

COMPOUND ANGLES

Definitions and Formulae :

- The algebraic sum of two or more angles is called a compound angle. i.e. $A + B$, $A - B$, $A + B + C$, $A + B - C$, $A - B + C$, $B + C - A$, etc. are called compound angles.
- If A and B are any two angles then
 - $\sin(A + B) = \sin A \cos B + \cos A \sin B$.
 - $\sin(A - B) = \sin A \cos B - \cos A \sin B$
 - $\cos(A + B) = \cos A \cos B - \sin A \sin B$
 - $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
- If A , B , $A + B$, $A - B$ are not odd multiples of $\pi/2$ then
 - $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.
 - $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.
- If A , B , $A + B$ and $A - B$ are not integral multiples of π , then
 - $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$
 - $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$.
- $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$
 - $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$
 - $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$
 - $\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$
 - $\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$.
- $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B$
 $= \cos^2 B - \cos^2 A$
 - $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B$
 $= \cos^2 B - \sin^2 A$
 - $\tan(A + B) \tan(A - B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$
 - $\cot(A + B) \cot(A - B) = \frac{\cot^2 A \cot^2 B - 1}{\cot^2 B - \cot^2 A}$

$$\begin{aligned} \text{v) } \tan(45^\circ + \theta) &= \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \\ &= \cot(45^\circ - \theta) = \frac{1 + \tan \theta}{1 - \tan \theta} \end{aligned}$$

$$\begin{aligned} \text{vi) } \tan(45^\circ - \theta) &= \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \\ &= \cot(45^\circ + \theta) = \frac{1 - \tan \theta}{1 + \tan \theta} \end{aligned}$$

and $\tan(45^\circ + \theta) \cdot \tan(45^\circ - \theta) = 1$.

7. i) $\sin(A + B + C)$
 $= \Sigma(\cos A \cos B \cos C) - \sin A \sin B \sin C$
 ii) $\cos(A + B + C)$
 $= \cos A \cos B \cos C - \Sigma(\sin A \sin B \sin C)$
 iii) $\tan(A + B + C) = \frac{\Sigma(\tan A) - \pi(\tan A)}{1 - \Sigma(\tan A \tan B)}$

VSAQ'S

Simplify the following

1. $\cos 100^\circ \cdot \cos 40^\circ + \sin 100^\circ \cdot \sin 40^\circ$

Sol. L.H.S. =
 $= \cos 100^\circ \cdot \cos 40^\circ + \sin 100^\circ \cdot \sin 40^\circ$
 $= \cos (100^\circ - 40^\circ) = \cos 60^\circ = \frac{1}{2} = \text{R.H.S.}$

2. $\tan\left(\frac{\pi}{4} + \theta\right) \cdot \tan\left(\frac{\pi}{4} - \theta\right)$

Sol. $\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \cdot \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta}$
 $= \frac{(1 + \tan \theta)}{(1 - \tan \theta)} \cdot \frac{(1 - \tan \theta)}{(1 + \tan \theta)} = 1$

3. $\tan 75^\circ + \cot 75^\circ$

Sol. $\tan 75^\circ = 2 + \sqrt{3}$
 $\cot 75^\circ = 2 - \sqrt{3}$
 $\therefore \tan 75^\circ + \cot 75^\circ = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$

4. Express $\frac{(\sqrt{3} \cos 25^\circ + \sin 25^\circ)}{2}$ as a sine of an angle.

Sol. $\frac{(\sqrt{3} \cos 25^\circ + \sin 25^\circ)}{2}$
 $= \frac{\sqrt{3}}{2} \cos 25^\circ + \frac{1}{2} \sin 25^\circ$
 $= \sin 60^\circ \cos 25^\circ + \cos 60^\circ \sin 25^\circ$
 $= \sin(60^\circ + 25^\circ) = \sin 85^\circ$

5. $\tan \theta$ in terms of $\tan \alpha$, if $\sin(\theta + \alpha) = \cos(\theta + \alpha)$.

Sol. $\sin(\theta + \alpha) = \cos(\theta + \alpha)$

$$\frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)} = 1$$

$$\tan(\theta + \alpha) = 1$$

$$\frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha} = 1$$

$$\tan \theta + \tan \alpha = 1 - \tan \theta \tan \alpha$$

$$\tan \theta + \tan \theta \tan \alpha = 1 - \tan \alpha$$

$$\tan \theta [1 + \tan \alpha] = 1 - \tan \alpha$$

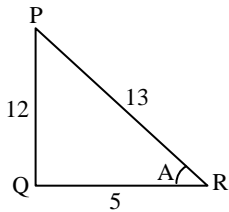
$$\therefore \tan \theta = \frac{1 - \tan \alpha}{1 + \tan \alpha}$$

6. If $\tan \theta = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$ and θ is in the third quadrant, find θ .

Sol. $\tan \theta = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$
 $= \frac{\cos 11^\circ \left[1 + \frac{\sin 11^\circ}{\cos 11^\circ} \right]}{\cos 11^\circ \left[1 - \frac{\sin 11^\circ}{\cos 11^\circ} \right]}$
 $= \frac{1 + \tan 11^\circ}{1 - \tan 11^\circ}$
 $= \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ} (\because \tan 45^\circ = 1)$
 $\tan \theta = \tan(45^\circ + 11^\circ)$
 $= \tan 56^\circ$
 $= \tan(180^\circ + 56^\circ)$
 $= \tan 236^\circ$
 $\theta = 236^\circ$

7. If $0^\circ < A, B < 90^\circ$, such that $\cos A = \frac{5}{13}$ and $\sin B = \frac{4}{5}$, find the value of $\sin(A - B)$.

Sol. $\cos A = \frac{5}{13}$ and $\sin B = \frac{4}{5}$

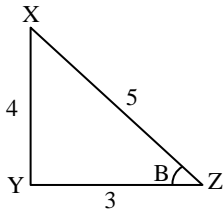


$$PQ^2 = PR^2 - QR^2$$

$$= (13)^2 - 5^2 = 169 - 25 = 144$$

$$PQ = 12$$

$$\cos A = \frac{5}{13}, \sin A = \frac{12}{13}$$



$$YZ^2 = XZ^2 - XY^2$$

$$= 5^2 - 4^2 = 25 - 16 = 9$$

$$YZ = 3$$

$$\sin B = \frac{4}{5}, \cos B = \frac{3}{5}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$= \frac{12}{13} \times \frac{3}{5} - \frac{5}{13} \times \frac{4}{5} = \frac{36}{65} - \frac{20}{65} = \frac{36 - 20}{65} = \frac{16}{65}$$

8. What is the value of $\tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ$?

Sol. $\tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ$

Consider $20^\circ + 40^\circ = 60^\circ$

$\tan(20^\circ + 40^\circ) = \tan 60^\circ$

$$\frac{\tan 20^\circ + \tan 40^\circ}{1 - \tan 20^\circ \tan 40^\circ} = \sqrt{3}$$

$$\tan 20^\circ + \tan 40^\circ = \sqrt{3}(1 - \tan 20^\circ \tan 40^\circ)$$

$$\tan 20^\circ + \tan 40^\circ = \sqrt{3} - \sqrt{3} \tan 20^\circ \tan 40^\circ$$

$$\tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ = \sqrt{3}$$

9. Find the value of $\tan 56^\circ - \tan 11^\circ - \tan 56^\circ \tan 11^\circ$.

Sol. We have $56^\circ - 11^\circ = 45^\circ$

$$\tan(56^\circ - 11^\circ) = \tan 45^\circ$$

$$\frac{\tan 56^\circ - \tan 11^\circ}{1 + \tan 56^\circ \tan 11^\circ} = 1$$

$$\tan 56^\circ - \tan 11^\circ = 1 + \tan 56^\circ \tan 11^\circ$$

$$\tan 56^\circ - \tan 11^\circ - \tan 56^\circ \tan 11^\circ = 1$$

10. Evaluate $\Sigma \frac{\sin(C-A)}{\cos C \sin A}$ if none of $\sin A$, $\sin B$, $\sin C$ is zero.

$$\text{Sol. } \Sigma \frac{\sin(C-A)}{\sin C \sin A} = \Sigma \frac{\sin C \cos A - \cos C \sin A}{\sin C \sin A}$$

$$= \Sigma \frac{\sin C \cos A}{\sin C \sin A} - \frac{\cos C \sin A}{\sin C \sin A}$$

$$= \Sigma \cot A - \cot C$$

$$= \cot A - \cot C + \cot B - \cot A + \cot C - \cot B = 0$$

11. $\tan 72^\circ = \tan 18^\circ + 2 \tan 54^\circ$

Sol. $72^\circ - 18^\circ = 54^\circ$

Take tan on both sides

$$\tan(72^\circ - 18^\circ) = \tan 54^\circ$$

$$\frac{\tan 72^\circ - \tan 18^\circ}{1 + \tan 72^\circ \tan 18^\circ} = \tan 54^\circ$$

$$\frac{\tan 72^\circ - \tan 18^\circ}{1 + \tan(90-18) \tan 18^\circ} = \tan 54^\circ$$

$$\frac{\tan 72^\circ - \tan 18^\circ}{1 + \tan 18^\circ \tan 18^\circ} = \tan 54^\circ$$

$$\frac{\tan 72^\circ - \tan 18^\circ}{1+1} = \tan 54^\circ$$

$$\frac{\tan 72^\circ - \tan 18^\circ}{2} = \tan 54^\circ$$

$$\tan 72^\circ - \tan 18^\circ = 2 \tan 54^\circ$$

$$\tan 72^\circ = \tan 18^\circ + 2 \tan 54^\circ$$

12. $\sin 750^\circ \cos 480^\circ + \cos 120^\circ \cos 60^\circ = \frac{-1}{2}$

Sol. $\sin 750^\circ = \sin(2 \cdot 360^\circ + 30^\circ) = \sin 30^\circ = \frac{1}{2}$

$$\cos 480^\circ = \cos(360^\circ + 120^\circ) = \cos 120^\circ$$

$$= \cos 120^\circ = \cos(180^\circ - 60^\circ)$$

$$= -\cos 60^\circ = -\frac{1}{2}$$

$$\cos 120^\circ = -\frac{1}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\therefore \text{L.H.S.} =$$

$$= \sin 750^\circ \cos 480^\circ + \cos 120^\circ \cos 60^\circ$$

$$= \frac{1}{2} \left(-\frac{1}{2} \right) + \left(-\frac{1}{2} \right) \frac{1}{2}$$

$$= -\frac{1}{4} - \frac{1}{4} = \frac{-2}{4} = \frac{-1}{2} = \text{R.H.S.}$$

$$13. \cos A + \cos \left(\frac{4\pi}{3} - A \right) + \cos \left(\frac{4\pi}{3} + A \right) = 0$$

$$\text{Sol. Consider } \cos \left(\frac{4\pi}{3} - A \right) + \cos \left(\frac{4\pi}{3} + A \right)$$

$$= \cos(240^\circ + A) + \cos(240^\circ - A)$$

$$= \cos 240^\circ \cos A - \sin 240^\circ \sin A$$

$$+ \cos 240^\circ \cos A + \sin 240^\circ \sin A$$

$$= 2 \cos 240^\circ \cos A$$

$$= 2 \cos(180^\circ + 60^\circ) \cos A$$

$$= -2 \cos 60^\circ \cos A$$

$$= -2 \times \frac{1}{2} \cos A = -\cos A$$

$$14. \cos^2 \theta + \cos^2 \left(\frac{2\pi}{3} + \theta \right) + \cos^2 \left(\frac{2\pi}{3} - \theta \right) = \frac{3}{2}$$

$$\text{Sol. } \cos^2 \left(\frac{2\pi}{3} + \theta \right) + \cos^2 \left(\frac{2\pi}{3} - \theta \right)$$

$$\begin{aligned}
 &= \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) \\
 &= [\cos 60^\circ \cos \theta - \sin 60^\circ \sin \theta]^2 + [\cos 60^\circ \cos \theta + \sin 60^\circ \sin \theta]^2 \\
 &= 2(\cos^2 60^\circ \cos^2 \theta + \sin^2 60^\circ \sin^2 \theta) [\because (a+b)^2 + (a-b)^2 = 2(a^2 + b^2)] \\
 &= 2 \left[\left(\frac{1}{2}\right)^2 \cos^2 \theta + \left(\frac{\sqrt{3}}{2}\right)^2 \sin^2 \theta \right] \\
 &= 2 \left[\frac{1}{4} \cos^2 \theta + \frac{3}{4} \sin^2 \theta \right] \\
 &= \frac{2}{4} [\cos^2 \theta + 3 \sin^2 \theta] \\
 &= \frac{1}{2} \cos^2 \theta + \frac{3}{2} \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{L.H.S.} &= \cos^2 \theta + \frac{1}{2} \cos^2 \theta + \frac{3}{2} \sin^2 \theta \\
 &= \frac{3}{2} \cos^2 \theta + \frac{3}{2} \sin^2 \theta \\
 &= \frac{3}{2} [\cos^2 \theta + \sin^2 \theta] \\
 &= \frac{3}{2} (\because \cos^2 \theta + \sin^2 \theta = 1) \\
 &= \text{R.H.S.}
 \end{aligned}$$

15. Evaluate $\sin^2 82\frac{1^\circ}{2} - \sin^2 22\frac{1^\circ}{2}$.

Sol. Put $A = \sin^2 82\frac{1^\circ}{2}$ and $B = \sin^2 22\frac{1^\circ}{2}$, then

$$\begin{aligned}
 &\sin^2 82\frac{1^\circ}{2} - \sin^2 22\frac{1^\circ}{2} \\
 &= \sin^2 A - \sin^2 B \\
 &= \sin(A+B) \sin(A-B) \\
 &= \sin 105^\circ \sin 60^\circ \\
 &= \sin(90^\circ + 15^\circ) \sin 60^\circ \\
 &= \cos 15^\circ \sin 60^\circ \\
 &= \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{4\sqrt{2}}
 \end{aligned}$$

16. Prove that $\sin^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ = \frac{\sqrt{3}+1}{4\sqrt{2}}$

17. $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)$

Sol. $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)$

$$\begin{aligned} & [\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)] \\ &= \sin\left(\frac{\pi}{8} + \frac{A}{2} + \frac{\pi}{8} - \frac{A}{2}\right) \sin\left(\frac{\pi}{8} + \frac{A}{2} - \frac{\pi}{8} + \frac{A}{2}\right) \\ &= \sin\left(\frac{2\pi}{8}\right) \sin\left(\frac{2A}{2}\right) \\ &= \sin\frac{\pi}{4} \sin A = \sin 45^\circ \sin A = \frac{1}{\sqrt{2}} \sin A \end{aligned}$$

18. $\cos^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

Sol. $\cos^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

$$\begin{aligned} & [\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)] \\ &= \cos\left(52\frac{1}{2}^\circ + 22\frac{1}{2}^\circ\right) \cos\left(52\frac{1}{2}^\circ - 22\frac{1}{2}^\circ\right) \\ &= \cos 75^\circ \cos 30^\circ \\ &= \frac{\sqrt{3}}{2} (\cos 75^\circ) \\ &= \frac{\sqrt{3}}{2} \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{3-\sqrt{3}}{4\sqrt{2}} \end{aligned}$$

19. $\cos^2 112\frac{1}{2}^\circ - \sin^2 52\frac{1}{2}^\circ = \cos\left(112\frac{1}{2}^\circ + 52\frac{1}{2}^\circ\right) \cos\left(112\frac{1}{2}^\circ - 52\frac{1}{2}^\circ\right)$

$$\begin{aligned} & (\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)) \\ &= \cos 165^\circ \cdot \cos 60^\circ = \frac{1}{2} \cos(180^\circ - 15^\circ) = -\frac{1}{2} \cos 15^\circ \\ &= -\frac{1}{2} \left\{ \frac{\sqrt{3}+1}{2\sqrt{2}} \right\} \end{aligned}$$

20. Prove that $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

Solution:

We know that $\tan 3A = \tan(2A + A)$

$$\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$\tan 3A - \tan 2A \tan A \tan 3 = \tan 2A + \tan A$$

$$\tan 3A - \tan 2 - \tan A = \tan A \tan 2A \tan 3A$$

21. Find the expansion of $\sin(A + B + C)$.

Sol. $\sin(A + B + C)$

$$= \sin[(A + B) - C]$$

$$= \sin(A + B) \cos C - \cos(A - B) \sin C$$

$$= (\sin A \cos B + \cos A \sin B) \cos C - [\cos(A - B) \cos B - \sin A \sin B] \sin C$$

$$= \sin A \cos B \cos C + \cos A \sin B \cos C - \cos A \cos B \sin C + \sin A \sin B \sin C$$

22. Find the expansion of $\cos(A - B - C)$.

Sol. $\cos(A - B - C) = \cos[(A - B) - C]$

$$= \cos(A - B) \cos C + \sin(A - B) \sin C$$

$$= (\cos A \cos B + \sin A \sin B) \cos C + (\sin A \cos B - \cos A \sin B) \sin C$$

$$= \cos A \cos B \cos C + \sin A \sin B \cos C + \sin A \cos B \sin C - \cos A \sin B \sin C$$

23. For what values of x in the first quadrant $\frac{2 \tan x}{1 - \tan^2 x}$ is positive?

Sol. $\frac{2 \tan x}{1 - \tan^2 x} > 0 \Rightarrow \tan 2x > 0$

$$\Rightarrow 0 < 2x < \frac{\pi}{2} \text{ (since x in the first quadrant)}$$

$$\Rightarrow 0 < x < \frac{\pi}{4}$$

Therefore, $\frac{2 \tan x}{1 - \tan^2 x}$ is positive for $0 < x < \frac{\pi}{4}$

SAQ'S

24. Prove that $\sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8} = \frac{3}{2}$

Solution:

$$\begin{aligned} & \sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8} \\ & \left(\sin \frac{\pi}{8} \right)^4 + \left\{ \sin^2 \frac{\pi}{2} - \frac{\pi}{8} \right\}^4 + \left\{ \sin^2 \left(\frac{\pi}{2} + \frac{\pi}{8} \right) \right\}^2 + \left\{ \sin \left(\pi - \frac{\pi}{8} \right) \right\}^4 \\ & \sin^4 \frac{\pi}{8} + \cos^4 \frac{\pi}{8} + \cos^4 \frac{\pi}{8} + \sin^4 \frac{\pi}{8} \\ & 2 \left\{ \sin^4 \frac{\pi}{8} + \cos^4 \frac{\pi}{8} \right\} = 2 \left\{ \left(\sin^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{8} \right)^2 - 2 \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \right\} \\ & = 2 - 4 \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \\ & = 2 - \left(2 \sin \frac{\pi}{8} \cos \frac{\pi}{8} \right)^2 = 2 - \sin^2 \frac{\pi}{4} = 2 - \frac{1}{2} = \frac{3}{2} \end{aligned}$$

25. Prove that $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$

26. Prove that (i) $\sin A \sin(60^\circ - A) \sin(60^\circ + A) = \frac{1}{4} \sin 3A$

(ii) $\cos A \cos(60^\circ + A) \cos(60^\circ - A) = \frac{1}{4} \cos 3A$

(iii) $\tan A \tan(60^\circ + A) \tan(60^\circ - A) = \tan 3A$

(iv) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

(v) $\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{3\pi}{9} \cos \frac{4\pi}{9} = \frac{1}{16}$

27. Prove that $\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta = \cot \theta$

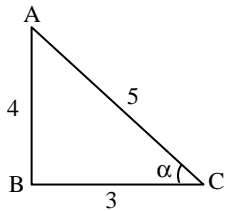
28. If $\cos \alpha = \frac{-3}{5}$ **and** $\sin \beta = \frac{7}{25}$, **where** $\frac{\pi}{2} < \alpha < \pi$ **and** $0 < \beta < \frac{\pi}{2}$, **then find the values of**
 $\tan(\alpha + \beta)$ and $\sin(\alpha + \beta)$.

Sol. $\cos \alpha = \frac{-3}{5}$, where $\frac{\pi}{2} < \alpha < \pi$

α in II quadrant

$\sin \beta = \frac{7}{25}$, where $0 < \beta < \frac{\pi}{2}$

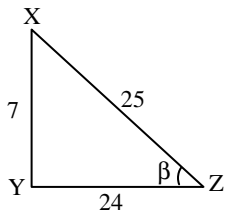
β in I Quadrant



$$\begin{aligned} AB^2 &= AC^2 - BC^2 \\ &= 5^2 - 3^2 = 25 - 9 = 16 \end{aligned}$$

$$\therefore AB = 4$$

$$\cos \alpha = -\frac{3}{5}, \sin \alpha = \frac{4}{5}, \tan \alpha = -\frac{4}{3}$$



$$\begin{aligned} YZ^2 &= XZ^2 - XY^2 \\ &= 25^2 - 7^2 \\ &= 625 - 49 = 576 \end{aligned}$$

$$YZ = 24$$

$$\sin \beta = \frac{7}{25}, \cos \beta = \frac{24}{25}, \tan \beta = \frac{7}{24}$$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{-4}{3} + \frac{7}{24}}{1 + \frac{4}{3} \times \frac{7}{24}} = \frac{\frac{-32+7}{24}}{1 + \frac{7}{18}}$$

$$= \frac{\frac{-25}{24}}{\frac{18+7}{18}} = \frac{-25}{24} \times \frac{18}{25} = -\frac{3}{4}$$

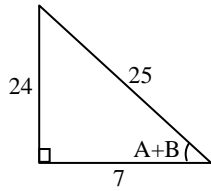
$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \frac{4}{5} \times \frac{24}{25} + \frac{-3}{5} \times \frac{7}{25}$$

$$= \frac{96}{125} - \frac{21}{125} = \frac{75}{125} = \frac{3}{5}$$

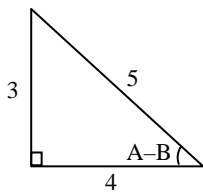
29. If $0 < A < B < \frac{\pi}{4}$ and $\sin(A+B) = \frac{24}{25}$ and $\cos(A-B) = \frac{4}{5}$, then find the value of $\tan 2A$.

Sol. $\sin(A+B) = \frac{24}{25}$



$$\tan(A+B) = \frac{24}{7}$$

$$\cos(A-B) = \frac{4}{5}$$



$$\tan(A-B) = \frac{3}{4}$$

Now $2A = (A+B) + (A-B)$

$$\tan 2A = \tan[(A+B) + (A-B)]$$

$$\begin{aligned} &= \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)} \\ &= \frac{\frac{24}{7} + \frac{3}{4}}{1 - \frac{24}{7} \times \frac{3}{4}} = \frac{96 + 21}{28 - 72} = \frac{-117}{44} \end{aligned}$$

30. If $A+B, A$ are acute angles such that $\sin(A+B) = \frac{24}{25}$ and $\tan A = \frac{3}{4}$ find the value of $\cos B$

Solution:

$$\sin(A+B) = \frac{24}{25} \quad \therefore \cos(A+B) = \frac{7}{25}$$

$$\tan A = \frac{3}{4} \quad \therefore \sin A = \frac{3}{5} \quad \& \quad \cos A = \frac{4}{5}$$

$$\cos B = \cos(A+B-A) = \cos(A+B)\cos A + \sin(A+B)\sin A$$

$$\frac{7}{25} \times \frac{4}{5} + \frac{24}{25} \times \frac{3}{5} = \frac{100}{125} = \frac{4}{5}$$

31. If $\tan \alpha - \tan \beta = m$ and $\cot \alpha - \cot \beta = n$, then prove that $\cot(\alpha - \beta) = \frac{1}{m} - \frac{1}{n}$.

Sol. We have $\tan \alpha - \tan \beta = m$

$$\frac{1}{\cot \alpha} - \frac{1}{\cot \beta} = m$$

$$\frac{\cot \beta - \cot \alpha}{\cot \alpha \cot \beta} = m$$

$$\therefore \frac{\cot \alpha \cot \beta}{\cot \beta - \cot \alpha} = \frac{1}{m} \quad \dots(1)$$

$$\cot \alpha - \cot \beta = n$$

$$-(\cot \beta - \cot \alpha) = n$$

$$\cot \beta - \cot \alpha = -n$$

$$\frac{1}{\cot \beta - \cot \alpha} = -\frac{1}{n} \quad \dots(2)$$

$$\text{L.H.S.} = \cot(\alpha - \beta) = \frac{\cot \alpha \cot \beta}{\cot \beta - \cot \alpha}$$

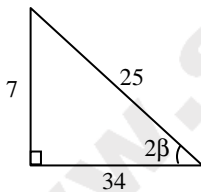
$$= \frac{\cot \alpha \cot \beta}{\cot \beta - \cot \alpha} + \frac{1}{\cot \beta - \cot \alpha}$$

$$= \frac{1}{m} - \frac{1}{n} \quad (\because \text{from (1) \& (2)}) = \text{R.H.S.}$$

32. If $\tan(\alpha - \beta) = \frac{7}{24}$ and $\tan \alpha = \frac{4}{3}$, where α and β are in the first quadrant prove that

$$\alpha + \beta = \pi/2.$$

Sol. $\tan(\alpha - \beta) = \frac{7}{24}$ and $\tan \alpha = \frac{4}{3}$



$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\Rightarrow \frac{\frac{4}{3} - \tan \beta}{1 + \frac{4}{3} \tan \beta} = \frac{7}{24}$$

$$\Rightarrow \frac{4 - 3 \tan \beta}{3 + 4 \tan \beta} = \frac{7}{24}$$

$$\Rightarrow \frac{4 - 3 \tan \beta}{3 + 4 \tan \beta} = \frac{7}{24}$$

$$\Rightarrow 24[4 - 3 \tan \beta] = 7(3 + 4 \tan \beta)$$

$$\Rightarrow 96 - 72 \tan \beta = 21 + 28 \tan \beta$$

$$\Rightarrow 96 - 21 = 28 \tan \beta + 72 \tan \beta$$

$$\Rightarrow 100 \tan \beta = 75$$

$$\therefore \tan \beta = \frac{75}{100} = \frac{3}{4}$$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{4}{3} + \frac{3}{4}}{1 - \frac{4}{3} \cdot \frac{3}{4}} = \alpha$$

$$\tan(\alpha + \beta) = \tan \frac{\pi}{2}$$

$$\therefore \alpha + \beta = \frac{\pi}{2}$$

33. If $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{a + b}{a - b}$, **then prove that** $a \tan \beta = b \tan \alpha$.

Sol. Given that $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{a + b}{a - b}$

By using componendo and dividendo, we get

$$\frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\sin(\alpha + \beta) - \sin(\alpha - \beta)} = \frac{a + b + a - b}{a + b - a + b} = \frac{2a}{2b} = \frac{a}{b}$$

$$\Rightarrow \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\sin(\alpha + \beta) - \sin(\alpha - \beta)} = \frac{a}{b}$$

$$\Rightarrow \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta - \sin \alpha \cos \beta + \cos \alpha \sin \beta} = \frac{a}{b}$$

$$\Rightarrow \frac{2 \sin \alpha \cos \beta}{2 \cos \alpha \sin \beta} = \frac{a}{b}$$

$$\Rightarrow \tan \alpha \cot \beta = \frac{a}{b}$$

$$\Rightarrow b \tan \alpha = a \tan \beta$$

hence, $a \tan \beta = b \tan \alpha$

34. If $A - B = \frac{3\pi}{4}$, then show that

$$(1 - \tan A)(1 + \tan B) = 2.$$

Sol. $A - B = \frac{3\pi}{4}$

$$A - B = 135^\circ$$

$$\tan(A - B) = \tan 135^\circ$$

$$= \tan(90^\circ + 45^\circ) = -\cot 45^\circ = -1$$

$$\therefore \frac{\tan A - \tan B}{1 + \tan A \tan B} = -1$$

$$\tan A - \tan B = -(1 + \tan A \tan B)$$

$$\tan A - \tan B = -1 - \tan A \tan B$$

$$\tan A - \tan B + \tan A \tan B = -1$$

$$\tan B - \tan A - \tan A \tan B = 1 \dots(1)$$

$$\text{L.H.S.} = (1 - \tan A)(1 + \tan B)$$

$$= 1 + (\tan B - \tan A - \tan A \tan B)$$

$$= 1 + 1 (\because \text{from}(1))$$

$$= 2 = \text{R.H.S.}$$

35. If $A + B + C = \frac{\pi}{2}$ and if none of A, B, C is an odd multiple of $\pi/2$, then prove that

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C.$$

Sol. $A + B + C = \frac{\pi}{2}$

$$A + B = \frac{\pi}{2} - C$$

$$\cot(A + B) = \cot\left(\frac{\pi}{2} - C\right)$$

$$\frac{\cot A \cot B - 1}{\cot B + \cot A} = \tan C$$

$$\frac{\cot A \cot B - 1}{\cot B + \cot A} = \frac{1}{\cot C}$$

$$\cot C[\cot A \cot B - 1] = \cot B + \cot A$$

$$\cot A \cot B \cot C - \cot C \cot A + \cot B$$

$$\cot A \cot B \cot C = \cot A + \cot B + \cot C$$

$$\therefore \cot A + \cot B + \cot C = \cot A \cot B \cot C$$

36. If $A + B + C = \frac{\pi}{2}$ and if none of A, B, C is an odd multiple of $\pi/2$, then prove that,

$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1.$$

Sol. $A + B + C = \frac{\pi}{2}$

$$A + B = \frac{\pi}{2} - C$$

$$\tan(A + B) = \tan\left(\frac{\pi}{2} - C\right)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \cot C$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{\tan C}$$

$$\Rightarrow \tan C[\tan A + \tan B] = 1 - \tan A \tan B$$

$$\Rightarrow \tan C \tan A + \tan C \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$$

37. $\Sigma \frac{\cos(B+C)}{\cos B \cos C} = 2.$

Sol. L.H.S. = $\Sigma \frac{\cos(B+C)}{\cos B \cos C}$

$$= \Sigma \frac{\cos B \cos C - \sin B \sin C}{\cos B \cos C}$$

$$= \Sigma \frac{\cos B \cos C}{\cos B \cos C} - \frac{\sin B \sin C}{\cos B \cos C}$$

$$= \Sigma(1 - \tan B \tan C)$$

$$= 1 - \tan B \tan C + 1 - \tan C \tan A + 1 - \tan A \tan B$$

$$= 3 - (\tan A \tan B + \tan B \tan C + \tan C \tan A)$$

$$= 3 - 1 \quad (\because \text{from (b)})$$

$$= 2 = \text{R.H.S.}$$

38. Prove that $\sin^2 \alpha + \cos^2 (\alpha + \beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta)$ is independent of α .

Sol. Given expression,

$$\sin^2 \alpha + \cos^2 (\alpha + \beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta)$$

$$= \sin^2 \alpha + 1 - \sin^2 (\alpha + \beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta)$$

$$= 1 + [\sin^2 \alpha - \sin^2 (\alpha + \beta)] + 2 \sin \alpha \sin \beta \cos(\alpha + \beta)$$

$$= 1 + \sin(\alpha + \alpha + \beta) \sin(\alpha - \alpha - \beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta)$$

$$\begin{aligned}
 &= 1 + \sin(2\alpha + \beta) \sin(-\beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta) \\
 &= 1 - \sin(2\alpha + \beta) \sin \beta + [2 \sin \alpha \cos(\alpha + \beta)] \sin \beta \\
 &= 1 - \sin(2\alpha + \beta) \sin \alpha + [\sin(\alpha + \alpha + \beta) + \sin(\alpha - \alpha - \beta)] \sin \beta \\
 &= 1 - \sin(2\alpha + \beta) \sin \alpha + [\sin(2\alpha + \beta) - \sin \beta] \sin \beta \\
 &= 1 - \sin(2\alpha + \beta) \sin \alpha + \sin(2\alpha + \beta) \sin \beta - \sin^2 \beta \\
 &= 1 - \sin^2 \beta = \cos^2 \beta
 \end{aligned}$$

Thus the given expression is independent of α .

39. Prove that $\cot \frac{\pi}{16} \cdot \cot \frac{2\pi}{16} \cdot \cot \frac{3\pi}{16} \dots \cot \frac{7\pi}{16} = 1$.

$$\begin{aligned}
 \text{Sol. } &\cot \frac{\pi}{16} \cdot \cot \frac{2\pi}{16} \cdot \cot \frac{3\pi}{16} \dots \cot \frac{7\pi}{16} \\
 &= \left(\cot \frac{\pi}{16} \cdot \cot \frac{7\pi}{16} \right) \left(\cot \frac{2\pi}{16} \cdot \cot \frac{6\pi}{16} \right) \left(\cot \frac{3\pi}{16} \cdot \cot \frac{5\pi}{16} \right) \cdot \cot \frac{4\pi}{16} \\
 &= \left[\cot \frac{\pi}{16} \cdot \cot \left(\frac{\pi}{2} - \frac{\pi}{16} \right) \right] \left[\cot \frac{2\pi}{16} \cdot \cot \left(\frac{\pi}{2} - \frac{2\pi}{16} \right) \right] \left[\cot \frac{3\pi}{16} \cdot \cot \left(\frac{\pi}{2} - \frac{3\pi}{16} \right) \right] \cdot \cot \frac{\pi}{4} \\
 &= \left(\cot \frac{\pi}{16} \cdot \tan \frac{\pi}{16} \right) \left(\cot \frac{2\pi}{16} \cdot \tan \frac{2\pi}{16} \right) \left(\cot \frac{3\pi}{16} \cdot \tan \frac{3\pi}{16} \right) \cdot 1 \\
 &= 1 \times 1 \times 1 \times 1 = 1
 \end{aligned}$$

40. Prove that $\tan 70^\circ - \tan 20^\circ = 2 \tan 50^\circ$.

$$\begin{aligned}
 \text{Sol. } \tan 50^\circ &= \tan(70^\circ - 20^\circ) \\
 &= \frac{\tan 70^\circ - \tan 20^\circ}{1 + \tan 70^\circ \tan 20^\circ} \\
 \Rightarrow \tan 70^\circ - \tan 20^\circ &= \tan 50^\circ (1 + \tan 70^\circ \cdot \tan 20^\circ) \\
 &= \tan 50^\circ (1 + \tan 70^\circ \cdot \tan(90^\circ - 70^\circ)) \\
 &= \tan 50^\circ [1 + \tan 70^\circ \cdot \cot 70^\circ] \\
 &= \tan 50^\circ [1 + 1] \\
 &= 2 \tan 50^\circ \\
 \therefore \tan 70^\circ - \tan 20^\circ &= 2 \tan 50^\circ
 \end{aligned}$$

41. If $A + B = 45^\circ$, then prove that

i) $(1 + \tan A)(1 + \tan B) = 2$

ii) $(\cot A - 1)(\cot B - 1) = 2$.

Sol. i) $A + B = 45^\circ$

$$\Rightarrow \tan(A + B) = \tan 45^\circ = 1$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan A + \tan B + \tan A \tan B = 1 \dots (1)$$

$$\text{Now, } (1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2$$

(from(1))

ii) $A + B = 45^\circ \Rightarrow \cot(A + B) = \cot 45^\circ = 1$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot B + \cot A} = 1$$

$$\Rightarrow \cot A \cot B - 1 = \cot A + \cot B$$

$$\Rightarrow \cot A \cot B - \cot A - \cot B = 1 \dots (2)$$

$$\text{Now, } (\cot A - 1)(\cot B - 1) = \cot A \cot B - \cot A - \cot B + 1 = 2$$

(from(2))

42. If A, B, C are the angles of a triangle and if none of them is equal to $\pi/2$, then prove that i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

ii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

Sol. i) Given $A + B + C = \pi$

$$\Rightarrow A + B = \pi - C$$

$$\Rightarrow \tan(A + B) = \tan(\pi - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \tan A + \tan B = -\tan C(1 - \tan A \tan B)$$

$$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$$

$$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

ii) Replacing $\tan A$ by $\frac{1}{\cot A}$ etc., in (i) above, we get

$$\frac{1}{\cot A} + \frac{1}{\cot B} + \frac{1}{\cot C} = \frac{1}{\cot A \cot B \cot C}$$

$$\Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

43. Let ABC be a triangle such that

$\cot A + \cot B + \cot C = \sqrt{3}$. Then prove that ABC is an equilateral triangle.

Sol. Given that $A + B + C = 180^\circ$

We get $\Sigma \cot A \cot B = 1$

Now, $\Sigma(\cot A - \cot B)^2 = \Sigma \cot^2 A + \cot^2 B - 2 \cot A \cot B$

$$= 2 \cot^2 A + 2 \cot^2 B + 2 \cot^2 C - 2 \cot A \cot B - 2 \cot B \cot C - 2 \cot C \cot A$$

(on expanding)

$$= 2\{(\cot A + \cot B + \cot C)^2 - 2(\cot A \cot B) - 2 \cot B \cot C - 2 \cot C \cot A\}$$

$$- 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)$$

$$= 2(\cot A + \cot B + \cot C)^2 - 6(\cot A \cot B + \cot B \cot C + \cot C \cot A)$$

$$= 2 \cdot 3 - 6 = 0$$

$$\Rightarrow \cot A = \cot B = \cot C$$

$$\Rightarrow \cot A = \cot B = \cot C = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

(since $\cot A + \cot B + \cot C = \sqrt{3}$)

$$\Rightarrow A + B + C = 60^\circ$$

(Since each angle lies in the interval $[0, 180^\circ]$)