the midpoint of QR

Co-ordinates of S are $\left(\frac{6+7}{2},\frac{-1+3}{2}\right)=\left(\frac{13}{2},1\right)$

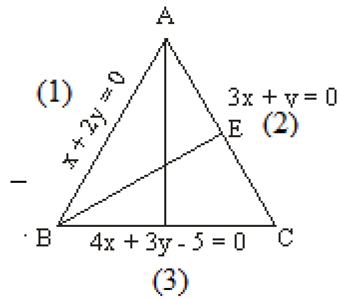
Slope of PS $=\frac{1-2}{\frac{13}{2}-2}=-\frac{1}{\left(\frac{9}{2}\right)}=-\frac{2}{9}$

Required line is parallel to PS and passing through $A(1,-1)$,

Equation of the line is $y+1=-\frac{2}{9}(x-1)$

$\Rightarrow 9y+9=-2x+2 \Rightarrow 2x+9y+7=0$

7. **Find the orthocentre of the triangle formed by the lines.** $x+2y=0, 4x+3y-5=0$ **and** $3x+y=0$.



Sol: Given equations are $x + 2y = 0$ ---(1)

$4x + 3y - 5 = 0$ ---(2)

$3x + y = 0$ ---(3)

Solving (1) and (2), vertex A = (0, 0)

Solving (1) and (3),

Vertex B (2, –1)

Equation of BC is $4x+3y-5=0$

AB is perpendicular to BC and passes through A(0, 0)

Equation of AB is $3x-4y=0$ ---(4)

BE is perpendicular to AC

$\therefore$ Equation of BE is $x-3y=k$

BE passes through $B(2,-1)$

$2+3=k \Rightarrow k=5$

Equation of BE is $x-3y-5=0$ ---(5)

Solving (4) and (5),

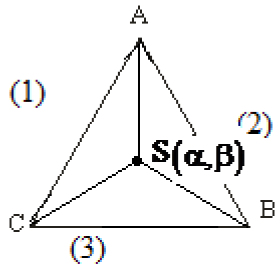
$\therefore$ Orthocentre is $O(-4,-3)$

8. **Find the circumference of the triangle whose sides are given by** $x+y+2=0$**,** $5x-y-2=0$ **and** $x-2y+5=0$.

Sol: Given lines are $x+y+2=0$ ---(1)

$5x-y-2=0$ ---(2)

$x-2y+5=0$ ---(3)



Point of intersection of (1) and (2) is $A=(0,-2)$

Point of intersection of (2) and (3) is $B=(1,3)$

Point of intersection of (1) and (3) is $C=(-3,1)$

Let $S=(\alpha,\beta)$ the orthocentre of ΔABC then $SA=SB=SC$

$\Rightarrow SA^2=SB^2=SC^2$

$\Rightarrow (\alpha-0)^2+(\beta+2)^2=(\alpha-1)^2+(\beta-3)^2=(\alpha+3)^2+(\beta-1)^2$

$\Rightarrow \alpha^2+\beta^2+4\beta+4=\alpha^2+\beta^2-2\alpha-6\beta+10=\alpha^2+\beta^2+6\alpha-2\beta+10$

$SA^2=SB^2 \Rightarrow \alpha^2+\beta^2+4\beta+4=\alpha^2+\beta^2-2\alpha-6\beta+10$

$\Rightarrow 2\alpha+10\beta-6=0 \Rightarrow \alpha+5\beta-3=0$ ---(4)

$SA^2=SC^2 \Rightarrow \alpha^2+\beta^2+4\beta+4=\alpha^2+\beta^2+6\alpha-2\beta+10$

$\Rightarrow 6\alpha-6\beta+6=0 \Rightarrow \alpha-\beta+1=0$ ---(5)

From (4) and (5)

$\alpha \qquad \beta \qquad 1$

$5 \quad -3 \quad 1 \quad 5$

$-1 \quad 1 \quad 1 \quad -1$

$$\frac{\alpha}{5-3}=\frac{\beta}{-3-1}=\frac{1}{-1-5} \Rightarrow \frac{\alpha}{2}=\frac{\beta}{-4}=\frac{1}{-6}$$

$$\alpha=-\frac{2}{6}=-\frac{1}{3}$$

$$\beta=-\frac{4}{-6}=\frac{2}{3}$$

$\therefore$ Circumcentre $S=\left(-\frac{1}{3},\frac{2}{3}\right)$

9. Find the equation of the straight lines passing through (1, 1) and which are at a distance of 3 units from (-2, 3).

Sol: let A(1, 1). Let m be the slope of the line.

Equation of the line is y - 1 = m(x – 1)

$\Rightarrow mx-y+(1-m)=0$ ---(1)

Give distance from (-2, 3) to (1) = 3

$\Rightarrow \frac{|-2m-3+1-m|}{\sqrt{m^2+1}} = 3$

$\Rightarrow (3m+2)^2 = 9(m^2+1)$

$\Rightarrow 9m^2+4+12m = 9m^2+9$

$\Rightarrow 12m = 5 \Rightarrow m = \frac{5}{12}$

Co-efficient of $m^2 = 0 \Rightarrow m = \infty$

Case i) $m = \infty$

line is a vertical line

Equation of the vertical line passing through A(1, 1) is x = 1

Case ii) $m = \frac{5}{12}$, point (1,1)

Equation of the line is $y-1 = \frac{5}{12}(x-1) = 0$

$\Rightarrow \quad 5x-12y+7=0$

10. If p and q are lengths of the perpendiculars from the origin to the straight lines $x\sec\alpha + y\operatorname{cosec}\alpha = a$ and $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$, prove that $4p^2+q^2=a^2$.

Sol: Equation of AB is $x\sec\alpha + y\operatorname{cosec}\alpha = a$

$\frac{x}{\cos\alpha} + \frac{y}{\sin\alpha} = a$

$\Rightarrow x\sin\alpha + y\cos\alpha = a\sin\alpha\cos\alpha$

$\Rightarrow x\sin\alpha + y\cos\alpha - a\sin\alpha\cos\alpha = 0$

p = length of the perpendicular from O to AB = $\frac{|0+0-a\sin\alpha\cos\alpha|}{\sqrt{\sin^2\alpha+\cos^2\alpha}}$

$= a\sin\alpha.\cos\alpha = a.\frac{\sin 2\alpha}{2}$

$\Rightarrow \quad 2p = a\sin 2\alpha \quad$ ---(1)

Equation of CD is $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$

$\Rightarrow x\cos\alpha - y\sin\alpha - a\cos 2\alpha = 0$

q = Length of the perpendicular from O to CD $\frac{|0+0-a\cos 2\alpha|}{\sqrt{\cos^2\alpha+\sin^2\alpha}} = a\cos 2\alpha$ ---(2)

Squaring and adding (1) and (2)

$4p^2+q^2 = a^2\sin^2 2\alpha + a^2\cos^2 2\alpha$

$= a^2\left(\sin^2 2\alpha + \cos^2 2\alpha\right) = a^2.1 = a^2$

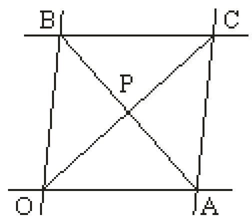
11. Two adjacent sides of a parallelogram are given by $4x+5y=0$ and $7x+2y=0$ and one diagonal is $11x+7y=9$. Find the equations of the remaining sides and the other diagonal.

Sol: Let $4x+5y=0$ ---(1) and

$7x+2y=0$ ---(2) respectively

denote the side $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ of the parallelogram OABC.

Equation of the diagonal $\overleftrightarrow{AB}$ is $11x+7y-9=0$ ---(3)



Solving (1) and (2) vertex O = (0, 0)

Solving (1) and (3), A $=\left(\frac{5}{3}, -\frac{4}{3}\right)$

Solving (2) and (3), $B=\left(-\frac{2}{3}, \frac{7}{3}\right)$

Midpoint of AB is $P\left(\frac{1}{2}, \frac{1}{2}\right)$. Slope of OP is 1

Equation to OC is y = (1) x $\Rightarrow$ x – y = 0

x = y.

Equation of AC is $4\left(x-\frac{5}{3}\right)+3\left(y+\frac{4}{3}\right)=0 \Rightarrow 4x+5y=9$

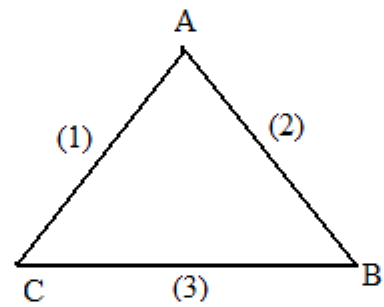
Equation of BC is $7\left(x+\frac{2}{3}\right)+2\left(y-\frac{7}{3}\right)=0 \Rightarrow 7x+2y=9$

12. Find the in centre of the triangle whose sides are given below.

i) $x+1=0,\ 3x-4y=5$ and $5x+12y=27$

ii) $x+y-7=0,\ x-y+1=0$ and $x-3y+5=0$

Sol: i) Sides are



$$x + 1 = 0 \quad \text{---(1)}$$
$$3x - 4y - 5 = 0 \quad \text{---(2)}$$
$$5x + 12y - 27 = 0 \quad \text{---(3)}$$

The point of intersection of (1), (2) is $A = (-1,,-2)$

The point of intersection of (2), (3), $B = (3,1)$

The point of intersection of (3), (1) is $C = \left(-1, \frac{8}{3}\right)$

$$a = BC = \sqrt{(3+1)^2 + \left(1+\frac{8}{3}\right)^2} = \sqrt{16 + \frac{25}{9}} = \sqrt{\frac{169}{9}} = \frac{13}{3}$$

$$b = CA = \sqrt{(-1+1)^2 + \left(-2-\frac{8}{3}\right)^2} = \sqrt{0 + \left(-\frac{14}{3}\right)^2} = \sqrt{\left(\frac{14}{3}\right)^2} = \frac{14}{3}$$

$$c = AB = \sqrt{(-1-3)^2 + (-2-1)^2} = \sqrt{16+9} = 5$$

Incentre = I =

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right) = \left(\frac{\frac{13}{3}(-1) + \frac{14}{3}(3) + 5(-1)}{\frac{13}{3} + \frac{14}{3} + 5}, \frac{\frac{13}{3}(-2) + \frac{14}{3}(1) + 5\left(\frac{8}{3}\right)}{\frac{13}{3} + \frac{14}{3} + 5}\right)$$

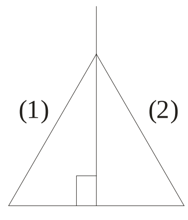
$$= \left(\frac{14}{42}, \frac{28}{42}\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$$

$\therefore$ Incentre = $\left(\frac{1}{3}, \frac{2}{3}\right)$

ii) Ans: $\left(3, 1+\sqrt{5}\right)$

13. A Δ^{le} is formed by the lines $ax + by + c = 0$, $lx + my + n = 0$ **and** $px + qy + r = 0$**. Given that the straight line** $\frac{ax + by + c}{ap + bq} = \frac{lx + my + n}{lp + mq}$ **passes through the orthocentre of the Δ^{le}.**

Sol:



Sides of the triangle are

$ax + by + c = 0$ ---(1)

$lx + my + n = 0$ ---(2)

$px + qy + r = 0$ ---(3)

Equation of the line passing through intersecting points of (1), (2) is

$ax + by + c + k(lx + my + n) = 0$ ---(4)

$(a + kl)x + (b + km)y + (c + nk) = 0$

If (4) is the altitude of the triangle then it is $\perp^{lr}$ to (3),

$p(a + kl) + q(b + km) = 0 \Rightarrow k = -\dfrac{ap + bq}{lp + mq}$

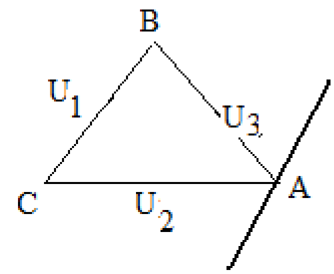
From (4)

$(ax + by + c) - \left(\dfrac{ap + bq}{lp + mq}\right)(lx + my + n) = 0$

$\therefore \dfrac{ax + by + c}{ap + bq} = \dfrac{lx + my + n}{lp + mq}$

is the required straight line equation which is passing through orthocenter. (it is altitude)

14. The Cartesian equations of the sides BC, CA, AB of a Δ^{le} are respectively $u_1 = a_1x + b_1y + c_1 = 0,\ u_2 = a_2x + b_2y + c_2 = 0.$ and $u_3 = a_3x + b_3y + c_3 = 0.$ Show that the equation of the straight line through A Parallel to the side $\overline{BC}$ is $\dfrac{u_3}{a_3b_1 - a_1b_3} = \dfrac{u_2}{a_2b_1 - a_1b_2}$.



Sol: A is the point of intersecting of the lines $u_2 = 0$ and $u_3 = 0$

$\therefore$ Equation to a line passing through A is

$u_2 + \lambda u_3 = 0 \Rightarrow (a_2x + b_2y + c_2) + \lambda(a_3x + b_3y + c_3)$ ---(1)

$\Rightarrow (a_2 + \lambda a_3)x + (b_2 + \lambda b_3)y + (c_2 + \lambda c_3) = 0$

If this is parallel to $a_1x + b_1y + c_1 = 0$,

$$\Rightarrow \frac{(a_2 + \lambda a_3)}{a_1} = \frac{(b_2 + \lambda b_3)}{b_1}$$

$\Rightarrow (a_2 + \lambda a_3)b_1 = (b_1 + \lambda b_3)a_1$

$\Rightarrow a_2b_1 + \lambda a_3b_1 = a_1b_2 + \lambda a_1b_3$

$\Rightarrow \lambda(a_3b_1 - a_1b_3) = -(a_2b_1 - a_1b_2)$

$$\Rightarrow \lambda = \frac{(a_2b_1 - a_1b_2)}{a_3b_1 - a_1b_3}$$

Substituting this value of λ in (1), the required equation is

$$(a_2x + b_2y + c_2) - \frac{(a_2b_1 - a_1b_2)}{(a_3b_1 - a_1b_3)}(a_3x + b_3y + c_3) = 0$$

$\Rightarrow (a_3b_1 - a_1b_3)(a_2x + b_2y + c_2) - (a_2b_1 - a_1b_2)(a_3x + b_3y + c_3) = 0$

$\Rightarrow (a_3b_1 - a_1b_3)u_2 - (a_2b_1 - a_1b_2)u_3 = 0$

$\Rightarrow (a_3b_1 - a_1b_3)u_2 = (a_2b_1 - a_1b_2)u_3$

$$\Rightarrow \frac{u_3}{(a_3b_1 - a_1b_3)} = \frac{u_2}{(a_2b_1 - a_1b_2)}.$$

PROBLEMS FOR PRACTICE

1. **Find the equation of the straight line passing through the point (2, 3) and making non-zero intercepts on the axes of co-ordinates whose sum is zero.**
2. **Find the equation of the straight line passing through the points $\left(at_1^2,\ 2at_1\right)$ and $\left(at_2^2,\ 2at_2\right)$.**

3. **Find the equation of the straight line passing through the point** $A(-1,3)$ **and**

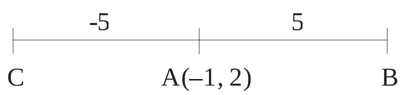
i) parallel

ii) perpendicular to the straight line passing through $B(2,-5)$ **and** $C(4,6)$**.**

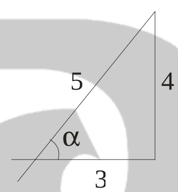
4. **Prove that the points** $(1,11)$**,** $(2,15)$ **and** $(-3,-5)$ **are collinear and find the equation of the line containing them.**

5. **A straight line passing through** $A(1,-2)$ **makes an angle** $\tan^{-1}\frac{4}{3}$ **with the positive direction of the X-axis in the anti clock-wise access. Find the points on the straight line whose distance from A is** ± 5**units.**

Sol:



Given $\alpha = \tan^{-1}\frac{4}{3} \Rightarrow \tan\alpha = \frac{4}{3}$



$\cos\alpha = \frac{3}{5},\ \sin\alpha = \frac{4}{5}$

$(x_1, y_1) = (1,-2) = x_1 = 1,\ y_1 = -2$

Case i): r = 5

$x = x_1 + r\cos\alpha = 1 + 5.\frac{4}{3} = 1 + 4 = 5$

$y = y_1 + r\sin\alpha = -2 + 5.\frac{3}{5} = -2 + 3 = 1$

Co-ordinate of B are (5, 1)

Case ii):

$x = x_1 + r\cos\alpha = 1 - 5.\frac{4}{5} = 1 - 4 = -3$

$y = y_1 + r\sin\alpha = -2 - 5.\frac{3}{4} = -2 - 3 = -5$

Co-ordinate of C are $(-3,-5)$

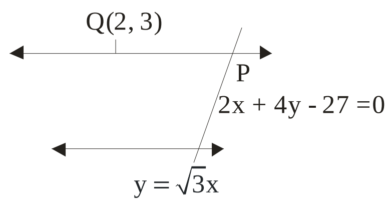
6. A straight line parallel to the line $y=\sqrt{3}x$ passes through $Q(2,3)$ and cuts the line $2x+4y-27=0$ at P. Find the length of PQ.

Sol: PQ is parallel to the straight line $y=\sqrt{3}x$

$\tan\alpha=\sqrt{3}=\tan 60^{o}$

$\alpha=60^{o}$

$Q(2,3)$ is a given point



Co-ordinates of any point P are

$(x_1+r\cos\alpha y_1+r\sin\alpha)=(2+r\cos 60^{o},\ 3+r\sin 60^{o})$

$=P\left(2+\frac{r}{2},3+\frac{\sqrt{3}}{2}r\right)$

P is a point on the line $2x+4y-27=0$

$\Rightarrow 2\left(2+\frac{r}{2}\right)+4\left(3+\frac{\sqrt{3}}{2}r\right)-27=0$

$\Rightarrow 4+r+12+2\sqrt{3}r-27=0$

$\Rightarrow r\left(2\sqrt{3}+1\right)=27-16=11$

$\Rightarrow r=\frac{11}{2\sqrt{3}+1}.\frac{2\sqrt{3}-1}{2\sqrt{3}-1}=\frac{11\left(2\sqrt{3}-1\right)}{11}$

7. Transform the equation $3x+4y+12=0$ into

i) slope – intercept form

ii) intercept form and

iii) normal form

8. If the area of the triangle formed by the straight line $x=0,\ y=0$ and $3x+4y=a\,(a>0)$, find the value of a.

9. Find the value of k, if the lines $2x-3y+k=0$, $3x-4y-13=0$ and $8x-11y-33=0$ are concurrent.

10. If the straight lines $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ are concurrent, then prove that $a^3+b^3+c^3=3abc$.

Sol: The equations of the given lines are

$ax+by+c=0$ ---(1)

$bx + cy + a = 0$ ---(2)

$cx + ay + b = 0$ ---(3)

Solving (1) and (2) points of intersection is got by

$$\begin{array}{cccccccc} & x & & y & & 1 & & \\ b & & c & & a & & b \\ c & & a & & b & & c \end{array}$$

$$\frac{x}{ab - c^2} = \frac{y}{bc - a^2} = \frac{1}{ca - b^2}$$

Point of intersection is $\left(\frac{ab - c^2}{ca - b^2}, \frac{bc - a^2}{ca - b^2} \right)$

$$c\left(\frac{ab - c^2}{ca - b^2} \right) + a\left(\frac{bc - a^2}{ca - b^2} \right) + b = 0$$

$$c\left(ab - c^2\right) + a\left(bc - a^2\right) + b\left(ca - b^2\right) = 0$$

$$abc - c^3 + abc - a^3 + abc - b^3 = 0$$

$$\therefore a^3 + b^3 + c^3 = 3abc.$$

11. A variable straight line drawn through the point of intersection of the straight lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ meets the co-ordinate axes at A and B. Show that the locus the mid point of $\overline{AB}$ is $2(a+b)xy = ab(x+y)$.

Sol: Equations of the given lines are $\frac{x}{a} + \frac{y}{b} = 1$

and $\frac{x}{b} + \frac{y}{a} = 1$

Solving the point of intersection $P\left(\frac{ab}{a+b}, \frac{ab}{a+b} \right)$

$Q(x_0, y_0)$ is any point on the locus

$\Leftrightarrow$ The line with x-intercept $2x_0$, y-intercept $2y_0$, passes through P

$\Leftrightarrow$ P lies on the straight line $\frac{x}{2x_0} + \frac{y}{2y_0} = 1$

i.e., $\frac{ab}{a+b}\left(\frac{1}{2x_0} + \frac{1}{2y_0} \right) = 1$

$$\Rightarrow \frac{ab}{a+b} \cdot \frac{x_0 + y_0}{2x_0 y_0} = 0$$

$ab(x_0+y_0)=2(a+b)x_0y_0$

$Q(x_0,y_0)$ lies on the curve $2(a+b)xy=ab(x+y)$

Locks the midpoint of AB is $2(a+b)xy=ab(x+y)$.

12. **If a, b, c are in arithmetic progression, then show that the equation** $ax+by+c=0$ **represents a family of concurrent lines and find the point of concurrency.**

13. **Find the value of k, if the angle between the straight lines** $4x-y+7$ **and** $kx-5y+9=0$ is 45^o.

14. **Find the equation of the straight line passing through** (x_0,y_0) **and**
   i) parallel
   ii) perpendicular to the straight line
   $ax+by+c=0$.

15. **Find the equation of the straight line perpendicular to the line** $5x-2y=7$ **and passing through the point of intersection of the lines** $2x+3y=1$ **and** $3x+4y=6$.

16. **If** $2x-3y-5=0$ **is the perpendicular bisectors of the line segment joining (3 -4) and** (α,β) **find** $\alpha+\beta$.

17. **If the four straight lines** $ax+by+p=0$, $ax+by+q=0$, $cx+dy+r=0$ **and** $cx+dy+s=0$ **form a parallelogram, show that the area of the parallelogram bc formed is.**

$$\left|\frac{(p-q)(r-s)}{bc-ad}\right|$$

18. **Find the orthocentre of the triangle whose vertices are** $(-5,-7)(13,2)$ **and** $(-5,6)$.

19. **If the equations of the sides of a triangle are** $7x+y-10=0$, $x-2y+5=0$ **and** $x+y+2=0$, **find the orthocentre of the triangle.**

20. **Find the circumcentre of the triangle whose vertices are** $(1,3),(-3,5)$ and $(5,-1)$.

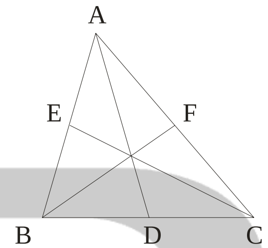
21. **Find the circumcentre of t\he triangle whose sides are** $3x-y-5=0$, $x+2y-4=0$ **and** $5x+3y+1=0$.

Sol: Let the given equations $3x-y-5=0$, $x+2y-4=0$ and $5x+3y+1=0$ represents the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ respectively of ΔABC. Solving the above equations two by two, we obtain the vertices $A(-2,3), B(1,-2)$ and $(2,1)$ of the given triangle.

The midpoints of the sides $\overline{BC}$ and $\overline{CA}$ are respectively $D=\left(\frac{3}{2},\frac{-1}{2}\right)$ and $E=(0,2)$.

22. Let 'O' be any point in the plane of ΔABC such that O does not lie on any side of the triangle. If the line joining O to the vertices A, B, C meet the opposite sides in D, E, F respectively, then prove that $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=1$ (Ceva's Theorem)

Sol: Without loss of generality take the point P as origin O. Let $A(x_1,y_1)B(x_2,y_2)C(x_3,y_3)$ be the vertices. Slope of AP is $\frac{y_1-0}{x_1-0}=\frac{y_1}{x_1}$



Equation of AP is $y-0=\frac{y_1}{x_1}(x-0)$

$\Rightarrow yx_1=xy_1 \Rightarrow xy_1-yx_1=0$

$\therefore \frac{BD}{DC}=\frac{-(x_2y_1-x_1y_2)}{x_3y_1-x_1y_3}=\frac{x_1y_2-x_2y_1}{x_3y_1-x_1y_3}$

Slope of $\overleftrightarrow{BP}$ is $\frac{y_2-0}{x_2-0}=\frac{y_2}{x_2}$

Equation of $\overleftrightarrow{BP}$ is $y-0=\frac{y_2}{x_2}(x-0)$

$\Rightarrow x_2y=y_2x \Rightarrow xy_2-x_2y=0$

$\therefore \frac{CE}{EA}=\frac{-(x_3y_2-x_2y_3)}{x_1y_2-x_2y_1}=\frac{x_2y_3-x_3y_2}{x_1y_2-x_2y_1}$

Slope of $\overleftrightarrow{CP}=\frac{y_3-0}{x_3-0}=\frac{y_3}{x_3}$

Equation of $\overleftrightarrow{CP}$ is $y-0=\frac{y_3}{x_3}(x-0)$

$\Rightarrow x_3y=xy_3 \Rightarrow xy_3-x_3y=0$

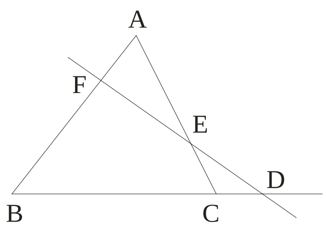
$$\therefore \frac{AF}{FB} = \frac{(x_1y_3 - x_3y_1)}{x_2y_3 - x_3y_2} = \frac{x_3y_1 - x_1y_3}{x_2y_3 - x_3y_2}$$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}$$

$$\frac{x_1y_2 - x_2y_1}{x_3y_1 - x_1y_3} \cdot \frac{x_2y_3 - x_3y_2}{x_1y_2 - x_2y_1} \cdot \frac{x_3y_1 - x_1y_3}{x_2y_3 - x_3y_2} = 1$$

23. If a transversal cuts the side $\overleftrightarrow{BC}, \overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ of ΔABC in D, E and F respectively. Then prove that $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$. (Meneclau's Theorem)

Sol:



Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$

Let the transversal be $ax + by + c = 0$

$\frac{BD}{DC} =$ The ratio in which $ax + by + c = 0$ divides.

$$\overline{BC} = \frac{-(ax_2 + by_2 + c)}{ax_3 + by_3 + c}$$

$\frac{CE}{EA} =$ The ratio in which $ax + by + c = 0$ divides.

$$\overline{CA} = \frac{-(ax_3 + by_3 + c)}{ax_1 + by_1 + c}$$

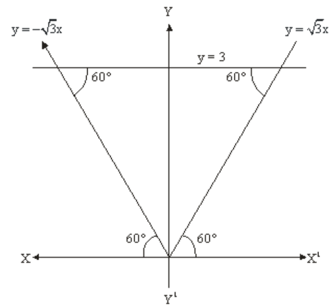
$\frac{AF}{FB} =$ The ratio in which $ax + by + c = 0$ divides.

$$\overline{AB} = \frac{-(ax_1 + by_1 + c)}{ax_2 + by_2 + c}$$

$$\therefore \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$$

24. Find the incentre of the triangle formed by straight lines $y=\sqrt{3}x$, $y=-\sqrt{3}x$ and $y=3$.

Sol:



The straight lines $y=\sqrt{3}x$ and $y=-\sqrt{3}x$ respectively make angles 60° and 120° with the positive directions of X-axis.

Since y = 3 is a horizontal line, the triangle formed by the three given lines is equilateral.

So in-centre is same and centriod.

Vertices of the triangle and $(0,0)$, $A(\sqrt{3},3)$ and $D(-\sqrt{3},3)$

$\therefore$ Incentre is $\left(\frac{o+\sqrt{3}-\sqrt{3}}{3}, \frac{0+3+3}{3}\right)$

= (0,2).