TRIGONOMETRIC EXPANSIONS

Very Short Answer Questions

1. **Show that** $\frac{\sin 4\theta}{\sin\theta} = 8\cos^3\theta - 4\cos\theta$

Solution: -

We know that ${}_1\cos^{n-1}\theta\sin\theta - nc_3\cos^{n-3}\theta\sin^3\theta + nc_5\cos^{n-5}\theta\sin^5\theta$.........

$\sin 4\theta = 3c_1\cos^3\theta\sin\theta - 4c_3\cos\theta\sin^3\theta$

$= 4c_1\sin\theta\left\{4\cos^3\theta - 4\cos\theta\sin^2\theta\right\}$

$\frac{\sin 4\theta}{\sin n\theta} = \left\{4\cos^3\theta - \cos\theta(1-\cos^2\theta)\right\}$

$\frac{\sin 4\theta}{\sin n\theta} = 8\cos^3\theta - 4\cos\theta$

2. **Show that** $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Solution : -

We know that $\cos n\theta = \cos^n\theta - nc_2\cos^{n-2}\theta\sin^2\theta + nc_4\cos^{n-2}\theta\sin^4\theta$.....

$\cos 7\theta = \cos^7\theta = 7c_2\cos^5\theta\sin^2\theta + 7c_4\cos^3\theta.\sin^4\theta - 7c_6\cos\theta\sin^6\theta$

$= \cos^7\theta - 21\left\{\cos^5\theta\left(1-\cos^2\theta\right) + 35\cos^3\theta\left(1-\cos^2\theta\right)^2 - 7\cos\theta\left\{1-\cos^2\theta\right\}^3\right\}$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta\left\{1+\cos^4\theta - 2\cos^2\theta\right\}$

$-7\cos\theta\left\{1-\cos^6\theta - 3\cos^2\theta + 3\cos^4\theta\right\}$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta + 35\cos^7\theta - 70\cos^5\theta$

$\_7\cos\theta + 7\cos^2\theta + 21\cos^3\theta - 21\cos^5\theta$

$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

3. **Show that** $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

Solution : -

We know that $\sin n\theta = nc_1\cos^{n-1}\theta\sin\theta - nc_3\cos^{n-3}\theta\sin^3\theta + ...$

$\sin 7\theta = 8c_1\cos^6\theta\sin\theta - 7c_3\cos^4\theta\sin^3\theta + 7c_5\cos^2\theta\sin^5\theta$

$= 7\sin\theta\left(1-\sin^2\theta\right)^3 - 35\sin^2\theta\left(1-\sin^2\theta\right)^2 + 21\sin^5\left(1-\sin^2\theta\right)$

$= 7\sin\theta\left\{1-\sin^6\theta - 3\sin^2\theta + 3\sin^4\theta\right\} - 35\sin^2\theta\left\{1-\sin^4\theta - 2\sin^2\theta\right\}$

$+ 21\sin^5\theta - 21\sin^7\theta$

$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

4. **Show that** $\tan 7\theta = \dfrac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta + \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$

Solution : -

We know that

$$\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + nc_5 \tan^5\theta - nc_7 \tan^7\theta}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta - nc_6 \tan^6\theta}$$

$$\tan 7\theta = \frac{7\tan\theta - 7c_3 \tan^3\theta + 7c_5 \tan^5\theta - 7c_7 \tan^7\theta}{1 - 7c_2 \tan^2\theta + 7c_4 \tan^4\theta - 7c_6 \tan^6\theta}$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta + \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

5. **Show that** $\tan 8\theta = \dfrac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}$

Solution : -

$$\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + nc_5 \tan^5\theta - nc_7 \tan^7\theta + \ldots..}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta - nc_6 \tan^6\theta + nc_8 \tan^8\theta + \ldots..}$$

$$\tan 8\theta = \frac{8c_1 \tan\theta - 8c_3 \tan^3\theta + 8c_5 \tan^5\theta - 8c_7 \tan^7\theta}{1 - 8c_2 \tan^2\theta + 8c_4 \tan^4\theta - 28\tan^6\theta + \tan^8\theta}$$

$$= \frac{8\tan\theta - 56\tan^3\theta + 56\tan^5\theta - 8\tan^7\theta}{1 - 28\tan^2\theta + 70\tan^4\theta - 28\tan^6\theta + \tan^8\theta}$$

SHORT ANSWER QUESTIONS :

1. **Show that** $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

Solution : -

Let $x = \cos\theta + i\sin\theta \quad \dfrac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta \Rightarrow \left(x + \frac{1}{x}\right)^6 = (\cos\theta)^6$$

$$x^6 + 6c_1 x^5\left(\frac{1}{x}\right) + 6c_2 x^4 \frac{1}{x^2} + 6c_3 x^3 \frac{1}{x^3} + 6c_4 x^2 \frac{1}{x^4} + 6c_5 x + \frac{1}{x^5} + 6c_6 \frac{1}{x^6} = 2^6 \cos^6\theta$$

$$\left(x^6 + \frac{1}{x^6}\right) + 6\left\{x^4 + \frac{1}{x^4}\right\} + 15\left(x^2 + \frac{1}{x^2}\right) + 20 = 64\cos^6\theta$$

$$2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 2\theta = 64\cos^6\theta$$

$$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 32\cos^6\theta$$

2. **Show that** $32\sin^6\theta = 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta$

Solution : -

Let $x = \cos\theta + i\sin\theta \quad \frac{1}{x} = \cos\theta - i\sin\theta$

$$x - \frac{1}{x} = 2i\sin\theta \Rightarrow \left(x - \frac{1}{x}\right)^6 (2i\sin\theta)^6$$

$$x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6} = 2^6 i^6 \sin^6\theta$$

$$\left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20 = -2\left\{2^5 \sin^6\theta\right\}$$

$$2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20 = -2\left\{32\sin^6\theta\right\}$$

$$10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta = 32\sin^6\theta$$

3. **Show that** $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

Solution : -

Let $x = \cos\theta = i\sin\theta \quad \frac{1}{x} = \cos\theta - \sin\theta$

$$x - \frac{1}{x} = 2i\sin\theta \Rightarrow \left(x - \frac{1}{x}\right)^5 = (2i\sin\theta)^5$$

$$x^5 - 5c_1 x^3 + 5c_2 x - \frac{5c_3}{x} + \frac{5c_4}{x^3} - \frac{5c_5}{x^5} = 2^5 i^5 \sin^5\theta$$

$$\left(x^5 - \frac{1}{x^5}\right) - 5\left\{x^3 - \frac{1}{x^3}\right\} + 10\left\{x - \frac{1}{x}\right\} = 32\, i\sin^5\theta$$

$$2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta) = 32i\sin^5\theta$$

$$\cancel{2i}\left\{\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right\} = \cancel{2i}\left\{16\sin^5\theta\right\}$$

$$\therefore 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

4. **Show that** $64\sin^5\theta\cos^2\theta = \sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta$

Solution : -

Let $x = \cos\theta + i\sin\theta \quad \frac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta \qquad x - \frac{1}{x} = 2i\sin\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\left(x + \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right)^5 = (2\cos\theta)^2 (2i\sin\theta)^5$$

$$\left\{\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right\}^2\left(x-\frac{1}{x}\right)^3=2^7\,i^5\sin^5\theta\cos^2\theta$$

$$\left(x^2-\frac{1}{x^2}\right)^2\left\{x^3-\frac{3x}{x}+\frac{3}{x}-\frac{3}{x^3}\right\}=2^7\,i\sin^5\theta\cos^2\theta$$

$$\left(x^4+\frac{1}{x^4}-2\right)\left\{x^3-3x+\frac{3}{x}-\frac{1}{x^3}\right\}=2^7\,i\sin^5\theta\cos^2\theta$$

$$x^7-3x^5+3x^3-\cancel{\frac{1}{2}}+\cancel{\frac{1}{x}}-\frac{3}{x^3}+\frac{3}{x^5}-\frac{1}{x^7}-2x^3+6x-\frac{6}{x}+\frac{2}{x^3}$$

$$2^7 i\sin^5\theta\cos^2\theta$$

$$\left(x^7-\frac{1}{x^7}\right)-3\left(x^5-\frac{1}{x^5}\right)+\left(x^3-\frac{1}{x^3}\right)+6\left(x-\frac{1}{x}\right)=2^7 i\sin^5\cos^2\theta$$

$$2i\sin7\theta-3(2i\sin5\theta)+2i\sin3\theta+6(2i\sin\theta)=2i\{64\sin^5\theta\cos^2\theta\}$$

$$\cancel{2i}\{\sin7\theta-3\sin5\theta+\sin3\theta+6\sin\theta\}=\cancel{2i}\{64\sin^5\theta\cos^2\theta\}$$

5. **Show that** $32\cos^2\theta\sin^4\theta=\cos6\theta-2\cos4\theta-\cos2\theta+2$

Solution : -

Let $x=\cos\theta+i\sin\theta$ $\quad\frac{1}{x}=\cos\theta-i\sin\theta$

$$x+\frac{1}{x}=2\cos\theta\qquad x-\frac{1}{x}=2i\sin\theta$$

$$x^n+\frac{1}{x^n}=2\cos n\theta\qquad x^n-\frac{1}{x^n}=2i\sin n\theta$$

$$\left(x+\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^4=(2\cos\theta)^2\,(2i\sin\theta)^4$$

$$\left\{\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right\}^2\left(x-\frac{1}{x}\right)^2=4\cos^2\theta\,16i^4\sin^4\theta$$

$$\left(x^2-\frac{1}{x^2}\right)^2\left(x-\frac{1}{x}\right)^2=64\cos^2\theta\sin^4\theta$$

$$\left(x^4+\frac{1}{x^4}-2\right)\left(x^2+\frac{1}{x^2}-2\right)=64\cos^2\theta\sin^4\theta$$

$$x^6+x^2-2x^4+\frac{1}{x^2}+\frac{1}{x^6}-\frac{2}{x^4}-2x^2-\frac{2}{x^2}+4=64\cos^2\theta\sin^4\theta$$

$$\left(x^6+\frac{1}{x^6}\right)-2\left(x^4+\frac{1}{x^4}\right)-\left(x^2+\frac{1}{x^2}\right)+4=64\cos^2\theta\sin^4\theta$$

$$2\cos6\theta-2(2\cos4\theta)-2\cos2\theta+4=64\cos^2\theta\sin^4\theta$$

$$\cos6\theta-2\cos4\theta-\cos2\theta+2=32\cos^2\theta\sin^4\theta$$

KEY CONCEPTS

1. $\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta \ldots\ldots$

2. $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta \ldots\ldots$

3. $\tan\theta = \dfrac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - {}^nC_7 \tan^7\theta + \ldots\ldots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta + \ldots\ldots\ldots}$