

Complex Numbers

ii)CONJUGATE OF COMPLEX NUMBERS, SQUARE ROOTS OF COMPLEX NUMBERS, Nth ROOTS OF COMPLEX NUMBERS

Very Short Answer Questions

1. If $\frac{4-2i}{1-2i}$ is expressed in the form of $a+ib$ find a

Solution: -

$$\text{Let } \frac{4-2i}{1-2i} = a+ib$$

$$\frac{(4-2i)(1+2i)}{(1-2i)(1+2i)} = a+ib \Rightarrow a+ib = \frac{4+6i-4i^2}{1-4i^2}$$

$$\therefore a+ib = \frac{8+6i}{5} \Rightarrow a = \frac{8}{5}, b = \frac{6}{5}$$

2. If $a = \cos \alpha + i \sin \alpha$ $b = \cos \beta + i \sin \beta$ then find $\frac{1}{2} \left(ab + \frac{1}{ab} \right)$

Solution : -

$$ab = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$$

$$= \cos \alpha \cos \beta + i \sin \beta \cos \alpha + i \sin \alpha \cos \beta + i^2 \sin \alpha \sin \beta$$

$$= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$

$$ab = \cos(\alpha+\beta) + i \sin(\alpha+\beta)$$

$$\frac{1}{ab} = \frac{\cos(\alpha+\beta) - i \sin(\alpha+\beta)}{\{\cos(\alpha+\beta) + i \sin(\alpha+\beta)\}\{\cos(\alpha+\beta) - i \sin(\alpha+\beta)\}}$$

$$\frac{1}{ab} = \cos(\alpha+\beta) - i \sin(\alpha+\beta)$$

$$\frac{1}{2} \left(ab + \frac{1}{ab} \right) = \frac{1}{2} \left[\cos(\alpha+\beta) + i \sin(\alpha+\beta) + \cos(\alpha+\beta) - i \sin(\alpha+\beta) \right]$$

$$= \cos(\alpha + \beta)$$

3. Find the square root of $3 + 4i$

Solution :-

$$\text{Let } \sqrt{3 + 4i} = x + iy$$

Squaring on both sides

$$3 + 4i = x^2 + i^2 y^2 + 2ixy$$

$$3 + 4i = (x^2 - y^2) + i(2xy)$$

$$x^2 - y^2 = 3 : 2xy = 4$$

$$x^2 + y^2 = \sqrt{(x^2 - y^2) + 4x^2 y^2}$$

$$= \sqrt{9 + 16} \Rightarrow x^2 + y^2 = 5 : x^2 - y^2 = 3 = \sqrt{9 + 16} \Rightarrow x^2 + y^2 = 5 : x^2 - y^2 = 3$$

$$x^2 = 4 \Rightarrow x = \pm 2$$

$$y^2 = 1 \Rightarrow y = \pm 1$$

$$\sqrt{3 + 4i} = \pm(2 + i)$$

4. Express $(5 - 3i)^3$ in the form $a + ib$

Solution :-

$$(5 - 3i)^3 = a + ib$$

$$(5)^3 - (3i)^3 - 3(5)(3i)(5 - 3i) = a + ib$$

$$125 - 27i^3 - 225i + 135i^2 = a + ib$$

$$125 + 27i - 225i - 135 = a + ib \Rightarrow a + ib = -10 - 198i$$

5. Express $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$ in the form of $a + ib$

Solution :-

$$\text{Let } (-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i) = a + ib$$

$$(-\sqrt{3} + i\sqrt{2})(2\sqrt{3} - i) = a + ib$$

$$-6i + \sqrt{3}i + 2i\sqrt{6} - i^2\sqrt{2} = a + ib$$

$$(\sqrt{2} - 6) + i(2\sqrt{6} + \sqrt{3}) = a + ib$$

SHORT ANSWER QUESTIONS

1. If $z = 3 - 5i$ then show that $z^3 - 10z^2 + 58z - 136 = 0$

$$z^3 = (3 - 5i)^3 = 3^3 - (5i)^3 - 3(3)(5i)(3 - 5i)$$

$$= 27 + 125i - 135i - 225 = -198 - 10i$$

$$z^2 = (3 - 5i)^2 = 9 + 25i^2 - 30i = -16 - 30i$$

$$\text{L.H.S} = z^3 - 10z^2 + 58z - 136$$

$$= -198 - 10i - 10\{-16 - 30i\} + 58\{3 - 5i\} - 136$$

$$334 - 334 + 10i - 10i = 0 = \text{RHS}$$

2. If $x + iy = \frac{1}{1 + \cos \theta + i \sin \theta} = \frac{1}{2 \cos \frac{\theta}{2} \left\{ \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right\}}$

Solution :-

$$x + iy = \frac{1}{2 \cos^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{1}{2 \cos \frac{\theta}{2} \left\{ \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right\}}$$

$$x + iy = \frac{\cos \frac{\theta}{2} - i \sin \frac{\theta}{2}}{2 \cos \frac{\theta}{2} \left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right] \left[\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right]}$$

$$= \frac{\cos \frac{\theta}{2} - i \sin \frac{\theta}{2}}{2 \cos \frac{\theta}{2} \left\{ \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right\}} = \frac{\cos \frac{\theta}{2}}{2 \cos \frac{\theta}{2}} - \frac{i \sin \frac{\theta}{2}}{2 \cos \frac{\theta}{2}}$$

$$x + iy = \frac{1}{2} - \frac{i}{2} \tan \frac{\theta}{2} \Rightarrow x = \frac{1}{2} \Rightarrow 2x = 1$$

$$4x^2 = 1 \quad \therefore 4x^2 - 1 = 0$$

3. If $x + iy = \frac{3}{2 + \cos \theta + i \sin \theta}$ show that $x^2 + y^2 = 4x - 3$

Solution :-

$$x + iy = \frac{3 \{ 2 + \cos \theta - i \sin \theta \}}{(2 + \cos \theta)^2 - (i \sin \theta)^2} \Rightarrow x + iy = \frac{3 \{ 2 + \cos \theta - i \sin \theta \}}{4 + \cos^2 \theta + 4 \cos \theta + \sin^2 \theta}$$

$$\therefore x + iy = \frac{3 \{ 2 + \cos \theta - i \sin \theta \}}{5 + 4 \cos \theta}$$

$$x = \frac{3(2 + \cos \theta)}{5 + 4 \cos \theta} \quad y = \frac{-3 \sin \theta}{5 + 4 \cos \theta}$$

$$\text{LHS } x^2 + y^2 = \left\{ \frac{3(2 + \cos \theta)}{5 + 4 \cos \theta} \right\}^2 + \left\{ \frac{-3 \sin \theta}{5 + 4 \cos \theta} \right\}^2$$

$$= \frac{9 \{ 4 + \cos^2 \theta + 4 \cos \theta + \sin^2 \theta \}}{(5 + 4 \cos \theta)^2}$$

$$= \frac{9 \{ 5 + 4 \cos \theta \}}{(5 + 4 \cos \theta)^2} = \frac{9}{5 + 4 \cos \theta}$$

$$\text{RHS } 4x - 3 = \frac{12(2 + \cos \theta)}{5 + 4 \cos \theta} - 3 = \frac{24 + 12 \cancel{\cos \theta} - 15 - 12 \cancel{\cos \theta}}{5 + 4 \cos \theta}$$

$$RHS = \frac{9}{5 + 4\cos\theta}$$

$\therefore LHS = RHS$

4. If $u + i\vartheta = \frac{2+i}{z+3}$ and $z = x + iy$ then find u, ϑ

Solution :-

$$u + i\vartheta = \frac{2+i}{x+iy+3}$$

$$u + i\vartheta = \frac{\cancel{(2+i)\{x+iy-3\}}}{\cancel{x+iy}} = \frac{(2+i)\{x+3-iy\}}{\{(x+3)+iy\}\{(x+3)-iy\}}$$

$$= \frac{2(x+3) - 2iy + i(x+3) + y}{(x+3)^2 - i^2y^2}$$

$$u + i\vartheta = \frac{(2x+y+6) + i\{x+3-2y\}}{(x+3)^2 + y^2}$$

$$\therefore u = \frac{2x+y+6}{(x+3)^2 + y^2} \quad \vartheta = \frac{x-2y+3}{(x+3)^2 + y^2}$$

5. Show that $\frac{2-i}{(1-2i)^2}$ and $\frac{-2-11i}{25}$ are conjugate to each other

Solution :-

$$\text{Let } a + ib + \frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i}$$

$$= \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-6+8i+3i-4i^2}{9-16i^2}$$

$$a + ib = \frac{-2+11i}{25}$$

Conjugate of $\frac{-2+11i}{25}$ is $\frac{-2-11i}{25}$

Hence proved

6. Write $\left(\frac{a+ib}{a-ib}\right)^2 - \left(\frac{a-ib}{a+ib}\right)^2$ in the form of $x+iy$

Solution :-

$$\begin{aligned} \left(\frac{a+ib}{a-ib}\right)^2 - \left(\frac{a-ib}{a+ib}\right)^2 &= \left\{ \frac{a+ib}{a-ib} + \frac{a-ib}{a+ib} \right\} \left\{ \frac{a+ib}{a-ib} - \frac{a-ib}{a+ib} \right\} \\ &\left\{ \frac{(a+ib)^2 + (a-ib)^2}{a^2 - i^2 b^2} \right\} \left\{ \frac{(a+ib)^2 - (a-ib)^2}{a^2 - i^2 b^2} \right\} \\ &= \frac{2\{a^2 + i^2 b^2\}\{4aib\}}{a^2 + b^2} = \frac{8iab(a^2 - b^2)}{a^2 + b^2} \\ \text{Given } \frac{-8iab(a^2 + b^2)}{a^2 + b^2} &= x + iy \\ \therefore x = 0; y = \frac{-8ab(a^2 - b^2)}{a^2 + b^2} & \end{aligned}$$

7. Find the square root of $-47i + i8\sqrt{3}$

Solution :-

$$\text{Let } \sqrt{-47 + i8\sqrt{3}} = x + iy$$

Squaring on both sides

$$-47 + i8\sqrt{3} = x^2 - y^2 + 2ixy$$

$$x^2 - y^2 = -47 \quad 2xy = 8\sqrt{3}$$

$$x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2}$$

$$= \sqrt{(-47)^2 + (8\sqrt{3})^2} = \sqrt{2209 + 192}$$

$$= \sqrt{2401} = 49$$

$$\begin{aligned}
 x^2 + y^2 &= 49 \\
 x^2 - y^2 &= -47 \\
 \hline
 x^2 &= 1 \Rightarrow x = \pm 1
 \end{aligned}
 \quad
 \begin{aligned}
 y^2 &= 48 \\
 y &= \pm 4\sqrt{3}
 \end{aligned}$$

$$\therefore \sqrt{-47 + i8\sqrt{3}} = \pm \{1 + 4i\sqrt{3}\}$$

8. If $(x - iy)^{\frac{1}{3}} = a - ib$ then show that $\frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2)$

Solution :-

$$(x - iy)^{\frac{1}{3}} = a - ib \quad \text{Cubing on both sides we have}$$

$$x - iy = (a - ib)^3 \Rightarrow x - iy = a^3 + b^3i - 3a^2ib - 3ab^2$$

$$x = a^3 - 3ab^2 \quad y = b^3 - 3a^2b$$

$$\frac{x}{a} = a^2 - 3b^2 \quad \frac{y}{b} = b^2 - 3a^2$$

$$\frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2)$$