9.(ii) In Circle and Ex-Circles of a Triangle

1. In circle

The circle that touches the three sides of a triangle ABC internally is called the "in circle" or inscribed" of its triangle. The centre of the circle is called Incentre denoted by I the radius of the circle is denoted by inradius denoted by Y

2. In a triangle ABC

(i) $r = \frac{\Delta}{S}$ (ii) $r = (s-a) = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{b}{\cot\frac{C}{2} + \cot\frac{A}{2}} = \frac{C}{\cot\frac{A}{2} + \cot\frac{B}{2}}$

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

3. **Excircle:** - The circle that touches the side BC (opposite to angle A) internally and the other two sides AB and AC externally is called Excircle. The centre of this circle is called excentre opposite to 'A'. denoted by I_1. The radius of this circle is called ex-radius, denoted by r_1

$|||^{ly}$ exradius opposite to angle B is denoted by r_2. The centre of this excircle is denoted by I_2 exradius opposite to angle C is denoted by r_3. The centre of this ex-circle is denoted by I_3

4. **In a triangle ABC**

(i) $r_1 = \frac{\Delta}{s-a}$ (ii) $r_2 = \frac{\Delta}{s-b}$ (iii) $r_3 = \frac{\Delta}{s-c}$

5. (i) $r_1 = s\tan A/2$ (ii) $r_2 = s\tan\frac{B}{2}$ (iii) $r_3 = s\tan\frac{C}{2}$

6. (i) $r_1 = (s-c)\cot\frac{B}{2}$ (ii) $r_2 = (s-a)\cot\frac{c}{2}$ (iii) $r_3 = (s-a)\cot\frac{B}{2}$

$= (s-b)\cot\frac{C}{2}$ $= (s-c)\cot\frac{A}{2}$ $= (s-c)\cot\frac{A}{2}$

7. (i) $r_1 = \dfrac{a}{\tan\frac{B}{2} + \tan\frac{C}{2}}$ (ii) $r_2 = \dfrac{b}{\tan\frac{C}{2} + \tan\frac{A}{2}}$ (iii) $r_3 = \dfrac{c}{\tan\frac{A}{2} + \tan\frac{B}{2}}$

8. (i) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (ii) $r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

(iii) $r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

SOME IMPORTANT THEOREMS

1. In a triangle ABC prove that $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

Solution : -

RHS $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{\frac{\Delta}{s-a}} + \dfrac{1}{\frac{\Delta}{s-b}} + \dfrac{1}{\frac{\Delta}{s-c}}$

$$\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{3s-a-b-c}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{\left(\frac{\Delta}{s}\right)} = \frac{1}{r}$$

2. In a triangle ABC prove that $r\,r_1\,r_2\,r_3 = \Delta^2$

Solution : -

LHS $r\,r_1\,r_2\,r_3 = \dfrac{\Delta}{s}\,\dfrac{\Delta}{s-a}\,\dfrac{\Delta}{s-b}\,\dfrac{\Delta}{s-c}$

$$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \frac{\Delta^4}{\Delta^2} = \Delta^2$$

3. In a triangle ABC prove that $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

Solution : -

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$$

$$\frac{\Delta^2(s-c) + \Delta^2(s-a) + \Delta^2(s-b)}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2\{3s-a-b-c\}}{(s-a)(s-b)(s-c)} = \frac{\Delta^2 s\{3s-2s\}}{s(s-a)(s-b)(s-c)} = \frac{\Delta^2(s)(s)}{\Delta^2} = s^2$$

4. In a triangle ABC prove that $r\{r_1 + r_2 + r_3\} = bc + ca + ab - s^2$

Solution : -

$$r(r_1 + r_2 + r_3) = \frac{\Delta}{s}\left\{\frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c}\right\}$$

$$= \frac{\Delta^2}{s}\left\{\frac{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right\}$$

$$= \Delta^2\left\{\frac{s^2 - bs - cs + bc + s^2 - cs - as + ac + s^2 - ab - bs + ab}{s(s-a)(s-b)(s-c)}\right\}$$

$$= \frac{\Delta^2\{3s^2 - 2as - 2bs - 2cs + bc + ca + ab\}}{\Delta^2}$$

$$= \{3s^2 - 2s(a+b+c) + bc + ca + ab\}$$

$$= bc + ca + ab + 3s^2 - 4s^2 \quad \because a+b+c = 2s$$

$$= bc + ca + ab - s^2$$

VERY SHORT ANSWER QUESTIONS

1. In a triangle ABC express $\sum r_1 \cot \frac{A}{2}$ in terms of S

Solution : -

$$\sum r_1 \cot A/2 = r_1 \cot A/2 + r_2 \cot B/2 + r_3 \cot \frac{C}{2}$$

$$r_1 = s \tan A/2 \quad r_2 = s \tan B/2 \quad r_3 = s \tan C/2$$

$$= s \tan \frac{A}{2} \cot A/2 + s \tan B/2 \cot B/2 + s \tan \frac{C}{2} \cot \frac{C}{2}$$

$$= s + s + s = 3s$$

2. Show that $\sum a \cot A = 2(R + r)$

LHS $\sum a \cot A = a \cot A + b \cos B + c \cot C$

$$2R \sin A \frac{\cos A}{\sin A} + 2R \sin B \frac{\cos B}{\sin B} + 2R \sin c \frac{\cos c}{\sin c}$$

$$= 2R\{\cos A + \cos B + \cos C\} = 2R\left\{1 + 4\sin\left(\frac{A}{2}\right)\sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$= 2R\left\{1 + \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right\} = 2R\left\{1 + \frac{r}{R}\right\}$$

$$= 2\{R + r\}$$

3. In a triangle ABC prove that

(i) $r_1 + r_2 + r_3 - r = 4R$ (ii) $r_1 - r_2 + r_3 + r = 4R \cos B$

(iii) $r_1 + r_2 - r_3 + r = 4R \cos C$ (iv) $r_3 + r_2 - r_1 + r = 4R \cos A$

Solution: - Here $r_1 = 4R \sin A/2 \cos B/2 \cos C/2 : r_2 = 4R \cos\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}$

$$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} \qquad r = 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(i) $r_1 + r_2 + r_3 = \{4R\sin A/2\cos B/2\cos C/2 + 4R\cos A/2\sin B/2\cos C/2\}$

$$+\left\{4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$= R\cos\frac{C}{2}\left\{\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right\} + 4R\sin\frac{C}{2}\left\{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$= R\cos\frac{C}{2}\sin\left(\frac{A}{2}+\frac{B}{2}\right) + 4R\sin\frac{C}{2}\cos\left(\frac{A}{2}+\frac{B}{2}\right)$$

$$= 4R\cos\frac{C}{2}\sin\left(90^0 - \frac{C}{2}\right) + 4R\sin\frac{C}{2}\cos\left(\frac{A}{2}+\frac{B}{2}\right)$$

$$= 4R\cos\frac{C}{2}\sin\left(90^0 - \frac{C}{2}\right) + 4R\sin\frac{C}{2}\cos\left(90^0 - \frac{C}{2}\right)$$

$$= 4R\cos^2\frac{C}{2} + 4R\sin^2\frac{C}{2} = 4R$$

(ii) $r_1 - r_2 + r_3 + r = (r_1 + r_3) - (r_2 - r)$

$$= \left\{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$-\left\{4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} - 4R\sin A/2\sin B/2\sin C/2\right\}$$

$$= 4R\cos\frac{B}{2}\left\{\sin\frac{A}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\sin\frac{C}{2}\right\}$$

$$-4R\sin\frac{B}{2}\left\{\cos\frac{A}{2}\cos\frac{C}{2} - \sin A/2\ \sin\frac{C}{2}\right\}$$

$$= 4R\cos\frac{B}{2}\sin\left(\frac{A+C}{2}\right) - 4R\sin B/2\cos\left(\frac{A}{2}+\frac{C}{2}\right)$$

$$4R\cos\frac{B}{2}\sin\left(90^0-\frac{B}{2}\right)-4R\sin\frac{B}{2}.\cos\left(90-\frac{B}{2}\right)$$

$$4R\left\{\cos^2\frac{B}{2}-\sin^2\frac{B}{2}\right\}=4R\cos B$$

(iii) $r_1+r_2-r_3+r=(r_1+r_2)-(r_3-r)$

$$=\left\{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right\}$$

$$-\left\{4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$=4R\cos\frac{C}{2}\left\{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$-4R\sin\frac{C}{2}\left\{\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$=4R\cos\frac{C}{2}\sin\left(\frac{A}{2}+\frac{B}{2}\right)-4R\sin\frac{C}{2}\cos\left(\frac{A}{2}+\frac{B}{2}\right)$$

$$=4R\cos\frac{C}{2}\sin\left(90^0-\frac{C}{2}\right)-4R\sin\frac{C}{2}\cos\left(90^0-\frac{C}{2}\right)$$

$$=4R\cos^2\frac{C}{2}-4R\sin^2\frac{C}{2}=4R\left\{\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right\}$$

$$=4R\cos C$$

(iv) $r_2+r_3-r_1+r=(r_2+r_3)-(r_1-r)$

$$=4R\cos^2\frac{A}{2}-4R\sin^2\frac{A}{2}=4R\left\{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right\}$$

$$=4RcosA$$

5. **It** $r_1 + r_2 = r_3 - r$ **then show that** $C = 90^0$

Solution : -

$$r_1 + r_2 = r_3 - r$$

$$4R\cos^2\frac{C}{2} = 4R\sin^2\frac{C}{2} \Rightarrow \tan^2\frac{C}{2} = 1 \Rightarrow \tan\frac{C}{2} = 1$$

$$\frac{C}{2} = 45^0 \Rightarrow C = 90^0$$

SHORT ANSWER QUESTIONS

1. **Prove that** $4(r_1r_2 + r_2r_3 + r_3r_1) = (a+b+c)^2$

In theorem 3 we have proved $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$\therefore LHS$ {The student should prove the above result in the examination}

$$= 4\{r_1r_2 + r_2r_3 + r_3r_1\} = 4s^2 = (2s)^2 = (a+b+c)^2$$

2. **Prove that** $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2s^2}$

Solution : -

$$\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \left(\frac{s}{\Delta} - \frac{s-a}{\Delta}\right)\left(\frac{s}{\Delta} - \frac{s-b}{\Delta}\right)\left(\frac{s}{\Delta} - \frac{s-c}{\Delta}\right)$$

$$= \frac{a}{\Delta} \times \frac{b}{\Delta} \times \frac{c}{\Delta} = \frac{abc}{\Delta^3} \text{ but } abc = 4R\Delta$$

$$= \frac{4R\Delta}{\Delta^3} = \frac{4Rs^2}{\Delta^2 s^2} = \frac{4R}{r^2s^2} \left(r = \frac{\Delta}{s}\right)$$

3. **Prove that** $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

Hint : Refer to Theorem 4

4. **Show that** $\sum \frac{r_1}{(s-b)(s-c)} = \frac{3}{r}$

Solution : -

$$\sum \frac{r_1}{(s-b)(s-c)} = \frac{r_1}{(s-b)(s-c)} + \frac{r_2}{(s-c)(s-a)} + \frac{r_3}{(s-a)(s-b)}$$

But $r_1 = \frac{\Delta}{s-a}\ r_2 = \frac{\Delta}{s-b}\ r_3 = \frac{\Delta}{s-c}$

$$= \frac{\Delta}{(s-a)(s-b)(s-c)} + \frac{\Delta}{(s-a)(s-b)(s-c)} + \frac{\Delta}{(s-a)(s-b)(s-c)}$$

$$= \frac{3\Delta s}{s(s-a)(s-b)(s-c)} = \frac{3\not{\Delta}s}{\Delta^2} = \frac{3}{\left(\frac{\Delta}{s}\right)} = \frac{3}{r}$$

5. **Show that** $(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = C$

Solution : - $(r_1 + r_2)\tan\frac{C}{2} = \left(\frac{\Delta}{s-a} + \frac{\Delta}{s-b}\right)\frac{(s-a)(s-b)}{\Delta}$

$$\frac{\not{\Delta}}{\not{(s-a)(s-b)}}\{s-b+s-a\}\frac{\not{(s-a)(s-b)}}{\Delta} = 2s-a-b = \not{a}+\not{b}+c-\not{a}-\not{b} = c$$

6. **Show that** $r_1\, r_2\, r_3 = r^3 \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$

Consider RHS $r_1\, r_2\, r_3 = r^3 \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$

$$= \frac{\Delta^3}{\not{s^3}} \times \frac{s\not{(s-a)}}{\not{(s-b)}(s-c)} \times \frac{s\not{(s-b)}}{\not{(s-c)}\not{(s-a)}} \times \frac{s\not{(s-c)}}{(s-a)(s-b)}$$

$$= \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c} = r_1r_2r_3$$

7. **Show that** $\frac{1}{r_2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$

$$\frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}=\frac{s^2+s^2-2as+a^2+s^2-2bs+b^2+s^2-2cs+c^2}{\Delta^2}$$

$$\frac{4s^2-2s(a+b+c)+a^2+b^2+c^2}{\Delta^2}=\frac{\cancel{4s^2}-\cancel{4s^2}+a^2+b^2+c^2}{\Delta^2}=\frac{a^2+b^2+c^2}{\Delta^2}$$

8. **If** $A=90^0$ **then show that** $2(r+R)=b+c$

Solution : -

$$2(r+R)=2r+2R$$

$$2(s-a)\tan A/2+2R\sin 90^0$$

$$2(s-a)\tan 45^0+2R\sin A$$

$$2(s-a)+a\Rightarrow 2s-2a+a=b+c$$

9. **Prove that** $\frac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_3}}=a$

Solution : -

LHS

$$\frac{\frac{\Delta}{s-a}\left\{\frac{\Delta}{s+b}+\frac{\Delta}{s-c}\right\}}{\sqrt{\frac{\Delta^2}{(s-a)(s-b)}+\frac{\Delta^2}{(s-b)(s-c)}+\frac{\Delta^2}{(s-c)(s-a)}}}=\frac{\frac{\Delta^2}{(s-a)(s-b)(s-c)}\{s-b+s-c\}}{\sqrt{\frac{\Delta^2}{(s-a)(s-b)(s-c)}}(s-a)(s-b)+(s-c)}$$

$$\frac{\Delta^{\cancel{2}}}{(s-a)(s-b)(s-c)}\times\frac{\sqrt{\cancel{(s-a)}\,\cancel{(s-b)}\,\cancel{(s-c)}}}{\cancel{\Delta}}\times\frac{a}{\sqrt{3s-2s}}$$

$$\frac{a\Delta}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{a\Delta}{\Delta}=a$$

10. In a triangle ABC if $(r_2 - r_1)(r_3 - r_1) = 2r_2 r_3$ then prove that $A = 90^0$

Solution: -

Given $(r_2 - r_1)(r_3 - r_1) = 2r_2 r_3$

$r_2 r_3 - r_1 r_2 - r_1 r_3 + r_1^2 = 2r_2 r_3$

$r_1^2 = r_1 r_2 + r_2 r_3 + r_3 r_1$

$$r_1^2 = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$$

$$r_1^2 = \frac{\Delta^2}{(s-a)(s-b)(s-c)}\{s-c+s-a+s-b\}$$

$$r_1^2 = \frac{\cancel{s(s-a)(s-b)(s-c)}}{\cancel{(s-a)(s-b)(s-c)}} \times s \Rightarrow r_1^2 = s^2$$

$$s^2 \tan^2\frac{A}{2} = s^2 \Rightarrow \tan\frac{A}{2} = 1$$

$A = 90^0 \quad \therefore$ Triangle is right angled

11. If $r : R = r_1 = 2:5:12$ then prove that triangle on right angled of A

Solution: Let $r = 2k \quad R = 5k \quad r_1 = 12k$

$r_1 - r = 4R\sin A/2\cos B/2\cos C/2 - 4R\sin A/2\sin B/2\sin C/2$

$$= 4R\sin A/2\left\{\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$= R\sin\frac{A}{2}\left\{\cos\left(\frac{B}{2}+\frac{C}{2}\right)\right\} = 4R\sin\frac{A}{2}\cos\left(90^0 - \frac{A}{2}\right)$$

$= 4R\sin^2 A/2$

$r_1 - r = 4R\sin^2 A/2$

$$12k - 2k = 4(5k)\sin^2 A/2 \Rightarrow \sin^2\left(\frac{A}{2}\right) = \frac{10\cancel{K}}{20\cancel{K}}$$

$$\sin\frac{A}{2}=\frac{1}{\sqrt{2}}\Rightarrow A=90^0$$

12. **If** $A,\ A_1, A_2, A_3$ **are the areas of incircle and excircle of a triangle respectively then prove that** $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$

Solution : -

Here $A=\pi r^2\quad A_1=\pi r_1^2\ ;\ A_2=\pi r_2^2\ ;\ A_3=\pi r_3^2$

LHS $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{\pi r_1^2}}+\frac{1}{\sqrt{\pi r_2^2}}+\frac{1}{\sqrt{\pi r_3^2}}$

$$\frac{1}{\sqrt{\pi}}\left\{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right\}=\frac{1}{\sqrt{\pi}}\left\{\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right\}$$

$$\frac{1}{\sqrt{\Delta}}\left\{\frac{3s-(a+b+c)}{\Delta}\right\}=\frac{1}{\sqrt{\pi}}\frac{s}{\Delta}=\frac{1}{\sqrt{\pi}}\times\frac{1}{\left(\frac{\Delta}{s}\right)}$$

$$\frac{1}{\sqrt{\pi}}\cdot\frac{1}{r}=\frac{1}{\sqrt{\pi r^2}}=\frac{1}{\sqrt{A}}$$

LONG ANSWER QUESTIONS

1. **In a triangle ABC show that** $\cos A+\cos B+\cos C=1+\frac{r}{R}$

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$2\cos\left(90^0-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}$$

$$1+2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\sin\frac{C}{2}\right\}$$

$$1+2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\sin 90^{0}-\left(\frac{A+B}{2}\right)\right\}$$

$$1+\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}$$

$$1+2\sin\frac{C}{2}\left\{2\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$1+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}=1+\frac{r}{R}$$

2. In a triangle ABC prove that

$$\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R}$$

$$\text{LHS } \frac{1+\cos A+1+\cos B+1+\cos C}{2}=\frac{3+\{\cos A+\cos B+\cos C\}}{2}$$

From above problem $\cos A+\cos B+\cos C=1+\frac{r}{R}$

$$\therefore=\frac{3+1+\frac{r}{R}}{2}=\frac{4+\frac{r}{R}}{2}=2+\frac{r}{2R}$$

3. In a triangle ABC $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}$

Solution : -

$$\text{LHS }=\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$$

$$\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\frac{1-\cos C}{2}$$

$$\frac{3+\{\cos A+\cos B+\cos c\}}{2}=\frac{3-\left\{1+\frac{r}{R}\right\}}{2}$$

$$\frac{3-1-\frac{r}{R}}{2}=\frac{2-\frac{r}{R}}{2}=1-\frac{r}{2R}$$

4.(i) In a triangle ABC prove that (i) $a=(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}$

(ii) $\Delta=r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$

Solution : -

$$\text{RHS }=(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}=\left(\frac{\Delta}{s-b}+\frac{\Delta}{s-c}\right)\sqrt{\frac{\frac{\Delta^2}{s(s-a)}}{\frac{\Delta^2}{(s-b)(s-c)}}}$$

$$=\frac{\Delta\{s-c+s-b\}}{(s-b)(s-c)}\sqrt{\frac{\Delta^2}{s(s-a)}\cdot\frac{(s-b)(s-c)}{\Delta^2}}$$

$$=\frac{\Delta.(2s-b-c)}{\left\{\sqrt{(s-b)(s-c)}\right\}^2}\frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}=\frac{\Delta\{a+b+c-b-c\}}{\sqrt{s(s-a)(s-b)(s-c)}}=a$$

(ii) $\Delta=r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$

$$r_1+r_2=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$=4R\cos\frac{C}{2}\left\{\sin\left(\frac{A}{2}+\frac{B}{2}\right)\right\}=4R\cos\frac{c}{2}\sin\left(90^0-\frac{c}{2}\right)$$

$$=4R\cos^2\frac{C}{2}$$

$$\text{RHS } r_1r_2\sqrt{\frac{4R-4R\cos^2\frac{C}{2}}{4R\cos^2\frac{C}{2}}}$$

$$r_1r_2\sqrt{\frac{4R\sin^2\frac{C}{2}}{4R\cos^2\frac{C}{2}}}=\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\times\tan\frac{C}{2}$$

$$=\frac{\Delta^2}{(s-a)(s-b)}\cdot\frac{(s-a)(s-b)}{\Delta}=\Delta$$

5. **Prove that** $r_1^2+r_2^2+r_3^2+r^2=16R^2=(a^2+b^2+c^2)$

Solution: -

From 3[rd] problem of very short answer questions

We have $r_1+r_2+r_3-r=4R$

Squaring on both sides

$$(r_1+r_2+r_3-r)^2=16R^2$$

$$(r_1+r_2)^2+(r_3-r)^2+2(r_1+r_2)(r_3-r)=16R^2$$

$$r_1^2+r_2^2+2r_1r_2+r_3^2+r^2-2r_3r+2r_1r_3-2r_1r+2r_2r_3-2r_2r=16R^2$$

$$r_1^2+r_2^2+r_3^2+2(r_1r_2+r_2r_3+r_3+r_3r_1)-2r(r_1+r_2+r_3)=16R^2$$

Here from theorem 3 & 4 we have

$$r_1r_2+r_2r_3+r_3r_1=s^2$$

$$r(r_1+r_2+r_3)=ab+bc+ca-s^2$$

$$\therefore r_1^2+r_2^2+r_3^2+r^2+2s^2-2\{ab+bc+ca-s^2\}=16R^2$$

$$r_1^2+r_2^2+r_3^2+r^2+2s^2-2ab-2bc-2ca+2s^2=16R^2$$

$$r_1^2 + r_2^2 + r_3^2 + r^2 + 4s^2 - 2ab - 2bc - 2ca = 16R^2$$

$$r_1^2 + r_2^2 + r_3^2 + r^2 + (a+b+c)^2 - 2ab - 2bc - 2ca = 16R^2$$

$$r_1^2 + r_2^2 + r_3^2 + r^2 + a^2 + b^2 + c^2 + \cancel{2ab} + \cancel{2bc} + \cancel{2ca} - \cancel{2ab} - \cancel{2bc} - \cancel{2ca} = 16R^2$$

$$r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 - \left(a^2 + b^2 + c^2\right)$$

6. If P_1, P_2, P_3 are altitudes drawn from vertices A, B, C to the opposite sides of a triangle respectively then show that

(i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r}$ **(ii)** $\frac{1}{P_1} - \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r_2}$

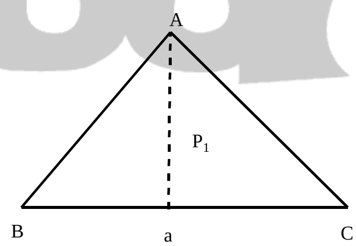
(iii) $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{1}{r_3}$ **(iv)** $\frac{1}{P_2} + \frac{1}{P_3} - \frac{1}{P_1} = \frac{1}{r_1}$

(v) $P_1P_2P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta R^3}{abc}$

(vi) $\frac{1}{P_1^2} + \frac{1}{P_2^2} + \frac{1}{P_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta^2}$

Solution: -

In a triangle ABC



$$\Delta = \frac{1}{2} \times a \times P_1 \Rightarrow P_1 = \frac{2\Delta}{a}$$

$$|||^{ly} \quad P_2 = \frac{2\Delta}{b} \quad P_3 = \frac{2\Delta}{c}$$

Solution (i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} = \frac{a+b+c}{2\Delta} = \frac{2s}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$

Solution (ii) $\frac{1}{P_1} - \frac{1}{P_2} + \frac{1}{P_3} = \frac{a-b+c}{2\Delta} = \frac{\cancel{2}(s-b)}{\cancel{2}\Delta} = \frac{s-b}{\Delta} = \frac{1}{r_2}$

Solution (iii) $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}$

Solution (iv) $\frac{1}{P_2}+\frac{1}{P_3}-\frac{1}{P_1}=\frac{b+c-a}{2\Delta}=\frac{2(s-a)}{2\Delta}=\frac{s-a}{\Delta}=\frac{1}{r_1}$

Solution (v) $P_1P_2P_3=\frac{84^3}{abc}$ but $\Delta=\frac{abc}{\Delta R}$

$$=\frac{8\left(\frac{abc}{4R}\right)^3}{abc}=\frac{8(abc)^3}{64R^3abc}=\frac{(abc)^2}{8R^3}$$

Solution (vi) LHS $\frac{1}{p_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}=\frac{a^2+b^2+c^2}{4\Delta^2}$

RHS $=(\cot A+\cot B+\cot C)/\Delta$

$$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{2bc\sin A}=\frac{b^2+c^2-a^2}{4\Delta}$$

$|||^{ly}$ $\cot B=\frac{a^2+c^2-b^2}{4\Delta}$ $\cot c=\frac{a^2+b^2-c^2}{4\Delta}$

$$\frac{\cot A+\cot B+\cot C}{\Delta}=\frac{1}{\Delta}\left[\frac{b^2+c^2-\cancel{a^2}+\cancel{c^2}+\cancel{a^2}-\cancel{b^2}+a^2+\cancel{b^2}-\cancel{c^2}}{4\Delta}\right]$$

$$=\frac{a^2+b^2+c^2}{4\Delta^2}=LHS$$

7. If a = 13 b = 14 c = 15 show that $R=\frac{65}{8}$ $r=4$ $r_1=\frac{21}{2}$ $r_2=12$ *and* $r_3=14$

Solution : -

Given $a=13, b=14, c=15$

$$s=\frac{a+b+c}{2}=21 \quad s-a=8 \quad s-b=7 \quad s-c=6$$

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{21\times8\times7\times6}$$

$$=\sqrt{7\times3\times2\times2\times2\times7\times3\times2} \quad =7\times3\times2\times2=84$$

$$R=\frac{abc}{4\Delta}=\frac{13\times \cancel{14}\times \cancel{15}^{5}}{4\times \cancel{84}}=\frac{65}{8}:r=\frac{\Delta}{S}=\frac{84}{21}=4$$

$$\cancel{6}\ 2$$

$$r_1=\frac{\Delta}{s-a}=\frac{84}{8}=\frac{21}{2} \qquad r_2=\frac{\Delta}{s-b}=\frac{84}{7}=12\,;\,r_3=\frac{\Delta}{s-c}=\frac{84}{6}=14$$

8. **If** $r_1=2\,r_2=3\,r_3=6$ **then prove that** $r=1\;\; a=3\;\; b=4\;c=5$

Solution: -

We know that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=\frac{1}{6}\Rightarrow\frac{3+2+1}{6}=\frac{1}{r}\Rightarrow r=1$$

We know that $rr_1r_2r_3=\Delta^2\Rightarrow 1\times 2\times 3\times 6=\Delta^2$

$$\therefore \Delta=6$$

$$r=\frac{\Delta}{s}\Rightarrow 1=\frac{6}{s}\Rightarrow s=6$$

$$r_1=\frac{\Delta}{s-a}\Rightarrow 2=\frac{6}{s-a}\Rightarrow s-a=3\Rightarrow 6-a=3\Rightarrow a=3$$

$$r_2=\frac{\Delta}{s-b}\Rightarrow 3=\frac{6}{s-b}\Rightarrow s-b=2\Rightarrow 6-b=2\Rightarrow b=4$$

$$r_s=\frac{\Delta}{s-c}\Rightarrow 6=\frac{6}{s-c}\Rightarrow s-c=1\Rightarrow 6-c=1\Rightarrow c=5$$

Problem for practice : In a triangle ABC if $r_1=8\,r_2=12\,r_3=24$ find a, b, c

In a triangle ABC prove that $\left(r_1+r_2\right)\sec^2\frac{C}{2}=\left(r_2+r_3\right)84^2\frac{A}{2}=\left(r_3+r_1\right)\sec^2\frac{B}{2}$

9. In a triangle ABC show that $\frac{ab-r_1r_2}{r_3}=\frac{bc-r_2r_3}{r_1}=\frac{ca-r_1r_3}{r_2}$

$$\frac{ab-r_1r_2}{r_3}=\frac{(2R\sin A)(2R\sin B)-\left(4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\left(4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right)}{4R\cos A/2\cos B/2\sin C/2}$$

$$=\frac{4R^2\left(2\sin\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\ 16R^2(\sin A/2\cos B/2\cos C/2)\left(\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right)}{4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$$

$$\frac{16R^2\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{B}{2}\left\{1-\cos^2\frac{C}{2}\right\}}{4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$$

$$4R\sin A/2\sin B/2\sin C/2=r$$

$|||^{ly}$ we have prove $\frac{bc-r_2r_3}{r_1}=r$ and $\frac{ca-r_1r_3}{r_2}=r$

10. In a triangle ABC prove that $\sum (r+r_1)\tan\left(\frac{B-C}{2}\right)=0$

Solution: -

$$r_1r_2=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\sum (r_1+r)\tan\left(\frac{B-C}{2}\right)=\sum 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$$

$$\sum 3R\, 2\sin\left(90^0 - \frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)$$

$$\sum 2R\left\{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)\right\}$$

$$\sum 2R\{\sin B - \sin C\} = \sum 2R\sin B - 2R\sin C$$

$$\sum b - c = b - c + c - a + a - b = 0$$

11. In a triangle ABC prove that $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$

Solution: - $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{ar_1 + br_2 + cr_3}{abc}$

$$= \frac{1}{abc}\sum ar_1 = \frac{1}{abc}\sum (2R\sin A)\left(s\tan\frac{A}{2}\right)$$

$$= \frac{2RS}{abc}\sum 2\sin\frac{A}{2}\cancel{\cos\frac{A}{2}}\frac{\sin A}{\cancel{\cos\frac{A}{2}}}$$

$$= \frac{2RS}{4R\Delta}\sum 2\sin^2\frac{A}{2}$$

$$= \frac{s}{2\Delta}\sum 1 - \cos A$$

$$= \frac{s}{2\Delta}\{1 - \cos A + 1 - \cos B + 1 - \cos C\}$$

$$= \frac{1}{2\left(\frac{\Delta}{s}\right)}\{3 - (\cos A + \cos B + \cos C)\}$$

$$\frac{1}{2r}\left\{3 - \left(1 + \frac{r}{R}\right)\right\} = \frac{1}{2r}\left\{2 - \frac{r}{R}\right\} = \frac{1}{r} - \frac{1}{2R}$$

$$\left\{\begin{array}{l} \text{from prob 1 of long answer questions} \\ \cos A+\cos B+\cos C=1+\dfrac{r}{R} \\ \text{The student has to prove this in} \\ \text{examination} \end{array}\right\}$$