

## DE MOIVRE'S THEOREM

### VERY SHORT ANSWER QUESTIONS

- 1.** If n is an integer then show that  $(1+i)^{2n} + (1-i)^{2n} = 2^{n+1} \cos \frac{n\pi}{2}$

**Solution :** -

$$\text{Let } 1+i = r\{\cos \theta + i \sin \theta\}$$

$$(r \cos \theta)^2 + (r \sin \theta)^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\cos \theta = \frac{1}{\sqrt{2}} \quad \sin \theta = \frac{1}{\sqrt{2}} \quad P.V \text{ of } \theta = \pi/4$$

$$\therefore 1+i = \sqrt{2} \left\{ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right\} \quad \text{similarly } (1-i) = \sqrt{2} \left\{ \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right\}$$

$$\begin{aligned} (1+i)^{2n} + (1-i)^{2n} &= (\sqrt{2})^{2n} \left\{ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right\}^{2n} + (\sqrt{2})^{2n} \left\{ \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right\}^{2n} \\ &= 2^n \left\{ \cos \frac{2n\pi}{4} + i \sin \frac{2n\pi}{4} \cos \frac{2n\pi}{4} - i \sin \frac{2n\pi}{4} \right\} \\ &= 2^{n+1} \cos \frac{n\pi}{2} \end{aligned}$$

- 2.** Find the values of the following

$$\begin{array}{lll} \text{(i)} (1+i\sqrt{3})^3 & \text{(ii)} (1-i)^8 & \text{(iii)} (1+i)^{16} \quad \text{(iv)} \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right)^5 - \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right)^3 \end{array}$$

**Solution :** -

$$1+i\sqrt{3} = 2 \left\{ \cos \frac{\pi}{3} + i \sin \pi/3 \right\} \quad \text{(by mod-amplitude form)}$$

$$(1+i\sqrt{3})^3 = 8 \left\{ \cos \frac{\pi}{3} + i \sin \pi/3 \right\}^3$$

$$= 8 \{ \cos \pi + i \sin \pi \} \quad \left\{ \because (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \right\}$$

$$= 8 \{-1 + 0\} = -8$$

$$\text{Solution (ii)} \quad (1-i)^8 = \left( \sqrt{2} \left\{ \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right\} \right)^8 = 2^4 \{ \cos 2\pi - i \sin 2\pi \}$$

$$\begin{aligned} \text{Solution (iii)} \quad (1+i)^{16} &= \left\{ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right\}^{16} = 2 \{ \cos 2\pi + i \sin 2\pi \} \\ &= 256 \end{aligned}$$

**Solution (iv)**  $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

$$\left\{\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right\}^5 - \left\{\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right\}^5$$

$$\cancel{\cos \frac{5\pi}{6}} + i \sin \frac{5\pi}{6} - \cancel{\cos \frac{5\pi}{6}} + 1 \sin \pi/6$$

$$2i \sin \frac{5\pi}{6} = (\not{2}i) \frac{1}{\not{2}} = i$$

### SHORT ANSWER QUESTIONS

1.  $\alpha, \beta$  are the roots of the equation  $x^2 - 2x + 4 = 0$  then for any  $n \in N$  show that

$$\alpha^n + \beta^n = 2^{n+1} \cos \frac{n\pi}{3}$$

**Solution:** -

$$x^2 - 2x + 4 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2}$$

$$\alpha = 2 \left\{ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right\}, \beta = 2 \left\{ \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right\}$$

$$\begin{aligned} \alpha^n + \beta^n &= \left\{ 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right\}^n + \left\{ 2 \left( \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right) \right\}^n \\ &= 2^n \left\{ \cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} + \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right\} \end{aligned}$$

$$= 2^n \left\{ 2 \cos \frac{n\pi}{3} \right\} = 2^{n+1} \cos \frac{n\pi}{3}$$

2.  $\cos \alpha + \cos \beta + \cos \vartheta = 0 = \sin \alpha + \sin \beta + \sin \vartheta = 0$  then show that

- (i)  $\cos 3\alpha + \cos 3\beta + \cos 3\vartheta = 3 \cos(\alpha + \beta + \vartheta)$
- (ii)  $\sin 3\alpha + \sin 3\beta + \sin 3\vartheta = 3 \sin(\alpha + \beta + \vartheta)$
- (iii)  $\cos(2\alpha - \beta - \vartheta) + \cos\{2\beta - \vartheta - \alpha\} + \sin(2\vartheta - \alpha - \beta) = 3$
- (iv)  $\sin(2\alpha - \beta - \vartheta) + \sin(2\beta - \vartheta - \alpha) + \sin(2\vartheta - \alpha - \beta) = 0$
- (v)  $\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$
- (vi)  $\sin 2\alpha + \sin 2\beta + \sin 2\vartheta = 0$
- (vii)  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \vartheta = 0$
- (viii)  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \vartheta = 3/2$
- (ix)  $\cos(\alpha + \beta) + \cos(\beta + \vartheta) + \cos(\vartheta + \alpha) = 0$
- (x)  $\sin(\alpha + \beta) + \sin(\beta + \vartheta) + \sin(\vartheta + \alpha) = 0$

**Solution :-**

$$\text{Let } x = \cos \alpha + i \sin \alpha \quad y = \cos \beta + i \sin \beta : z = \cos \vartheta + i \sin \vartheta$$

$$x + y + z = (\cos \alpha + \cos \beta + \cos \vartheta) + i(\sin \alpha + \sin \beta + \sin \vartheta)$$

$$x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 = 3xyz$$

**Proof of (i) & (ii)**

$$(\cos \alpha + i \sin \alpha)^3 + (\cos \beta + i \sin \beta)^3 + (\cos \vartheta + i \sin \vartheta)^3 = 3 \operatorname{cis} \alpha \operatorname{cis} \beta \operatorname{cis} \vartheta$$

$$\operatorname{cis} 3\alpha + \operatorname{cis} 3\beta + \operatorname{cis} 3\vartheta = 3 \operatorname{cis}(\alpha + \beta + \vartheta)$$

$$(\cos 3\alpha + i \sin 3\alpha) + (\cos 3\beta + i \sin 3\beta) + (\cos 3\vartheta + i \sin 3\vartheta) = 3 \cos(\alpha + \beta + \vartheta) + 3i \sin(\alpha + \beta + \vartheta)$$

By comparing real and imaginary parts on both sides

$$\cos 3\alpha + \cos 3\beta + \cos 3\vartheta = 3 \cos(\alpha + \beta + \vartheta)$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\vartheta = 3 \sin(\alpha + \beta + \vartheta)$$

**Proof of (iii) & (iv)**

We know that

$$\frac{x^3 + y^3 + z^3}{xyz} = 3 \Rightarrow \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3$$

$$\frac{\operatorname{cis} 2\alpha}{\operatorname{cis} \beta \operatorname{cis} \vartheta} + \frac{\operatorname{cis} 2\beta}{\operatorname{cis} \vartheta \operatorname{cis} \alpha} + \frac{\operatorname{cis} 2\vartheta}{\operatorname{cis} \alpha \operatorname{cis} \beta} = 3$$

$$\operatorname{cis}(2\alpha - \beta - \vartheta) + \operatorname{cis}(2\beta - \vartheta - \alpha) + \operatorname{cis}(2\vartheta - \alpha - \beta) = 3$$

$$\{\cos(2\alpha - \beta - \vartheta) + i \sin(2\alpha - \beta - \vartheta)\} + \cos(2\beta - \vartheta - \alpha) + i \sin(2\beta - \vartheta - \alpha)$$

$$+ \cos(2\vartheta - \alpha - \beta) + i \sin(2\vartheta - \alpha - \beta) = 3$$

Comparing real and imaginary parts on both sides

$$\cos(2\alpha - \beta - \vartheta) + \cos(2\beta - \vartheta - \alpha) + \cos(2\vartheta - \alpha - \beta) = 3$$

$$\sin(2\alpha - \beta - \vartheta) + \sin(2\beta - \vartheta - \alpha) + \sin(2\vartheta - \alpha - \beta) = 0$$

**Proof of V & VI**

We know that  $x + y + z = 0$

$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{\cos \alpha + i \sin \alpha} + \frac{1}{\cos \beta + i \sin \beta} + \frac{1}{\cos \vartheta + i \sin \vartheta}$$

$$= \cos \alpha - i \sin \alpha + \cos \beta - i \sin \beta + \cos \vartheta - i \sin \vartheta$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

$$x + y + z = 0 \Rightarrow (x + y + z)^2 = 0 \Rightarrow x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 0$$

$$\begin{aligned}
 & x^2 + y^2 + z^2 + 2xyz \left\{ \frac{1}{z} + \frac{1}{x} + \frac{1}{y} \right\} = 0 \\
 & (cis\alpha)^2 + (cis\beta)^2 + (cis\vartheta)^2 + 2(cis\alpha cis\beta cis\vartheta)(0) \\
 & \left\{ \because \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \right\} \\
 & cis2\alpha + cis2\beta + cis2\vartheta = 0 \Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\vartheta) + i(\sin 2\alpha + \sin 2\beta + \sin 2\vartheta) = 0 \\
 & \text{By comparing real and imaginary parts on both sides} \\
 & \cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0 \\
 & \sin 2\alpha + \sin 2\beta + \sin 2\vartheta = 0
 \end{aligned}$$

#### Proof of (vii)

From the above problem we can prove

$$\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$$

$$2\cos^2 \alpha - 1 + 2\cos^2 \beta - 1 + 2\cos^2 \vartheta - 1 = 0$$

$$2\{\cos^2 \alpha + \cos^2 \beta + \cos^2 \vartheta\} = 3$$

$$\therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \vartheta = \frac{3}{2}$$

#### Proof viii

$$\cos 2\alpha + \cos 2\beta + \cos 2\vartheta = 0$$

$$1 - 2\sin^2 \alpha + 1 - 2\sin^2 \beta + 1 - 2\sin^2 \vartheta = 0$$

$$3 = 2\{\sin^2 \alpha + \sin^2 \beta + \sin^2 \vartheta\}$$

#### Proof of (ix) and (x)

In proving iv & v we proved

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

$$\therefore yz + zx + xy = 0$$

$$\therefore cis\alpha cis\beta + cis\beta cis\vartheta + cis\vartheta cis\alpha = 0$$

$$= cis(\alpha + \beta) + \cos(\beta + \vartheta) + cis(\vartheta + \alpha) = 0$$

$$\{\cos(\alpha + \beta) + i\sin(\alpha + \beta)\} + \{\cos(\beta + \vartheta)\} + \{\cos(\vartheta + \alpha) + i\sin(\vartheta + \alpha)\} = 0$$

By comparing real and imaginary parts on both sides

$$\cos(\alpha + \beta) + \cos(\beta + \vartheta) + \cos(\vartheta + \alpha) = 0$$

$$\sin(\alpha + \beta) + \sin(\beta + \vartheta) + \sin(\vartheta + \alpha) = 0$$

3. If n is an integer and  $z = cis\theta$  then show that  $\frac{z^{2n}-1}{z^{2n}+1} = i \tan n\theta$

#### Solution :-

$$\frac{z^{2n}-1}{z^{2n}+1} = \frac{(\cos \theta + i \sin \theta)^{2n} - 1}{(\cos \theta + i \sin \theta)^{2n} + 1}$$

$$\begin{aligned}
 &= \frac{\cos 2n\theta + i \sin 2n\theta - 1}{\cos 2n\theta + i \sin 2n\theta + 1} \\
 &= \frac{-(1 - \cos 2n\theta) + i \sin 2n\theta}{(1 + \cos 2n\theta) + i \sin 2n\theta} \\
 &= \frac{i^2 (2 \sin^2 n\theta) + 2i \sin n\theta \cos n\theta}{2 \cos^2 n\theta + 2i \sin n\theta \cos n\theta} \left\{ \because -1 = i^2 \right\} \\
 &= \frac{i \sin n\theta \overline{\{\cos n\theta + i \sin n\theta\}}}{\overline{\cos n\theta \{\cos n\theta + i \sin n\theta\}}} = i \tan n\theta
 \end{aligned}$$

## KEY CONCEPTS

1. If 'n' is an integer, then  $(\text{cis}\theta)^n = \text{cis}(n\theta)$  [De-Moivres theorem for integral index]
2. If 'n' is a rational number, then one of the values of  $(\text{cis}\theta)^n$  is  $\text{cis } n\theta$  [De Moivre's theorem for rational index]
3. If  $z_0 = r_0 \text{ cis } \theta_0 \neq 0$ , then the nth roots of  $z_0$  are  $a_k = r_0^{\frac{1}{n}} \text{ cis} \left( \frac{2k\pi + \theta_0}{n} \right)$   
 $k = 0, 1, 2, \dots, (n-1)$
4. The nth roots of unity are  $\text{cis} \frac{2k\pi}{n}$ ,  $k = 0, 1, 2, 3, \dots, (n-1)$
5. Cube roots of unity are  $1, \omega = \frac{-1+i\sqrt{3}}{2}, \omega^2 = \frac{-1-i\sqrt{3}}{2}$

### 12. (i) nth Root of a Complex Number and nth Roots of unity

#### Very Short Answer Questions

1. Find all values of  $(1 - i\sqrt{3})^{\frac{1}{3}}$ 

$$\begin{aligned}
 (1 - i\sqrt{3})^{\frac{1}{3}} &= \left\{ 2 \left( \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right) \right\}^{\frac{1}{3}} \\
 &= 2^{\frac{1}{3}} \left\{ \cos \left( \frac{-\pi}{3} \right) + i \sin \left( \frac{-\pi}{3} \right) \right\}^{\frac{1}{3}} \\
 &= 2^{\frac{1}{3}} \left\{ \cos \left( \frac{2k\pi - \pi}{3} \right) + i \sin \left( \frac{2k\pi - \pi}{3} \right) \right\} \quad k = 0, 1, 2
 \end{aligned}$$

$$= 3\sqrt{2} \operatorname{cis}(6k - 1)\frac{\pi}{9} \quad k = 0, 1, 2$$

**2. Find all values of  $(-i)^{\frac{1}{6}}$**

**Solution :-**

$$\begin{aligned} (-i)^{\frac{1}{6}} &= \left\{ \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \right\}^{\frac{1}{6}} \\ &= \operatorname{cis}\left(\frac{2k\pi - \pi/2}{6}\right) \quad k = 0, 1, 2, 3, 4, 5 \\ \therefore (-1)^{\frac{1}{6}} &= \operatorname{cis}(4k - 1)\frac{\pi}{12} \quad k = 0, 1, 2, 3 \end{aligned}$$

**3. Find all values of  $(1+i)^{\frac{2}{3}}$**

$$\begin{aligned} (1+i)^{\frac{2}{3}} &= \left[ \left\{ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right\}^2 \right]^{\frac{1}{3}} \\ &= \left\{ 2 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \right\}^{\frac{1}{3}} \\ &= 2^{\frac{1}{3}} \operatorname{cis}\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right) \quad k = 0, 1, 2 \\ &= 2^{\frac{1}{3}} \operatorname{cis}(4k + 1)\frac{\pi}{6} \quad k = 0, 1, 2 \end{aligned}$$

**4. Find all the values of  $(-16)^{\frac{1}{4}}$**

$$\begin{aligned} (-16)^{\frac{1}{4}} &= (2^4)^{\frac{1}{4}} (-1)^{\frac{1}{4}} \\ &= 2 (\operatorname{cis}\pi)^{\frac{1}{4}} = 2 \operatorname{cis}\left(\frac{2k\pi + \pi}{4}\right) \quad k = 0, 1, 2, 3 \\ &= 2 \operatorname{cis}(2k + 1)\frac{\pi}{4} \quad k = 0, 1, 2, 3 \end{aligned}$$

**5. Find all values of  $(-32)^{\frac{1}{5}}$**

$$(-32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} (-1)^{\frac{1}{5}} = 2 \{ \cos \pi + i \sin \pi \}^{\frac{1}{5}}$$

6. If  $1, \omega, \omega^2$  are the cube roots of units then prove that  $\frac{1}{2+\omega} + \frac{1}{1+2\omega} = \frac{1}{1+\omega}$

**Solution :-**

$$\begin{aligned} \text{L.H.S } & \frac{1}{2+\omega} + \frac{1}{1+2\omega} \\ & \frac{1+2\omega+2+\omega}{(2+\omega)(1+2\omega)} = \frac{3(1+\omega)}{2+4\omega+\omega+2\omega^2} \\ & = \frac{3(1+\omega)}{2(1+\omega^2)+5\omega} \\ & = \frac{3(-\omega^2)}{-2\omega+5\omega} \quad \because 1+\omega=-\omega^2 \\ & 1+\omega^2=\omega \\ & = \frac{-3\omega^2}{3\omega} = -\omega \\ & = -\frac{1}{\omega^2} = \frac{1}{1+\omega} \end{aligned}$$

7. If  $1, \omega, \omega^2$  are the cube roots of unity then prove that

$$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})=49$$

**Solution :-**

$$\begin{aligned} & \{(2-\omega)(2-\omega^2)\}\{(2-\omega)(2-\omega^2)\} \quad \{\because \omega^{10}=\omega \quad \omega^{11}=\omega^2\} \\ & \{4-2(\omega+\omega^2+\omega^3)\}\{4-2(\omega+\omega^2)+\omega^3\} \\ & (4+2+1)(4+2+1)=49 \end{aligned}$$

8. If  $1, \omega, \omega^2$  are the cube roots of units then prove that

$$(x+y+z)(x+y\omega+z\omega)(x+y\omega^2+2\omega)=x^3+y^3+z^3-3xyz$$

**Solution:-**

$$\begin{aligned} & (x+y+z)\{x+y\omega+z\omega^2\}\{x+y\omega^2+z\omega\} \\ & (x+y+z)\{x^2+xy\omega^2+xz\omega+xy\omega+xy\omega+y^2\omega^3+yz\omega^2+xz\omega^2+yz\omega+z^2\omega^3\} \\ & (x+y+z)\{x^2+y^2+z^2+xy(\omega^2+\omega)+yz(\omega+\omega^2)+zx(\omega+\omega^2)\} \\ & (x+y+z)\{x^2+y^2+z^2-xy-yz-zx\} \\ & x^3+y^3+z^3-3xyz \end{aligned}$$

### Short Answer Questions

#### 1. Solve the following equations

(i)  $x^4 - 1 = 0$  (ii)  $x^5 + 1 = 0$  (iii)  $x^9 - x^5 + x^4 - 1 = 0$  (iv)  $x^4 + 1 = 0$

**Solution :-**

$$\begin{aligned} \text{(i)} \quad & x^4 - 1 = 0 \Rightarrow x^4 = 1 \\ \therefore x &= (1)^{\frac{1}{4}} = (\cos 0 + i \sin 0)^{\frac{1}{4}} \\ &= \left\{ \cos \frac{2k\pi}{4} + \frac{i \sin 2k\pi}{4} \right\} k = 0, 1, 2, 3 \\ &= \cos 0 + i \sin 0 \quad cis \frac{\pi}{2} \quad cis \pi \quad cis \frac{3\pi}{2} \\ &= 1, \text{I}, -1, -i \\ &= \pm 1, \pm i \end{aligned}$$

**Solution :-**

$$\begin{aligned} \text{(ii)} \quad & x^5 + 1 = 0 \Rightarrow (\cos \pi + i \sin \pi)^{\frac{1}{5}} \\ x &= \cos \left( \frac{2k\pi + \pi}{5} \right) k = 0, 1, 2, 3, 4 \\ \therefore x &= cis \frac{\pi}{5}, cis \frac{3\pi}{5}, cis \pi, cis \frac{7\pi}{5}, cis \frac{9\pi}{5} \end{aligned}$$

**Solution :-**

$$\begin{aligned} \text{(iii)} \quad & x^9 - x^5 + x^4 - 1 = 0 \\ x^5(x^4 - 1) + 1(x^4 - 1) &= 0 \\ (x^4 - 1) &= 0 : (x^5 + 1) = 0 \end{aligned}$$

Do (i), (ii) to get the solution of (iii)

**Solution :-**

$$\begin{aligned} \text{(iv)} \quad & x^4 + 1 = 0 \Rightarrow x = (-1)^{\frac{1}{4}} \\ x &= cis \left( \frac{2k\pi + \pi}{4} \right) k = 0, 1, 2, 3 \\ x &= cis \frac{\pi}{4}, cis \frac{3\pi}{4}, cis \frac{5\pi}{4}, cis \frac{7\pi}{4} \end{aligned}$$

#### 2. If n is a positive integer then show that

$$(p + iq)^{\frac{1}{n}} + (p - iq)^{\frac{1}{n}} = 2(p^2 + q^2)^{\frac{1}{2n}} \cos \left\{ \frac{1}{n} \text{arc. tan} \frac{q}{p} \right\}$$

**Solution :-**

$$\text{Let } p + iq = r \{ \cos \theta + i \sin \theta \}$$

$$r \cos \theta = p \quad r \sin \theta = q \Rightarrow r^2 = p^2 + q^2$$

$$\therefore r = \sqrt{p^2 + q^2}$$

$$\begin{aligned}
 \cos \theta &= \frac{p}{\sqrt{p^2 + q^2}} \quad \sin \theta = \frac{q}{\sqrt{p^2 + q^2}} \\
 \tan \theta &= \frac{q}{p} \Rightarrow \theta = \tan^{-1} \left( \frac{p}{q} \right) \\
 (p + iq)^{\frac{1}{n}} + (p - q)^{\frac{1}{n}} &= \left\{ r(\cos \theta + i \sin \theta) \right\}^{\frac{1}{n}} + \left\{ r(\cos \theta - i \sin \theta) \right\}^{\frac{1}{n}} \\
 &= r^{\frac{1}{n}} \left\{ \cos \frac{\theta}{n} + i \cancel{\sin \frac{\theta}{n}} + \cos \frac{\theta}{n} - i \cancel{\sin \frac{\theta}{n}} \right\} \\
 &= \left( \sqrt{p^2 + q^2} \right)^{\frac{1}{n}} \left\{ 2 \cos \frac{1}{n} \theta \right\} \\
 &= 2(p^2 + q^2)^{\frac{1}{2n}} \cos \left( \frac{1}{n} \tan^{-1} \frac{q}{p} \right)
 \end{aligned}$$

3. Show that  $\left\{ \frac{1 + \sin \frac{\pi}{8} + i \cos \frac{\pi}{8}}{1 + \sin \frac{\pi}{8} - i \cos \frac{\pi}{8}} \right\}^{8/3} = -1$

Solution :-

$$\begin{aligned}
 \text{LHS} &= \left\{ \frac{1 + \sin \frac{\pi}{8} + i \cos \frac{\pi}{8}}{1 + \sin \frac{\pi}{8} - i \cos \frac{\pi}{8}} \right\}^{8/3} \\
 &= \left\{ \frac{1 + \cos \left( \frac{\pi}{2} - \pi/8 \right) + i \sin \left( \frac{\pi}{2} - \pi/8 \right)}{1 + \cos \left( \frac{\pi}{2} - \pi/8 \right) - i \sin \left( \frac{\pi}{2} - \pi/8 \right)} \right\}^{8/3} \\
 &= \left\{ \frac{1 + \cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8}}{1 + \cos \frac{3\pi}{8} - i \sin \frac{3\pi}{8}} \right\}^{8/3} = \left\{ \frac{2 \cos^2 \frac{3\pi}{16} + 2i \sin \frac{3\pi}{16} \cos \frac{3\pi}{16}}{2 \cos^2 \frac{3\pi}{16} - 2i \sin \frac{3\pi}{16} \cos \frac{3\pi}{16}} \right\}^{8/3} \\
 &= \left[ \frac{2 \cos \frac{3\pi}{16} \left\{ \cos \frac{3\pi}{16} + i \sin \frac{3\pi}{16} \right\}}{2 \cos \frac{3\pi}{16} \left( \cos \frac{3\pi}{16} - i \sin \frac{3\pi}{16} \right)} \right]^8
 \end{aligned}$$

$$\left[ \frac{\left( \cos \frac{3\pi}{16} + i \sin \frac{3\pi}{16} \right) \left( \cos \frac{3\pi}{16} + i \sin \frac{3\pi}{16} \right)}{\left( \cos \frac{3\pi}{16} - i \sin \frac{3\pi}{16} \right) \left( \cos \left( \frac{3\pi}{16} + i \sin \frac{3\pi}{16} \right) \right)} \right]^{8/3}$$
$$\left[ \frac{\left( \cos \frac{3\pi}{16} + i \sin \frac{3\pi}{16} \right)^2}{\cos^2 \frac{3\pi}{16} + \sin^2 \frac{3\pi}{16}} \right]^{8/3}$$
$$\left( \cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8} \right)^{8/3}$$
$$\cos \pi + i \sin \pi = -1$$

