Inverse Trigonometric Function

Key points :

1. If $\sin\theta = x$, we write $\theta = \sin^{-1} x$.

2. $\sin(\sin^{-1} x) = x$, $\sin^{-1}(\sin\theta) = \theta$ if ‘θ‘ $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos(\cos^{-1} x)=x$, $\cos^{-1}(\cos\theta) = \theta$ if $\theta \in [0, \pi]$

$\tan(\tan^{-1} x) = x$, $\tan^{-1}(\tan\theta) = \theta$ if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

3. That value of $\sin^{-1} x$ lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is called the principal value of $\sin^{-1} x$.

That value of $\cos^{-1} x$ lying between 0 and π is called the principal value of $\cos^{-1} x$.

That value of $\tan^{-1} x$ lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is called the principal value of tan-^{1} x.

4. If $-1 \le x \le 1$, then i) $\sin^{-1}(-x) = -\sin^{-1} x$

ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

5. If $x \varepsilon R$, then i) $\tan^{-1}(-x) = -\tan^{-1} x$

ii) $\cot^{-}(-x) = \pi - \cot^{-1} x$

6. If $x \le -1$ or $x \ge 1$, then i) $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1} x$

ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

7. $\mathrm{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$ (if $x \ne 0$).

$\sec^{-1} x = \cos^{-1}\frac{1}{x}$ (if $x \ne 0$).

$\cot^{-1} x = \tan^{-1}\frac{1}{x}$ (if $x > 0$).

$= \pi + \tan^{-1}\frac{1}{x}$ (if $x < 0$).

8. $\sin^{-1} x + \cos^{-1} x = \pi/2$, $\tan^{-1} x + \cot^{-1} x = \pi/2$, $\sec^{-1} x + \mathrm{cosec}^{-1} x = \pi/2$.

9. If $\sin^{-1} x + \sin^{-1} y = \pi/2$, then $x^2 + y^2 = 1$.

10. $\sin(\cos^{-1} x) = \sqrt{1-x^2}$, $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

11. $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$ for $0 \le x \le 1$

$= -\cos^{-1}\sqrt{1-x^2}$ for $-1 \le x < 0$

12. $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$ for $o \le x \le 1$

$= \pi - \sin^{-1}\sqrt{1-x^2}$ for $-1 \le x < 0$

$= \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ for $0 < x \le 1$, $\pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ for $-1 \le x < 0$

13. $\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$ for $x \ge 0$

$= \cos^{-1}\frac{1}{\sqrt{1+x^2}}$ for $x \ge 0$

14. $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ for $\frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$= \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ for $\frac{1}{\sqrt{2}} < x \le 1$

$= \sin^{-1}\left(2x\sqrt{1-x^2}\right) - \pi$ for $-1 \le x < \frac{-1}{\sqrt{2}}$

15. $2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right)$ for $0 \le x \le 1$

$= \pi - \cos^{-1}\left(2x^2 - 1\right)$ for $-1 \le x < 0$

16. $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ for $-1 < x < 1$

$= \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ for $x > 1$

$= -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ for $x < -1$

17. $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2}$ for $-1 \le x \le 1$

$= \cos^{-1}\frac{1-x^2}{1+x^2}$ for $0 \le x < \infty$

$= \tan^{-1}\frac{2x}{1-x^2}$ for $-1 < x < 1$

18. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$ for $\frac{-1}{2} \le x \le \frac{1}{2}$

$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$ for $\frac{1}{2} \le x \le 1$

$3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ for $\frac{-1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}$

19. (a) If $-1 < x < 1$, $-1 < y < 1$ and $xy < 1$, then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

(b) If $x > 0$, $y > 0$ and $xy > 1$, then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy} + \pi$

(c) If $x < 0$, $y < 0$ and $xy > 1$, then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy} - \pi$

20. (a) If $xy > -1$, then $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

(b) If $x > 0$, $y < 0$ and $xy < -1$, then $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} + \pi$

(c) If $x < 0$, $y > 0$ and $xy < -1$, then $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} - \pi$

21. $\tan^{-1}\frac{m}{n} + \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$ Or $\frac{-3\pi}{4}$

22. $\tan^{-1}\frac{m}{n} - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$ Or $-\frac{3\pi}{4}$

23. $\sin(2\tan^{-1}x) = \frac{2x}{1+x^2}$

$\cos(2\tan^{-1}x) = \frac{1-x^2}{1+x^2}$

$\tan(2\tan^{-1}x) = \frac{2x}{1-x^2}$

24. $\sin(3\sin^{-1}x) = 3x - 4x^3$

$\cos(3\cos^{-1}x) = 4x^3 - 3x$

$\tan(3\tan^{-1}x) = \frac{3x - x^3}{1 - 3x^2}$

25. $\sin(4\tan^{-1}x) = \dfrac{4x(1-x^2)}{(1+x^2)^2}$

$\cos(4\tan^{-1}x) = \dfrac{1-6x^2+x^4}{(1+x^2)^2}$

PROBLEMS

Very short Answer Questions:

1. **Evaluate the following**

(i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **(ii)** $\cos^{-1}\left(\frac{1}{2}\right)$ **(iii)** $\sec^{-1}(-\sqrt{2})$ **(iv)** $\cot^{-1}(-\sqrt{3})$

Solution:

(i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}\{\because \sin^{-1}(-x) = \sin^{-1}x\}$

Solution :

(ii) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Solution :

(iii) $\sec^{-1}(-\sqrt{2}) = \pi - \sec^{-1}\sqrt{2} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Solution :

(iv) $\cot^{-1}(-\sqrt{3}) = \pi - \cot^{-1}\sqrt{3} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

2. Evaluate the following

(i) $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right\}$ (ii) $\sin\left\{\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}$ (iii) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

(iv) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$

Solution :

(i) $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right\} = \sin\left\{\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right\} = \sin\left\{\frac{\pi}{3} + \frac{\pi}{6}\right\} = \sin\frac{\pi}{2} = 1$

Solution :

(ii) $\sin\left\{\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}=\sin\left\{\frac{\pi}{2}+\sin^{-1}\frac{\sqrt{3}}{2}\right\}=\sin\left\{\frac{\pi}{2}+\frac{\pi}{6}\right\}=\sin 120^0=\frac{\sqrt{3}}{2}$

Solution :

(iii) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)=\sin^{-1}\left(\frac{1}{2}\right)\left\{\because \sin\frac{5\pi}{6}=\frac{1}{2}\right\}$

$$=\frac{\pi}{6}\because \sin\frac{\pi}{6}=\frac{1}{2}$$

Solution :

(iv) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)=\cos^{-1}\left\{-\frac{1}{\sqrt{2}}\right\}=\pi-\cos^{-1}\frac{1}{\sqrt{2}}=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

3. **Find the values of** (i) $\sin\left\{\cos^{-1}\frac{3}{2}\right\}$ (ii) $\tan\left\{\operatorname{cosec}^{-1}\frac{65}{63}\right\}$ (iii) $\sin\left\{2\sin^{-1}\frac{4}{5}\right\}$

Solution :

(i) $\sin\left\{\cos^{-1}\frac{3}{5}\right\}$ $\quad$ *let* $\cos^{-1}\frac{3}{5}=\alpha\Rightarrow\cos\alpha=\frac{3}{5}$

$$\sin\left\{\cos^{-1}\frac{3}{5}\right\} = \sin\alpha=\frac{4}{5}\ \left\{\because \cos\alpha=\frac{3}{5}\right\}$$

Solution :

(ii) $\tan\left\{\operatorname{cosec}^{-1}\frac{65}{63}\right\}$ *let* $\operatorname{cosec}^{-1}\frac{65}{63}=\alpha\Rightarrow\operatorname{cosec}\alpha=\frac{65}{63}$

Solution :

(iii) $\sin\left\{2\sin^{-1}\frac{4}{5}\right\}$ $\quad$ *let* $\sin^{-1}\frac{4}{5}=\alpha \Rightarrow \sin\alpha=\frac{4}{5}:\cos\alpha=\frac{3}{5}$

$$\sin\left\{2\sin^{-1}\frac{4}{5}\right\} = \sin^2\alpha=2\sin\alpha\cos\alpha=2\times\frac{4}{5}\times\frac{3}{5}=\frac{24}{25}$$

4. **Evaluate (i)**: $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (ii) $\sin^{-1}\sin\left(\frac{33\pi}{7}\right)$ (iii) $\cos^{-1}\cos\left(\frac{17\pi}{6}\right)$

Solution :

(i) $Tan^{-1}\left\{\tan\frac{3\pi}{4}\right\}=Tan^{-1}\{-4\}=Tan^{-1}(1)$

$$=-\frac{\pi}{4}$$

Solution :

(ii) $\sin^{-1}\sin\left(\frac{33\pi}{7}\right)=\sin^{-1}\left\{\sin\left\{5\pi-\frac{2\pi}{7}\right\}\right\}=\sin^{-1}\left\{+\sin\left(\frac{2\pi}{7}\right)\right\}=\frac{2\pi}{7}$

Solution :

(iii) $\cos^{-1}\left\{\cos\frac{17\pi}{6}\right\}=\cos^{-1}\left\{\cos 3\pi-\frac{\pi}{6}\right\}=\cos^{-1}\left\{-\cos\frac{\pi}{6}\right\}=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

Short Answer Questions

(i) **Prove the following** $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$

(ii) $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$

Solution :

(i) L.H.S $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$

Let $\sin^{-1}\frac{3}{5}=\alpha$ $\quad\sin^{-1}\frac{8}{17}=\beta$

$\sin\alpha=\frac{3}{5}$ $\quad\sin\beta=\frac{8}{17}$

$\cos\alpha=\frac{4}{5}$ $\quad\cos\beta=\frac{15}{17}$

We know that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$=\frac{4}{5}\times\frac{15}{17}-\frac{3}{5}\times\frac{8}{17}=\frac{36}{85}$

$\therefore\ \alpha+\beta=\cos^{-1}\frac{36}{85}\Rightarrow\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$

Solution :

(ii) *let* $\sin^{-1}\frac{3}{5}=\alpha$ $\quad\cos^{-1}\frac{12}{13}=\beta$

$\sin\alpha=\frac{3}{5}$ $\quad\cos\beta=\frac{12}{13}$

$\cos\alpha\frac{4}{5}$ $\quad\sin\beta=\frac{5}{13}$

Cole know that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$\cos(\alpha+\beta)=\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}=\frac{33}{65}$$

$$\therefore \alpha+\beta=\cos^{-1}\frac{33}{65}$$

$$\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$$

Find the values of (i) $\sin\left\{\cos^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}\right\}$ (ii) $Tan\left\{\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{3}{\sqrt{34}}\right\}$

(iii) $\cos\left\{\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right\}$

Solution :

(i) Let $\cos^{-1}\frac{3}{5}=\alpha$ *and* $\cos^{-1}\frac{12}{13}=\beta$

$$\cos\alpha=\frac{3}{5} \text{ and } \cos\beta=\frac{12}{13}$$

$$\sin\alpha\frac{4}{5} \text{ and } \sin\beta=\frac{5}{13}$$

$$\sin\left\{\cos^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}\right\}=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$=\frac{4}{5}\times\frac{12}{13}+\frac{3}{5}\times\frac{5}{13}$$

$$=\frac{63}{65}$$

Solution :

(ii) Let $\sin^{-1}\frac{3}{5}=\alpha$ *and* $\cos^{-1}\frac{5}{\sqrt{34}}=\beta$

$$\sin\alpha=\frac{3}{5} \quad \cos\beta=\frac{5}{\sqrt{34}}$$

$$\cos\alpha=\frac{4}{5} \quad \sin\beta=\frac{3}{\sqrt{34}}$$

$$Tan(\alpha+\beta)=\frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}=\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{9}{20}}=\frac{\frac{27}{20}}{\frac{20-9}{20}}=\frac{27}{11}$$

Solution :

(iii) Let $\sin^{-1}\frac{3}{5}=\alpha$ *and* $\sin^{-1}\frac{5}{13}=\beta$

$$\sin\alpha=\frac{3}{5} \quad and \quad \sin\beta=\frac{5}{13}$$

$$\cos\alpha\frac{4}{5} \quad and \ \cos\beta=\frac{12}{13}$$

$$\cos\left\{\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right\}=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$=\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}=\frac{33}{65}$$

3. **Show that** $\sec^2(Tan^{-1}2)+\cos ec^2(\cot^{-1}2)=10$

Solution:

Let $Tan^{-1}2=\alpha$ *and* $\cot^{-1}2=\beta$

$Tan\alpha=2$ *and* $\cot\beta=2$

LHS = $\sec^2\alpha+\cos ec^2\beta=1+Tan^2\alpha+1+\cot^2\beta=44+1+4=10$

4. **Show that** (i) $Tan^{-1}\frac{1}{7}+Tan^{-1}\frac{1}{13}+Tan^{-1}\frac{2}{4}=0$ (ii) $Tan^{-1}\frac{1}{2}+Tan^{-1}\frac{1}{5}+Tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

(iii) $Tan^{-1}\frac{3}{4}+Tan^{-1}\frac{3}{5}-Tan^{-1}\frac{8}{19}=\frac{\pi}{4}$ (iv) $Tan^{-1}\frac{1}{7}+Tan^{-1}\frac{1}{8}=\cot^{-1}\frac{201}{43}+\cot^{-1}(18)$

Solution :

(i) Let $Tan^{-1}\frac{1}{7}=\alpha$ $Tan^{-1}\frac{1}{13}=\beta$ $Tan^{-1}\frac{2}{9}=\gamma$

$Tan\alpha=\frac{1}{7}$ $Tan\beta\frac{1}{13}$ $Tan\gamma=\frac{2}{9}$

$$Tan(\alpha+\beta)=\frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}=\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}=\frac{\frac{20}{91}}{\frac{90}{91}}=\frac{2}{9}$$

$$Tan(\alpha+\beta-\gamma)=\frac{Tan(\alpha+\beta)-Tan\gamma}{1+Tan(\alpha+\beta)Tan\gamma}=\frac{\frac{2}{9}-\frac{2}{9}}{1+\frac{4}{8}}=0$$

$$\because \alpha+\beta-\gamma=0 \Rightarrow Tan^{-1}\frac{1}{7}+Tan^{-1}\frac{1}{13}-Tan^{-1}\frac{2}{9}=0$$

Solution :

(ii) $Tan^{-1}\frac{1}{2}+Tan^{-1}\frac{1}{5}+Tan^{-1}\left(\frac{1}{8}\right)$

Let $Tan^{-1}\frac{1}{2}=\alpha \;\; Tan^{-1}\frac{1}{5}=\beta \;\; Tan^{-1}\frac{1}{8}=\delta$

$$Tan\alpha=\frac{1}{2} \;\; Tan\beta=\frac{1}{5} \;\; Tan\gamma=\frac{1}{8}$$

$$Tan(\alpha+\beta)=\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{10}}=\frac{7}{9}$$

$$Tan(\alpha+\beta+\gamma)=\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}=\frac{56+9}{72}\times\frac{72}{72-7}=1$$

$$\alpha+\beta+\delta=\frac{\pi}{4}\Rightarrow Tan^{-1}\frac{1}{2}+Tan^{-1}\frac{1}{5}+Tan^{-1}\frac{1}{8}=\frac{\pi}{4}$$

Solution :

(iii) Let $Tan^{-1}\frac{3}{4}=\alpha \;\; Tan^{-1}\frac{3}{5}=\beta \;\; and\ Tan^{-1}\frac{8}{19}=\delta$

$$Tan\alpha=\frac{3}{4} \quad Tan\beta=\frac{3}{5} \quad and\ Tan\delta=\frac{8}{19}$$

$$Tan(\alpha+\beta)=\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{9}{20}}=27\Rightarrow Tan(\alpha+\beta-\delta)=\frac{Tan(\alpha+\beta)+Tan\delta}{1-Tan(\alpha+\beta)Tan\delta}$$

$$=\frac{\frac{27}{11}+\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}=\frac{\frac{513-88}{209}}{\frac{209+216}{209}}=1$$

$$\because \alpha+\beta-\delta=\frac{\pi}{4}$$

Solution :

(iv) Let $Tan^{-1}\frac{1}{7}=\alpha \quad Tan^{-1}\frac{1}{8}=\beta$

$$Tan\alpha=\frac{1}{7} \quad and \quad Tan\beta=\frac{1}{8}$$

$$Tan(\alpha+\beta)=\frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}=\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{50}}=\frac{15}{55}=\frac{3}{11}$$

$$\text{LHS } \alpha+\beta=Tan^{-1}\frac{3}{11}=\cot^{-1}\left(\frac{11}{3}\right)$$

$$\text{Let } \cot^{-1}\frac{201}{43}=\delta \quad \cot^{-1}18=\delta$$

$$\cot\delta=\frac{201}{43} \quad \cot\delta=18$$

$$\cot(\gamma+\delta)=\frac{\cot\gamma\cot\delta-1}{\cot\delta+\cot\gamma}=\frac{\frac{201\times18}{43}-1}{\frac{201}{43}}=\frac{\frac{3618-43}{43}}{\frac{201+774}{43}}$$

$$=\frac{3575}{975}=\frac{11}{3}$$

$$\gamma+\delta=\cot^{-1}\frac{11}{3} \quad \because \text{LHS=RHS}$$

5.Find the value of $Tan\left\{\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right\}$

Solution:

$$\text{Let } \cos^{-1}\frac{4}{5}=\alpha \quad and \quad \tan^{-1}\frac{2}{3}=\beta$$

$$\cos\alpha=\frac{4}{5} \quad and \quad \tan\beta=\frac{2}{3}$$

$$Tan\alpha=\frac{3}{4} \quad and \quad \tan\beta=\frac{2}{3}$$

$$Tan\left\{\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right\}=Tan(\alpha+\beta)=\frac{Tan\alpha+Tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}=\frac{\frac{17}{12}}{\frac{6}{12}}=\frac{17}{6}$$

6. **Prove that** $Tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+Tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}$

Solution:

Let $\cos^{-1}\frac{a}{b}=\alpha \Rightarrow \cos\alpha=\frac{a}{b}$

L.H.S $\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

$$\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}+\frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}}=\frac{\left(1+\tan\frac{\alpha}{2}\right)^2+\left(1-\tan\frac{\alpha}{2}\right)^2}{1-\tan^2\frac{\alpha}{2}}$$

$$\frac{2\left\{1+\tan^2\frac{\alpha}{2}\right\}}{1-\tan^2\frac{\alpha}{2}}=2\sec\alpha=\frac{2b}{a}$$

7. **Solve (i)** $\cos\left(2\sec^{-1}x\right)=\frac{1}{9}$ **(ii)** $\cos^{-1}\frac{3}{5}-\sin^{-1}\frac{4}{5}=\cos^{-1}x$

(iii) $\sin^{-1}(1-x)+\sin^{-1}x=\cos^{-1}x$ (iv) $\sin\left\{\sin^{-1}\frac{1}{5}+\cos^{-1}x\right\}=1$

Solution:

Let $\sin^{-1}x=\alpha \Rightarrow x=\sin\alpha$

$$\cos\left(2\sin^{-1}x\right)=\frac{1}{9}\Rightarrow\cos^2\alpha=\frac{1}{9}\Rightarrow 1-2\sin^2\alpha=\frac{1}{9}$$

$$1-2x^2=\frac{1}{9}\Rightarrow 2x^2=\frac{8}{9}$$

$$x^2=\frac{4}{9}\Rightarrow x=\pm\frac{2}{3}$$

Verification: For $x=\frac{2}{3}$ $\sin^{-1}\frac{2}{3}=\alpha\Rightarrow\sin\alpha=\frac{2}{3}$

$$\cos\left\{2\sin^{-1}\frac{2}{3}\right\} = \cos^2\alpha = 1-2\sin^2\alpha = 1-2\times\frac{4}{9}=\frac{1}{9}$$

For $x=\frac{2}{3}$ $\cos\left\{2\sin^{-1}\left(\frac{2}{3}\right)\right\}=1-2\sin^2\beta=1-2\left(\frac{2}{3}\right)^2=\frac{1}{9}$

$\therefore$ The Equation gets satisfied for $x=\frac{2}{3},-\frac{2}{3}$

Solution :

(ii) $\cos^{-1}\frac{3}{5}-\sin^{-1}\frac{4}{5}=\cos^{-1}x$

$Let\ \cos^{-1}\frac{3}{5}=\alpha \qquad \sin^{-1}\frac{4}{5}=\beta$

$\cos\alpha=\frac{3}{5} \qquad \sin\beta=\frac{4}{5}$

Given $\alpha-\beta=\cos^{-1}x \Rightarrow x=\cos(\alpha-\beta)$

$x=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\frac{3}{5}\times\frac{3}{5}+\frac{4}{5}\times\frac{4}{5}=1\ x=1$

Solution :

(iii) $\sin^{-1}(1-x)+\sin^{-1}x=\cos^{-1}x \quad \therefore (1-x)=\cos\left(2\sin^{-1}x\right)$

$\therefore \sin^{-1}(1-x)=\cos^{-1}x-\sin^{-1}x \qquad Let\ \sin^{-1}x=\alpha \Rightarrow \sin\alpha=x$

$but\ \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x \qquad \therefore 1-x=\cos^2\alpha$

$\sin^{-1}(1-x)=\frac{\pi}{2}-\sin^{-1}x-\sin^{-1}x \qquad 1-x=1-2\sin^2\alpha \Rightarrow 1-x=1-2x^2$

$(1-x)=\sin\left\{\frac{\pi}{2}-2\sin^{-1}x\right\} \qquad 2x^2-x=0 \Rightarrow x=0=\frac{1}{2}$

Solution :

(iv) $\sin\left\{\sin^{-1}\frac{1}{5}+\cos^2 x\right\}=1 \Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}(1)$

$\cos^{-1}x\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$

$x=\cos\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{5}\right)=\sin\left(\sin^{-1}\frac{1}{5}\right)=\frac{1}{5}$

Long Answer Question

1 **Prove that** $2\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{5}{13}=\cos^{-1}\left(\frac{323}{325}\right)$

Let $\sin^{-1}\frac{3}{5}=\alpha \quad \cos^{-1}\frac{5}{13}=\beta$

$\sin\alpha=\frac{3}{5} \quad \cos\beta=\frac{5}{13}$

$\cos\alpha=\frac{4}{5} \quad \sin\beta=\frac{12}{13}$

$\sin^2\alpha=2\sin\alpha\cos\alpha=\frac{24}{25} \quad \cos^2\alpha=1-2\sin^2\alpha$

$=1-2\left(\frac{9}{25}\right)=\frac{7}{25}$

We know that $\cos(2\alpha-\beta)=\cos^2\alpha\cos\beta+\sin^2\alpha\sin\beta$

$=\frac{7}{25}\times\frac{5}{13}+\frac{24}{25}\times\frac{12}{13}=\frac{35+288}{325}=\frac{323}{325}$

Solution :

(ii) $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

Let $\sin^{-1}\frac{4}{5}=\alpha \;\tan^{-1}\frac{1}{3}=\beta$

$\sin\alpha=\frac{4}{5} \quad \tan\beta=\frac{1}{3}$

$\cos\alpha=\frac{3}{5} \quad \sin^2\beta=\frac{2\tan\beta}{1+\tan^2\beta}=\frac{\frac{2}{3}}{1+\frac{1}{9}}=\frac{2}{3}\times\frac{9}{10}=\frac{3}{5}$

$\cos^2\beta=\frac{4}{5}$

We know that $\cos(\alpha+2\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$=\frac{3}{4}\times\frac{4}{5}-\frac{4}{5}\times\frac{3}{5}=0$

$$\alpha + 2\beta = \frac{\pi}{2}$$

Solution :

(iii) $4\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{99} - \tan^{-1}\frac{1}{70} = \frac{\pi}{4}$

Let $\tan^{-1}\frac{1}{5} = \alpha$ $\qquad \tan^{-1}\frac{1}{99} = \beta$ $\qquad \tan^{-1}\frac{1}{70} = \delta$

$\tan\alpha = \frac{1}{5}$ $\qquad \tan\beta = \frac{1}{99}$ $\qquad \tan\delta = \frac{1}{70}$

$$Tan2\alpha = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{2}{5}\times\frac{25}{24} = \frac{5}{12} \qquad Tan4\alpha = \frac{2Tan2\alpha}{1-Tan^2 2\alpha} = \frac{2\times\frac{5}{12}}{1-\frac{25}{144}} = \frac{120}{119}$$

$$Tan(4\alpha+\beta) = \frac{Tan4\alpha + Tan\beta}{1-Tan4\alpha Tan\beta} = \frac{\frac{120}{119}-\frac{1}{99}}{1-\frac{120}{119}\times\frac{1}{94}} = \frac{\frac{11880+119}{119\times 99}}{11781-120} = \frac{11999}{11661}$$

$$Tan(4\alpha+\beta-\delta) = \frac{Tan(4\alpha+\beta)-Tan\delta}{1+\tan(4\alpha+\beta)Tan\delta} = \frac{\frac{11999}{11661}-\frac{1}{70}}{1+\frac{11999}{11661}\times\frac{1}{70}} = \frac{828269}{828269} = 1$$

$$\therefore 4\alpha+\beta-\delta = \frac{\pi}{4} \qquad \therefore 4\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{99} - \tan^{-1}\frac{1}{70} = \frac{\pi}{4}$$

2. **It** $\cos^{-1}p + \cos^{-1}q + \cos^{-1}r = \pi$ **prove that** $p^2+q^2+r^2+2pqr = 1$

Solution:

Let $\cos^{-1}p = \alpha$ $\qquad \cos^{-1}q = \beta$ $\qquad \cos^{-1}r = \delta$

$p = \cos\alpha$ $\qquad q = \cos\beta$ $\qquad r = \cos\delta$

Given $\alpha+\beta+\gamma = \pi$ $\qquad \cos(\alpha+\beta) = \cos(\pi-\gamma)$

$\cos\alpha\cos\beta - \sin\alpha\sin\gamma = \cos\gamma$

$$pq = -r + \sqrt{1-p^2}\ \sqrt{1-q^2} \Rightarrow pq + r = \sqrt{1-p^2}\sqrt{1-q^2}$$

Squaring on both such

$$p^2q^2 + r^2 + 2pqr = 1 - p^2 - q^2 + p^2q^2 \Rightarrow p^2 + q^2 + r^2 + 2pqr = 1$$

3. **If** $\sin^{-1}\left(\frac{2p}{1+p^2}\right) - \cos^{-1}\left(\frac{1-q^2}{1+q^2}\right) = Tan^{-1}\frac{2x}{1-x^2}$ **then prove that** $x = \frac{p-q}{1+pq}$

Solution: Let $p = Tan\alpha \qquad q = Tan\beta$

$$\therefore \sin^{-1}\left(\frac{2Tan\alpha}{1+Tan^2\alpha}\right) - \cos^{-1}\left(\frac{1-Tan^2\beta}{1+Tan^2\beta}\right) = Tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\sin^{-1}\left(\sin^2\alpha\right) = \cos^{-1}\left(\cos^2\beta\right) = Tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Let $x = Tan\gamma$

$$2\alpha - 2\beta = Tan^{-1}\left(\frac{2Tan\gamma}{1-Tan\gamma}\right)$$

$$2\alpha - 2\beta = 2\gamma$$

$$\tan\gamma = \frac{Tan\alpha - Tan\beta}{1+Tan\alpha Tan\beta} \Rightarrow \gamma = \frac{p-q}{1+pqr}$$

4. **If a, b, c are distinct non-zero real numbers having the same sign prove that**

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc-1}{b-c}\right) + \cot^{-1}\left(\frac{ac+1}{c-a}\right) = \pi \ \ or \ \ 2\pi$$

Solution: Since a, b, c have same sign two cases will arise

(i) Two of the three number a-b, b-c,c-a, are positive and there is negative

(ii) One of the number is positive and the other two are negative

Case(i) Suppose a-b>0 b-c>0 and c-a<0

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ac+1}{a-c}\right)$$

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \pi - \cot^{-1}\left(\frac{ac+1}{a-c}\right)$$

$$\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \pi - \tan^{-1}\left(\frac{a-c}{1+ac}\right) = \pi$$

$$\because Tan^{-1}\left(\frac{x-y}{1+xy}\right)=Tan^{-1}x-Tan^{-1}y$$

$$a-b<0,\ b-c<0 \ \ and \ \ c-a>0$$

Case (ii) Suppose

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ac+1}{a-c}\right)$$

$$=\cot^{-1}\left\{\frac{-(ab+1)}{b-a}\right\}+\cot^{-1}\left\{\frac{-(bc+1)}{c-b}\right\}+\cot^{-1}\left\{\frac{(ca+1)}{c-a}\right\}$$

$$=\pi-\cot^{-1}\left(\frac{ab+1}{b-a}\right)+\pi-\cot^{-1}\left(\frac{bc+1}{c-b}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$=2\pi-\tan^{-1}\left(\frac{b-a}{1+ab}\right)-\pi\tan^{-1}\left(\frac{c-b}{1+bc}\right)+\tan^{-1}\left(\frac{c-a}{1+ac}\right)$$

$$=2\pi-\tan^{-1}b+\tan^{-1}a-\tan^{-1}c+\tan^{-1}b+\tan^{-1}a \ = \ 2\pi$$

5. **If** $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$ **then prove that** ` $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$

Solution:

Let $\sin^{-1}x=\alpha \quad \sin^{-1}y=\beta \quad \sin^{-1}z=\gamma$

$\sin\alpha=x \quad \sin\beta=y \quad \sin\gamma=z$

Given $\alpha+\beta+\gamma=\pi\Rightarrow\alpha+\beta=\pi-\gamma$

$$\sin(\alpha+\beta)=\sin\gamma and \cos\gamma=-\cos(\alpha+\beta)$$

$$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}$$

$$\sin\alpha\sqrt{1-\sin^2\alpha}+\sin\beta\sqrt{1-\sin^2\beta}+\sin\gamma\sqrt{1-\sin^2\gamma}$$

$$\frac{1}{2}\left[\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right] \quad = \quad \frac{1}{2}\left[2\sin(\alpha+\beta)\cos(\alpha-\beta)+\sin^2\gamma\right]$$

$$\frac{1}{2}\left[2\sin\gamma\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\right]$$

$$\sin\gamma\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]=2\sin\alpha\sin\beta\sin\gamma=2xyz$$

6. **If (i)** $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\pi$ **then prove that** $x+y+z=xyz$

(ii) $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\frac{\pi}{2}$ then prove that $xy+yz+zx=1$

Solution: Let $\tan^{-1}x=\alpha \quad \tan^{-1}y=\beta \quad \tan^{-1}z=\gamma$

$x=Tan\alpha \quad y=Tan\beta \quad z=Tan\gamma$

Given $\alpha+\beta+\gamma=\pi \Rightarrow \alpha+\beta=\pi-x$

$Tan(\alpha+\beta)=Tan(\pi-\gamma)\Rightarrow \frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}=Tan\gamma$

$Tan\alpha+Tan\beta=-Tan\gamma+Tan\alpha Tan\beta Tan\gamma \Rightarrow Tan\alpha+Tan\beta+Tan\gamma\ Tan\alpha Tan\beta Tan\gamma$

$=x+y+z\ =xyz$

Solution :

(ii) $\alpha+\beta+\gamma=\frac{\pi}{2} \Rightarrow \alpha+\beta=\frac{\pi}{2}-\gamma$

$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\cot\gamma \Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1}{\tan\gamma}$

$\tan\alpha\tan\gamma+\tan\beta\tan\gamma=1-\tan\alpha\tan\beta$

$\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$

$xy+yz+zx=1$

7. **If** $\alpha=Tan^{-1}\frac{\sqrt{1-x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$ **then prove that** $x^2=\sin^2\alpha$

Solution:

$\tan\alpha\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} \Rightarrow \frac{1}{\tan\alpha}=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

Using componendo and dividend we have

$\frac{1+\tan\alpha}{1-\tan\beta}=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}+\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}-\sqrt{1+x^2}+\sqrt{1-x^2}}$

$$\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha} = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} \Rightarrow \text{ squaring on both sloves}$$

$$\frac{\cos^2+\sin^2\alpha+2\cos\alpha\sin\alpha}{\cos^2+\sin^2\alpha-2\cos\alpha\sin\alpha}+\frac{1+x^2}{1-x^2} \Rightarrow \frac{1+\sin 2\alpha}{1-\sin 2\alpha}=\frac{1+x^2}{1-x^2}$$

$$\frac{1+\sin 2\alpha+1-\sin 2\alpha}{1+\sin 2\alpha-1+\sin 2\alpha}=\frac{1+x^2+1-x^2}{1+x^2-1+x^2} \Rightarrow \frac{2}{2\sin^2\alpha}=\frac{2}{2\sin^2}$$

$$\Rightarrow x^2=\sin 2\alpha$$

8. **Prove that** $\cos^{-1}\left\{\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right\}=2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right\}$

$R.H.S\ \ 2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right\}$ $\quad Let\ \tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right\}=\theta$

$$\therefore \tan\theta=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$R.H.S=2\theta$ We know that $\cos 2\theta\frac{1-Tan^2\theta}{1+Tan^2\theta}$

$$\therefore \cos 2\theta=\frac{1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}} = \frac{\dfrac{\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}}}{\dfrac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\cdot\dfrac{\sin^2\frac{\beta}{2}}{\cos^2\frac{\beta}{2}}}$$

$$=\frac{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}-\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}=\frac{\left(\frac{1+\cos\alpha}{2}\right)\left(\frac{1+\cos\beta}{2}\right)-\left(\frac{1-\cos\alpha}{2}\right)\left(\frac{1-\cos\beta}{2}\right)}{\left(\frac{1+\cos\alpha}{2}\right)\left(\frac{1+\cos\beta}{2}\right)+\left(\frac{1-\cos\alpha}{2}\right)\left(\frac{1-\cos\beta}{2}\right)}$$

$$\cos 2\theta=\frac{1+\cos\alpha+\cos\beta+\cos\alpha\cos\beta-1+\cos\alpha+\cos\beta-\cos\alpha\cos\beta}{1+\cos\alpha+\cos\beta+\cos\alpha\cos\beta+1-\cos\alpha-\cos\beta+\cos\alpha\cos\beta}=\frac{\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}$$

$$\cos 2\theta=\frac{\cos\alpha+\cos\beta}{1+\cos\beta}\Rightarrow 2\theta=\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right)$$

$$\therefore 2Tan^{-1}\left(Tan\frac{\alpha}{2}Tan\frac{\beta}{2}\right)=\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right)$$

Solve the following Equations for x

(i) $$Tan^{-1}\left(\frac{x-1}{x-2}\right)+Tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

Let $Tan^{-1}\left(\frac{x-1}{x-2}\right)=\alpha$ $\qquad Tan^{-1}\frac{x+1}{x+2}=\beta$

$$Tan\alpha=\frac{x-1}{x-2}\qquad Tan\beta=\frac{x+1}{x+2}$$

Given that $\alpha+\beta=\frac{\pi}{4}\Rightarrow\frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}=1$

$$\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{(x^2-1)}{(x^2-4)}}=1\Rightarrow\frac{(x-1)(x+2)+(x+1)(x-2)}{x^2-4-x^2+1}=1$$

$$\frac{x^2+x-2+x^2-x-2}{-3}=1\ \Rightarrow\ 2x^2-4=-3$$

$$2x^2=1\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

ii) $$Tan^{-1}\left(\frac{1}{2x+1}\right)+Tan^{-1}\left(\frac{1}{4x+1}\right)=Tan^{-1}\left(\frac{2}{x^2}\right)$$

$$Tan^{-1}\frac{1}{2x+1}=\alpha\Rightarrow\quad Tan\alpha=\frac{1}{2x+1}$$

$$Tan^{-1}\left(\frac{1}{4x+1}\right)=\beta\Rightarrow Tan\beta=\frac{1}{4x+1}$$

$$\alpha+\beta=Tan^{-1}\frac{2}{x^2}$$

$$\frac{Tan\alpha+Tan\beta}{1-Tan\alpha Tan\beta}=\frac{2}{x^2}$$

$$\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{(2x+1)(4x+1)}}=\frac{2}{x^2}$$

$$\frac{4x+1+2x+1}{8x^2+6x+x-y}=\frac{2}{x^2}$$

$$6x^3+2x^2=16x^2+12x$$

$$6x^3-14x^2-12x=0$$

$$2x\{3x^2-7x-6\}=0$$

$$2x\{3x^2-9x+2x-6\}=0$$

$$2x(3x+2)(x-3)=0$$

$$1-\frac{5x^2}{4}=\frac{1}{4}-\frac{x^2}{2}\Rightarrow 1-\frac{1}{4}=\frac{5x^2}{4}-\frac{x^2}{2}$$

$$\frac{3}{4}=\frac{5x^2-2x^2}{4}\Rightarrow x=\frac{1}{\sqrt{3}}$$

$$Tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+Tan^{-1}\sqrt{\frac{b(a+b+c)}{ac}}+Tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi$$

$$x=\sqrt{\frac{a(a+b+c)}{bc}\times\frac{b(a+b+c)}{ac}}\Rightarrow xy=\frac{a+b+c}{c}>1$$

$$x>0:y>0 \ and \ \ xy>1$$

$$\therefore Tan^{-1}x+Tan^{-1}y=\pi+Tan^{-1}\left(\frac{x-y}{1-xy}\right)$$

$$\therefore Tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\sqrt{\frac{b(a+b+c)}{ac}}=\pi+Tan^{-1}\left\{\frac{\sqrt{\frac{a(a+b+c)}{bc}\times\frac{b(a+b+c)}{ac}}}{1-\frac{a+b+c}{c}}\right\}$$

$$=\pi+Tan^{-1}\left\{\frac{\sqrt{\frac{a(a+b+c)}{bc}\times\frac{b(a+b+c)}{ac}}}{1-\frac{a-b-c}{c}}\right\}$$

$$=\pi+Tan^{-1}\frac{\left(\sqrt{(a+b+c)}(a+b)\right)}{\sqrt{ab}\sqrt{c}}\times\frac{c}{-(a-b)}$$

$$=\pi+Tan^{-1}\left\{-\frac{\sqrt{c}\sqrt{(a+b+c)}}{\sqrt{ab}}\right\}$$

$$=\pi-Tan^{-1}\left\{-\frac{\sqrt{c(a+b+c)}}{\sqrt{ab}}\right\}$$

$$\therefore Tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+Tan^{-1}\sqrt{\frac{b(a+b+c)}{ac}}=\pi-Tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$Tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+Tan^{-1}\sqrt{\frac{b(a+b+c)}{ac}}+Tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi$$

(iii) $3\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$

Let $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$3\sin^{-1}-\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}-4\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)+2\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^1\theta}\right)=\frac{\pi}{3}$$

$$3\sin^{-1}-\{\sin 2\theta\}-4\cos^{-1}(\cos 2\theta)+2\tan^{-1}(\tan 2\theta)=\frac{\pi}{3}$$

$$2\theta=\frac{\pi}{3}\Rightarrow \theta=\frac{\pi}{6}\Rightarrow \tan^{-1}=\frac{\pi}{6}\Rightarrow x=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

(iv) $\tan\left\{arc\cos\frac{1}{x}\right\}=\sin\left\{arc\cot\frac{1}{2}\right\}$

Let $arc\cos\frac{1}{x}$ i.e $\cos\frac{1}{x}=\alpha\Rightarrow\cos\alpha=\frac{1}{x}$

$\cot^{-1}\frac{1}{2}=\beta\Rightarrow \quad \cot\beta=\frac{1}{2}$

$\therefore \tan\alpha=\sin\beta$

$\frac{1-x^2}{x^2}=\frac{4}{5}$

$\sqrt{x^2-1}=\frac{2}{v5}\Rightarrow x^2-1=\frac{4}{5}\Rightarrow x^2=\frac{9}{5}=x=\frac{3}{v5}$

$x=-\frac{3}{v5}$ does not satisfy the equation

(v) $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2} \quad\Rightarrow\quad \sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$

$(1-x)=\sin\left\{\frac{\pi}{2}+2\sin^{-1}x\right\}$

$(1-x)=\cos\left(2\sin^{-1}x\right)$ Let $\sin^{-1}x=\alpha\Rightarrow x-\sin\alpha$

$1-x=\cos^2\alpha\Rightarrow 1-x=1-2x^2$

$2x^2-x=0\Rightarrow x(x-1)=0\Rightarrow x=0=\frac{1}{2}$

(vi) $\cos^{-1}x+\sin^{-1}\frac{x}{2}=\frac{\pi}{6}$

Let $\cos^{-1}x=\alpha$ $\quad \sin^{-1}\frac{x}{2}=\beta$

$\cos\alpha=x$ $\quad \sin\beta=\frac{x}{2}$

$\alpha+\beta=\frac{\pi}{6}$ $\quad \beta=\frac{\sqrt{3}}{2}$

$\sin(\alpha+\beta)=\sin\frac{\pi}{6}$

$\sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{1}{2}$

$\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}+x\times\frac{x}{2}=\frac{1}{2}$

$\left(\sqrt{1-x^2}\right)\left(1-\frac{x^2}{4}\right)=\left(\frac{1}{2}\times\frac{x^2}{2}\right)^2$

$1-x^2-\frac{x^2}{4}+\frac{x^4}{4}=\frac{1}{4}+\frac{x^4}{4}-\frac{x^2}{2}$

11.If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$ **then prove that** $x^4+y^4+z^4+4x^2y^2z^2=2\left\{x^2y^2+y^2z^2+z^2x^2\right\}$

Let $\sin^{-1}x=\alpha$ $\quad \sin^{-1}y=\beta$ $\quad \sin^{-1}z=\gamma$

$\sin\alpha=x$ $\quad \sin\beta=y$ $\quad \sin\gamma=z$

$\alpha+\beta+\gamma=\pi\Rightarrow\cos(\alpha+\beta)=\cos(\pi-\gamma)$

$\sqrt{1-x^2}\sqrt{1-y^2}-xy=-\sqrt{1-z^2}$

$\sqrt{1-x^2}\sqrt{1-y^2}=xy-\sqrt{1-z^2}$

$(1-x)^2(1-y)^2=x^2y^2+1-z^2-2xy\sqrt{1-z^2}$

$1-x^2-y^2+x^2y^2=x^2y^2+1-z^2-2xy\sqrt{1-z^2}$

Squaring on both sides we have

$z^2-x^2-y^2=-2xy\sqrt{1-z^2}$

$z^4+x^4+y^4-2x^2z^2+2x^2y^2-2y^2z^2=4x^2y^2(1-z)^2$

$x^4+y^4+z^4-4x^2y^2z^2=2x^2y^2+2y^2z^2+2x^2z^2$

$\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$ then prove that $\frac{x^2}{az}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

Let $\cos^{-1}\frac{x}{a}=\theta$ *and* $\cos^{-1}\frac{y}{b}=\phi$

$\cos\theta=\frac{x}{a}\quad \cos\phi=\frac{y}{b}=\theta$

Given $\theta+\phi=\alpha$

$\cos(\theta+\phi)=\cos\alpha\Rightarrow\cos\theta\cos\phi-\sin\theta\sin\phi$

$$\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\ \sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}}\ \sqrt{1-\frac{y^2}{b^2}}$$

Squaring on both sides we have

$$\frac{x^2y^2}{a^2b^2}+\cos^2\alpha=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha$$

$$\frac{x^2y^2}{a^2b^2}+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2y^2}$$

$$\frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha$$

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha$$

$$\therefore\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\sin^2\alpha$$

PROBLEMS FOR PRACTICE

1. **Find the value of** $\cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3}$

2. **Prove that** $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{117}{125}$

3. **Prove that** $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{25}=\frac{\pi}{2}$

4. **Prove that** $\cot^{-1}9+\cos ec^{-1}\frac{\sqrt{41}}{4}=\frac{\pi}{4}$

5. **Show that** $\cot\left\{\sin^{-1}\sqrt{\frac{13}{17}}\right\}=\sin\left\{\tan^{-1}\frac{2}{3}\right\}$

6. **Prove that** $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

7. **Prove that** $\cos\left\{2\tan^{-1}\frac{1}{7}\right\}=\sin\left\{4\tan^{-1}\frac{1}{3}\right\}$

8. **Solve** $\sin\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}(x>0)$

9. **Solve** $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$

10. **Solve** $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$

11. **If** $\sin\left[2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]=0$

12. **Prove that** $\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]=\sqrt{\frac{x^2+1}{x^2+2}}$