Multiple and Sub Multiple Angles

If A is an angle, then 2A, 3A, 4A, ….. etc. are called multiple angles of A and A/2, A/3, ….. etc. are called sub-multiple angles of A.

Formulae :

i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$.

ii) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = \dfrac{2\tan A/2}{1+\tan^2 A/2}$.

iii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

iv) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1$

$= 1 - 2\sin^2\dfrac{A}{2} = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

v) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

vi) $\tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2} = \operatorname{cosec} 2A - \cot 2A$

vii) $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

viii) $\cot A = \dfrac{\cot^2 (A/2) - 1}{2\cot A/2} = \operatorname{cosec} 2A + \cot 2A$

ix) $\tan\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right) = \dfrac{1+\tan A/2}{1-\tan A/2}$

$= \dfrac{\cos A/2 + \sin A/2}{\cos A/2 - \sin A/2} = \dfrac{1+\sin A}{\cos A} = \dfrac{\cos A}{1-\sin A}$

$= \sec A + \tan A = \cot\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right)$.

x) $\tan\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right) = \dfrac{1-\tan A/2}{1+\tan A/2}$

$= \dfrac{\cos A/2 - \sin A/2}{\cos A/2 + \sin A/2} = \sqrt{\dfrac{1-\sin A}{1+\sin A}} = \dfrac{1-\sin A}{\cos A}$

$= \sec A - \tan A = \cot\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right)$

II. i) $\sin 3A = 3\sin A - 4\sin^3 A$

ii) $\cos 3A = 4\cos^3 A - 3\cos A$

iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

iv) $\sin^3 A = \dfrac{1}{4}[3\sin A - \sin 3A]$

v) $\cos^3 A = \dfrac{1}{4}[\cos 3A + 3\cos A]$

III. $\sin^2 A = \frac{1-\cos 2A}{2}$; $\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}$

$\cos^2 A = \frac{1+\cos 2A}{2}$; $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$

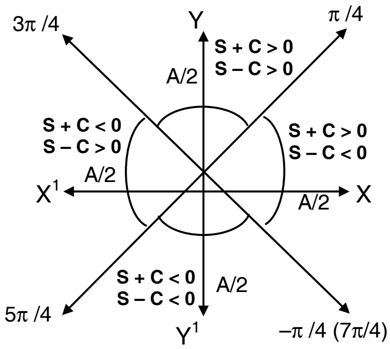
$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$; $\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$

$\sin^2 \frac{A}{2} = \frac{1-\cos A}{2}$; $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

$\cos^2 \frac{A}{2} = \frac{1+\cos A}{2}$; $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

$\tan^2 \frac{A}{2} = \frac{1-\cos A}{1+\cos A}$; $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

IV. If $S = \sin\frac{A}{2}$ and $C = \cos\frac{A}{2}$ then



V) $S + C = \pm\sqrt{1+\sin A}$

ii) $S - C = \pm\sqrt{1-\sin A}$

iii) $2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm\sqrt{1-\sin A}$

iv) $2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm\sqrt{1-\sin A}$

v) a) $S + C > 0$, $S - C > 0$ if

$\frac{\pi}{4} < \frac{A}{2} < \frac{3\pi}{4}$

b) $S + C < 0$, $S - C > 0$ if

$\frac{3\pi}{4} < \frac{A}{2} < \frac{5\pi}{4}$

c) $S + C < 0$, $S - C < 0$ if

$\frac{5\pi}{4} < \frac{A}{2} < \frac{7\pi}{4}$

d) $S + C > 0$, $S - C < 0$ if

$\frac{-\pi}{4} < \frac{A}{2} < \frac{\pi}{4}$

VI.

	18°	36°	54°	72°
Sin	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$
Cos	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$

VSAQ'S

1. Simplify $\frac{\sin 2\theta}{1+\cos 2\theta}$

Sol. $\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1}$

$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

2. Evaluate $\sin^2 42^\circ - \sin^2 12^\circ$**.**

Sol. $\sin(42^\circ+12^\circ)\sin(42^\circ-12^\circ)$

$$= \sin 54^\circ \sin 30^\circ$$

$$= \frac{\sqrt{5}+1}{4}\cdot\frac{1}{2} = \frac{\sqrt{5}+1}{8}$$

3. Express $\frac{1-\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta}$ **in terms of** $\tan\frac{\theta}{2}$**.**

Sol. $\frac{1-\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$

$$= \frac{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\left(1-2\sin^2\frac{\theta}{2}\right)}{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}-1}$$

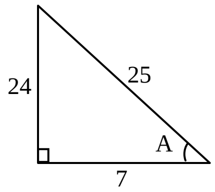
$$= \frac{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+2\sin^2\frac{\theta}{2}-1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \tan\frac{\theta}{2}$$

4. If $\cos A = \frac{7}{25}$ **and** $\frac{3\pi}{2} < A < 2\pi$**, then find the value of** $\cot\frac{A}{2}$**.**

Sol. $\cos A = \frac{7}{25}$, where $\frac{3\pi}{2} < A < 2\pi$



$\sin A = -\frac{24}{25}, \tan A = -\frac{24}{7}, \cos A = \frac{7}{25}$

$\cot\frac{A}{2} = \frac{\sin A}{1-\cos A} = \frac{-\frac{24}{25}}{1-\frac{7}{25}}$

$= \frac{-24}{25} \times \frac{25}{18} = \frac{-24}{18} \times \frac{-4}{3} = \frac{16}{9}$

5. If $0 < \theta < \frac{\pi}{8}$**, show that** $\sqrt{2+\sqrt{2+2+2\cos 4\theta}} = 2\cos\frac{\theta}{2}$**.**

Sol. $\sqrt{2+\sqrt{2+2+2\cos 4\theta}}$

$1+\cos 4\theta = 2\cos^2 2\theta$

$2(1+\cos 4\theta) = 4\cos^2 2\theta$

$\sqrt{2(1+\cos 4\theta)} = 2\cos 2\theta$

$2+\sqrt{2(1+\cos 4\theta)} = 2+2\cos 2\theta$

$= 2(1+\cos 2\theta) = 2(2\cos^2\theta) = 4\cos^2\theta$

$\sqrt{2+\sqrt{2(1+\cos 4\theta)}} = \sqrt{\cos^2\theta} = 2\cos\theta$

$2+\sqrt{2+\sqrt{2(1+\cos 4\theta)}} = 2+2\cos\theta$

$= 2(1+\cos\theta) = 2\left(2\cos^2\frac{\theta}{2}\right) = 4\cos^2\frac{\theta}{2}$

6. Prove that $\frac{\cos 3A + \sin 3A}{\cos A - \sin A} = 1+2\sin 2A$**.**

Sol. L.H.S. $= \frac{\cos 3A + \sin 3A}{\cos A - \sin A}$

$$= \frac{4\cos^3 A - 3\cos A + 3\sin A - 4\sin^3 A}{\cos A - \sin A}$$

$$= \frac{4(\cos^3 A - \sin^3 A) - 3(\cos A - \sin A)}{\cos A - \sin A}$$

$$= \frac{4[(\cos A - \sin A)(\cos^2 A + \cos A \sin A + \sin^2 A)] - 3(\cos A - \sin A)}{\cos A - \sin A}$$

$$= \frac{(\cos A - \sin A)[(4 + 4\sin A\cos A) - 3]}{(\cos A - \sin A)}$$

$$= 1 + 4\sin A\cos A$$

$$= 1 + 2\sin 2A = \text{R.H.S.}$$

7. Prove that $\cot\left(\frac{\pi}{4} - \theta\right) = \frac{\cos 2\theta}{1 - \sin 2\theta}$ **and hence find the value of cot 15°.**

Sol. R.H.S. = $\frac{\cos 2\theta}{1 - \sin 2\theta}$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$$

$$= \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)^2}$$

$$= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{\sin\theta\left[\frac{\cos\theta}{\sin\theta} + 1\right]}{\sin\theta\left[\frac{\cos\theta}{\sin\theta} - 1\right]}$$

$$= \frac{\cot\theta + 1}{\cot\theta - 1}$$

$$= \frac{\cot\theta \cdot \cot\frac{\pi}{4} + 1}{\cot\theta - \cot\frac{\pi}{4}}$$

$$= \cot\left(\frac{\pi}{4} - \theta\right) = \text{L.H.S.}$$

Put θ = 30° ⇒ cot 15° = $\frac{\cos 60°}{1 - \sin 60°}$

$$= \frac{\frac{1}{2}}{1 - \frac{\sqrt{3}}{2}} = \frac{1}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{2 + \sqrt{3}}{4 - 3} = 2 + \sqrt{3}$$

8. Prove that $\dfrac{1}{\cos 290^\circ} + \dfrac{1}{\sqrt{3}\sin 250^\circ} = \dfrac{4}{\sqrt{3}}$**.**

SOL

$$\cos 290^\circ = \cos(270^\circ + 20^\circ) = \sin 20^\circ$$

$$\sin 250^\circ = \sin(270^\circ - 20^\circ) = \cos 20^\circ$$

$$\text{L.H.S.} = \frac{1}{\sin 20^\circ} - \frac{1}{\sqrt{3}\cos 20^\circ} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sqrt{3}\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\frac{\sqrt{3}}{2}(2\sin 20^\circ \cos 20^\circ)} = \frac{4}{\sqrt{3}}\left[\frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 40^\circ}\right]$$

$$= \frac{4}{\sqrt{3}}\frac{\sin(60^\circ - 20^\circ)}{\sin 40^\circ} = \frac{4}{\sqrt{3}} = \text{R.H.S.}$$

9. Prove that $\dfrac{\sin 2x}{(\sec x + 1)} \cdot \dfrac{\sec 2x}{(\sec 2x + 1)} = \tan\left(\dfrac{x}{2}\right)$**.**

Sol. L.H.S. = $\dfrac{\sin 2x}{(\sec x + 1)} \cdot \dfrac{\sec 2x}{(\sec 2x + 1)}$

$$= \frac{\sin 2x}{\frac{1}{\cos x} + 1} \times \frac{\frac{1}{\cos 2x}}{\frac{1}{\cos x} + 1}$$

$$= \frac{\sin 2x}{\frac{1 + \cos x}{\cos x}} \times \frac{1}{\cos 2x} \times \frac{\cos 2x}{1 + \cos 2x}$$

$$= \frac{\sin 2x \cdot \cos x}{1 + \cos x} \cdot \frac{1}{1 + \cos 2x}$$

$$= \frac{2\sin x \cos^2 x}{1 + \cos x} = \frac{1}{2\cos^2 x}$$

$$= \frac{\sin x}{1 + \cos x} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1 + 2\cos^2\frac{x}{2} - 1}$$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2} = \text{R.H.S.}$$

10. If $\cos\alpha = \frac{3}{5}$ *and* $\cos\beta = \frac{5}{13}$ α, β **are acute angles then**

Prove that (i) $\sin^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{65}$ (ii) $\cos^2\left(\frac{\alpha+\beta}{2}\right) = \frac{16}{65}$

Solution:

$$\cos\alpha = \frac{3}{5} \quad \cos\beta = \frac{5}{13}$$

$$\sin\alpha = \frac{3}{5} \quad \sin\beta = \frac{5}{13}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{3}{4}\times\frac{5}{13} + \frac{4}{5}\times\frac{12}{13} = \frac{63}{65}$$

$$2\sin^2\left(\frac{\alpha-\beta}{2}\right) - 1 - \cos(\alpha-\beta) = 1 - \frac{63}{65} \Rightarrow 2\sin^2\left(\frac{\alpha-\beta}{2}\right) = \frac{2}{65}$$

$$\therefore \sin^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{65}$$

$$\cos^2\left(\frac{\alpha+\beta}{2}\right) = \frac{1+\cos(\alpha+\beta)}{2} \Rightarrow 2\cos^2\left(\frac{\alpha+\beta}{2}\right) = 1 - \frac{33}{65}$$

$$2\cos^2\left(\frac{\alpha+\beta}{2}\right) = \frac{32}{65} \Rightarrow \cos^2\left(\frac{\alpha+\beta}{2}\right) = \frac{16}{65}$$

11. Show that $\cos A = \frac{\cos 3A}{(2\cos 2A - 1)}$ **. Hence find the value of cos 15°.**

Sol. R.H.S. = $\frac{\cos 3A}{(2\cos 2A - 1)}$

$$= \frac{4\cos^2 A - 3\cos A}{2(2\cos^2 A - 1) - 1}$$

$$= \frac{\cos(4\cos^2 A - 3)}{(4\cos^2 A - 3)} = \cos A = \text{L.H.S.}$$

$$\cot 15° = \cot(45° - 30°)$$

$$= \frac{\cot 45° \cdot \cot 30° + 1}{\cot 30° - \cot 45°}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2}{3-1}$$

$$= \frac{3+1+2\sqrt{3}}{2} = \frac{4+2\sqrt{3}}{2}$$

$$= \frac{2[2+\sqrt{3}]}{2} = 2+\sqrt{3}$$

$$\therefore \cot 15° = 2+\sqrt{3}$$

12. Show that $\cos A = \frac{\sin 3A}{1+2\cos 2A}$. **Hence find the value of sin 15°.**

Sol. R.H.S. = $\frac{\sin 3A}{1+2\cos 2A}$

$$= \frac{3\sin A - 4\sin^3 A}{1+2\left(1-2\sin^2 A\right)}$$

$$= \frac{\sin[3-4\sin^2 A]}{[1+2-4\sin^2 A]}$$

$$= \frac{\sin A[3-4\sin^2 A]}{[3-4\sin^2 A]}$$

$$= \sin A = \text{L.H.S.}$$

$$\sin 15° = \frac{\sin 45°}{1+2\cos 30°}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1+2\cdot\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{2}(1+\sqrt{3})} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{2}(3-1)} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

13. Prove that $\tan\alpha = \frac{\sin 2\alpha}{1+\cos 2\alpha}$ **and hence deduce the values of tan 15° and tan** $22\frac{1}{2}°$.

Sol. R.H.S. $= \frac{\sin 2\alpha}{1+\cos 2\alpha}$

$$= \frac{2\sin\alpha\cos\alpha}{1+2\cos^2\alpha-1} = \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{L.H.S.}$$

$$\alpha = 15° \Rightarrow \tan 15° = \frac{\sin 30°}{1+\cos 30°}$$

$$= \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}} = \frac{1}{2}\times\frac{2}{2+\sqrt{3}}$$

$$= \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

$$\alpha = 22\frac{1}{2}^\circ \Rightarrow \tan 22\frac{1}{2}^\circ = \frac{\sin 45^\circ}{1+\cos 45^\circ}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$$

14. Prove that $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.

Sol. L.H.S. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ}$

$$= \frac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$$

$$= \frac{2\left(\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ\right)}{\frac{1}{2}(2\sin 10^\circ \cos 10^\circ)}$$

$$= 4\frac{[\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ]}{\sin 20^\circ}$$

$$= 4\frac{\sin(30^\circ - 10^\circ)}{\sin 20^\circ}$$

$$= 4\frac{\sin 20^\circ}{\sin 20^\circ} = 4 = \text{R.H.S.}$$

15. Prove that $\sqrt{3}\csc 20^\circ - \sec 20^\circ = 4$

Sol. L.H.S. = $\sqrt{3}\csc 20^\circ - \sec 20^\circ$

$$= \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\cdot\left(\frac{\sqrt{3}}{2}\sin 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\frac{1}{2}(2\sin 20^\circ \cos 20^\circ)}$$

$$= \frac{4(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{\sin 40^\circ}$$

$$= 4\frac{\sin(60^\circ - 20^\circ)}{\sin 40^\circ} = 4\cdot\frac{\sin 40^\circ}{\sin 40^\circ}$$

$$= 4 = \text{R.H.S.}$$

16. $\tan 9° - \tan 27° - \cot 27° + \cot 9° = 4$.

Sol. Consider,

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1}{\sin A \cos A}$$

$$= \frac{2}{\sin 2A} = 2\csc 2A$$

$$\tan 81° = \tan(90° - 9°) = \cot 9°$$

$$\tan 63° = \tan(90° - 27°) = \cot 27°$$

$$A = 9° \Rightarrow \tan 9° + \cot 9° = 2\csc 18°$$

$$A = 27° \Rightarrow \tan 27° + \cot 27° = 2\csc 54°$$

$$\text{L.H.S.} = 2(\csc 17° - \csc 54°)$$

$$= 2\left(\frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}\right)$$

$$= 2 \times 4\left(\frac{1}{\sqrt{5}-1} - \frac{1}{\sqrt{5}+1}\right)$$

$$= 8\left(\frac{\sqrt{5}+1-\sqrt{5}+1}{5-1}\right)$$

$$= \frac{8 \times 2}{4} = 4 = \text{R.H.S.}$$

SAQ'S

17. If $\frac{\sin\alpha}{a}=\frac{\cos\alpha}{b}$, then prove that $a\sec 2\alpha + b\cos 2\alpha = b$.

Sol. Given that $\frac{\sin\alpha}{a}=\frac{\cos\alpha}{b}$

$$\frac{\sin\alpha}{\cos\alpha}=\frac{a}{b}$$

$$\therefore \tan\alpha = \frac{a}{b}$$

L.H.S. = $a\sec 2\alpha + b\cos 2\alpha$

$$= a\left[\frac{2\tan\alpha}{1+\tan^2\alpha}\right]+b\left[\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right]$$

$$= a\left[\frac{2\times\frac{a}{b}}{1+\left(\frac{a}{b}\right)^2}\right]+b\left[\frac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2}\right]$$

$$=\left[\frac{\frac{2a^2}{b}}{\frac{b^2+a^2}{b^2}}\right]+b\left[\frac{\frac{b^2-a^2}{b^2}}{\frac{b^2+a^2}{b^2}}\right]$$

$$=\frac{2a^2b}{a^2+b^2}+\frac{b(b^2-a^2)}{a^2+b^2}$$

$$=\frac{2a^2b+b^3-ba^2}{a^2+b^2}$$

$$=\frac{b^3+a^2b}{a^2+b^2}=\frac{b(b^2+a^2)}{a^2+b^2}=b=\text{R.H.S.}$$

18. In a ΔABC, if $\tan\frac{A}{2}=\frac{5}{6}$ and $\tan\frac{B}{2}=\frac{20}{37}$, then show that $\tan\frac{C}{2}=\frac{2}{5}$.

Sol. A + B + C = 180°

$$\frac{A+B}{2}=\frac{180^\circ - C}{2}$$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(90^\circ-\frac{C}{2}\right)$$

$$\Rightarrow \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\frac{C}{2}$$

$$\Rightarrow \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{C}{2}$$

$$\Rightarrow \frac{\frac{5}{6}+\frac{20}{37}}{1-\frac{5}{6}\cdot\frac{20}{37}}=\cot\frac{C}{2}$$

$$\Rightarrow \frac{\frac{185+120}{222}}{\frac{222-100}{222}}=\cot\frac{C}{2}$$

$$\Rightarrow \frac{305}{122}=\frac{1}{\tan(C/2)}$$

$$\Rightarrow \tan\frac{C}{2}=\frac{122}{305}$$

$$\Rightarrow \tan\frac{C}{2}=\frac{2\times 61}{5\times 61}$$

$$\therefore \tan\frac{C}{2}=\frac{2}{5}$$

19. If α, β are the solution of the $a\cos\theta+b\sin\theta=c$ then prove that

$\cos\alpha\cos\beta=\frac{2ac}{a^2+b^2}$ **and (ii)** $\cos\alpha\cos\beta=\frac{c^2-b^2}{a^2+b^2}$

Solution:

$b\sin\theta=c-a\cos\theta \Rightarrow b^2\sin^2\theta=c^2+a^2\cos^2\theta-2ac\cos\theta+\left(c^2-b^2\right)=0$

Since α, β are solution $\cos\alpha, \cos\beta$ are roots

$\therefore \cos\alpha\cos\beta=\frac{2ac}{a^2+b^2}$ $\qquad$ $\cos\alpha.\cos\beta=\frac{c^2-b^2}{a^2+b^2}$

(Prove that $\sin\alpha+\sin\beta=\frac{2bc}{a^2+b^2}$ $\quad \sin\alpha\sin\beta=\frac{c^2-a^2}{a^2+b^2}$ TRY YOUR SELF)

20. If $\cos\theta = \frac{5}{13}$ and 270° < θ < 360°, evaluate sin(θ/2) and cos(θ/2).

Sol. $\cos\theta = \frac{5}{13}$ where 270° < θ < 360°

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{5}{13}}{2}}$$

$$= \sqrt{\frac{13-5}{2\times 13}} = \sqrt{\frac{8}{2\times 13}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}}$$

$$= \sqrt{\frac{13+5}{2\times 13}} = \sqrt{\frac{18}{2\times 13}} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}}$$

Since θ lies in IV quadrant and θ/2 lies in II quadrant.

Hence, $\sin\frac{\theta}{2} = \frac{2}{\sqrt{13}}$ and $\cos\frac{\theta}{2} = -\frac{3}{\sqrt{13}}$.

21. Prove that $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2$.

Sol. L.H.S. = $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8}$

$$= \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\left(\pi - \frac{3\pi}{8}\right) + \cos^2\left(\pi - \frac{\pi}{8}\right)$$

$$= \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{\pi}{8}$$

$$= 2\left(\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8}\right)$$

$$= 2\left(\cos^2\frac{\pi}{8} + \cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right)$$

$$= 2\left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right) = 2(1) = 2 = \text{R.H.S.}$$

22. If $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$**, show that tan 3x = 1.**

Sol. Consider,

$$\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$$

$$\tan x+\frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}+\frac{\tan x+\tan\frac{2\pi}{3}}{1-\tan x\tan\frac{2\pi}{3}}=3$$

$$\Rightarrow \tan x+\frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}+\frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}=3$$

$$\Rightarrow \tan x+\frac{(\tan x+\sqrt{3})(1+\sqrt{3}\tan x)+(\tan x-\sqrt{3})(1-\sqrt{3}\tan x)}{1-3\tan^2 x}=3$$

$$\Rightarrow \tan x+\frac{\tan x+\sqrt{3}\tan^2 x+\sqrt{3}+3\tan x+\tan x-\sqrt{3}\tan^2 x-\sqrt{3}+3\tan x}{1-3\tan^2 x}=3$$

$$\Rightarrow \tan x+\frac{8\tan x}{1-3\tan^2 x}=3$$

$$\Rightarrow \frac{\tan x(1-3\tan^2 x)+8\tan x}{1-3\tan^2 x}=3$$

$$\Rightarrow \frac{\tan x-3\tan^3 x+8\tan x}{1-3\tan^2 x}=3$$

$$\Rightarrow \frac{9\tan x-3\tan^3 x}{1-3\tan^2 x}=3$$

$$\Rightarrow \frac{3(3\tan x-\tan^3 x)}{1-3\tan^2 x}=3$$

$$\Rightarrow \frac{3\tan x-\tan^3 x}{1-3\tan^2 x}=1\Rightarrow \tan 3x=1$$

23. Prove that $\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5}\cdot\sin\frac{3\pi}{5}\cdot\sin\frac{4\pi}{5}=\frac{5}{16}$

Sol. L.H.S. = $\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5}\cdot\sin\frac{3\pi}{5}\cdot\sin\frac{4\pi}{5}$

$=\sin 36°\cdot\sin 72°\cdot\sin 108°\cdot\sin 144°$

$=\sin 36°\cdot\sin(90°-18°)\sin(90°+18°)\sin(180°-36°)$

$=\sin 36°\cdot\cos 18°\cdot\cos 18°\cdot\sin 36°$

$=\sin^2 36°\cdot\cos^2 18°$

$=\frac{10-2\sqrt{5}}{16}\cdot\frac{10+2\sqrt{5}}{16}$

$=\frac{100-20}{16\times16}=\frac{80}{16\times16}=\frac{5}{16}=$ R.H.S.

24. Show that $\cos^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{5\pi}{5}\right)+\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{9\pi}{10}\right)=2$.

Sol. L.H.S. $=\cos^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{5\pi}{5}\right)+\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{9\pi}{10}\right)$

$=\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}+\cos^2\left(\pi-\frac{2\pi}{5}\right)+\cos^2\left(\pi-\frac{\pi}{10}\right)$

$=\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}+\cos^2\frac{2\pi}{5}+\cos^2\frac{\pi}{10}$

$=2\left(\cos^2\frac{\pi}{10}+\cos^2\frac{2\pi}{5}\right)$

$=2(\cos^2 18°+\cos^2 72°)$

$=2[\cos^2 18°+\cos^2(90°-18°)]$

$=2[\cos^2 18°+\sin^2 18°]$

$=2(1)=2$

25. Prove that $\frac{1-\sec 8\alpha}{1-\sec 4\alpha}=\frac{\tan 8\alpha}{\tan 2\alpha}$

Sol. L.H.S. = $\frac{1-\sec 8\alpha}{1-\sec 4\alpha}$

$=\frac{1-\frac{1}{\cos 8\alpha}}{1-\frac{1}{\cos 4\alpha}}=\frac{\frac{\cos 8\alpha-1}{\cos 8\alpha}}{\frac{\cos 4\alpha-1}{\cos 4\alpha}}$

$$= \frac{\cos 8\alpha - 1}{\cos 8\alpha} \times \frac{\cos 4\alpha}{\cos 4\alpha - 1}$$

$$= \frac{-2\sin^2 4\alpha \cos 4\alpha}{-2\sin^2 2\alpha \cos 8\alpha}$$

$$= \frac{2\sin 4\alpha \cos 4\alpha \sin 4\alpha}{2\sin^2 2\alpha \cos 8\alpha}$$

$$= \frac{\sin 8\alpha \cdot \sin 4\alpha}{(2\sin^2 2\alpha)\cos 8\alpha}$$

$$= \frac{\sin 8\alpha}{\cos 8\alpha} \cdot \frac{2\sin 2\alpha \cos 2\alpha}{2\sin^2 2\alpha}$$

$$= \tan 8\alpha \cdot \frac{\cos 2\alpha}{\sin 2\alpha}$$

$$= \tan 8\alpha \cdot \cot 2\alpha$$

$$= \frac{\tan 8\alpha}{\tan 2\alpha} = \text{R.H.S.}$$

26. Prove that $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right) = \frac{1}{16}$.

Sol. L.H.S. =

$$\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$$

$$= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left[1+\cos\left(\pi-\frac{3\pi}{10}\right)\right]\left[1+\cos\left(\pi-\frac{\pi}{10}\right)\right]$$

$$= \left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{3\pi}{10}\right)\left(1-\cos\frac{\pi}{10}\right)$$

$$= \left(1-\cos^2\frac{\pi}{10}\right)\left(1-\cos^2\frac{3\pi}{10}\right)$$

$$= \sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10} = \left[\sin\frac{\pi}{10}\right]^2\left[\sin\frac{3\pi}{10}\right]^2$$

$$= \sin^2 18^\circ \sin^2 54^\circ$$

$$= \left[\frac{\sqrt{5}-1}{4}\right]^2\left[\frac{\sqrt{5}+1}{4}\right]^2 = \frac{(\sqrt{5}-1)^2}{16} \times \frac{(\sqrt{5}+1)^2}{16}$$

$$= \frac{[(\sqrt{5}-1)(\sqrt{5}+1)]^2}{16\times16} = \frac{(5-1)^2}{16\times16} = \frac{4^2}{16\times16} = \frac{16}{16\times16} = \frac{1}{16}$$

27. Prove that $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}=\frac{1}{32}$**.**

Sol. Let C = $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}$

$$S=\sin\frac{\pi}{11}\sin\frac{2\pi}{11}\sin\frac{3\pi}{11}\sin\frac{4\pi}{11}\sin\frac{5\pi}{11}$$

$$C\cdot S=\left(\sin\frac{\pi}{11}\cos\frac{\pi}{11}\right)\left(\sin\frac{2\pi}{11}\cos\frac{2\pi}{11}\right)\left(\sin\frac{3\pi}{11}\cos\frac{3\pi}{11}\right)\left(\sin\frac{4\pi}{11}\cos\frac{4\pi}{11}\right)\left(\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}\right)$$

$$=\frac{1}{32}\left(2\sin\frac{\pi}{11}\cos\frac{\pi}{11}\right)\left(2\sin\frac{2\pi}{11}\cos\frac{2\pi}{11}\right)\left(2\sin\frac{3\pi}{11}\cos\frac{3\pi}{11}\right)\left(2\sin\frac{4\pi}{11}\cos\frac{4\pi}{11}\right)\left(2\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}\right)$$

$$C\cdot S=\frac{1}{32}\sin\frac{2\pi}{11}\sin\frac{4\pi}{11}\sin\frac{6\pi}{11}\sin\frac{8\pi}{11}\sin\frac{10\pi}{11}$$

$$=\frac{1}{32}\sin\frac{2\pi}{11}\sin\frac{4\pi}{11}\sin\left(\pi-\frac{5\pi}{11}\right)\sin\left(\pi-\frac{3\pi}{11}\right)\sin\left(\pi-\frac{\pi}{11}\right)$$

$$=\frac{1}{32}\sin\frac{\pi}{11}\sin\frac{2\pi}{11}\sin\frac{3\pi}{11}\sin\frac{4\pi}{11}\sin\frac{5\pi}{11}$$

$$=\frac{1}{32}\cdot S$$

$$C=\frac{1}{32}$$

$$\therefore\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}=\frac{1}{32}$$

28. If A is not an integral multiple of π, prove that

(i) $\cos A\cdot\cos 2A\cdot\cos 4A\cdot\cos 8A=\frac{\sin 16A}{16\sin A}$ **and hence deduce that**

$\cos\frac{2\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{8\pi}{15}\cdot\cos\frac{16\pi}{15}=\frac{1}{16}$**.**

Sol. (i) L.H.S. : cos A cos 2A cos 4A cos 8A

$$=\frac{1}{2\sin A}(2\sin A\cos A)\cos 2A\cos 4A\cos 8A$$

$$=\frac{1}{2\sin A}\sin 2A\cos 2A\cos 4A\cos 8A$$

$$=\frac{1}{2^2\sin A}(2\sin 2A\cos 2A)\cos 4A\cos 8A$$

$$=\frac{1}{2^2\sin A}\sin 4A\cos 4A\cos 8A$$

$$=\frac{1}{2^3\sin A}2\sin 4A\cos 4A\cos 8A$$

$$=\frac{1}{2^3\sin A}\sin 8A\cos 8A$$

$$=\frac{1}{2^4\sin A}2\sin 8A\cos 8A$$

$$=\frac{\sin 16A}{16\sin A}=\text{R.H.S.}$$

ii) Put $n = \frac{2\pi}{15}$ in above result,

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$$

$$= \frac{\sin\frac{32\pi}{15}}{16\sin\frac{2\pi}{15}} = \frac{\sin\left(2\pi + \frac{2\pi}{15}\right)}{16\sin\frac{2\pi}{15}}$$

$$= \frac{\sin\frac{2\pi}{15}}{16\sin\frac{2\pi}{15}} = \frac{1}{16} = \text{R.H.S.}$$