ADDITIONAL EXERCISE

I. Very Short Answer Questions:

1. If the planes $\vec{r}.(\vec{i}+\vec{j}-3\vec{k})=7$ and $\vec{r}.(\vec{i}-t\vec{j}+\vec{k})=9$ are perpendicular to each other find the value of t.

Sol: since planes are perpendicular. The vectors $\vec{i}+\vec{j}-3\vec{k}$ and $\vec{i}-t\vec{j}+\vec{k}$ are perpendicular to each other.

$\therefore (\vec{i}+\vec{j}-3\vec{k}).(\vec{i}-t\vec{j}+\vec{k})=0$

$1-t-3=0 \Rightarrow t=-2$

2. If $|\vec{a}|=2$ $|\vec{b}|=5$ and $\vec{a}.\vec{b}=6$ then find $|\vec{a}\times\vec{b}|$.

Sol: $|\vec{a}\times\vec{b}|=\sqrt{|\vec{a}\times\vec{b}|^2}=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\times\vec{b})^2}$

$\sqrt{4\times 25-36}=8$

3. If $|\vec{a}\times\vec{b}|=\vec{a}.\vec{b}$ then find the angle between the vectors $\vec{a}$ and $\vec{b}$.

Sol: Given $|\vec{a}\times\vec{b}|=\vec{a}.\vec{b}$

$|\vec{a}||\vec{b}|\sin\theta=|\vec{a}||\vec{b}|\cos\theta$

$\tan\theta=1\Rightarrow\theta=\frac{\pi}{4}$

4. Find the value of $[\vec{i}+\vec{j}+\vec{k},\ \vec{i}-\vec{j},\ i+2\vec{j}-\vec{k}]$

Sol: $[\vec{i}+\vec{j}+\vec{k},\ \vec{i}-\vec{j},\ i+2\vec{j}-\vec{k}]=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 2 & -1\end{vmatrix}$

= 1(1 – 0) – 1(-1 – 0) +1(2 + 1) 4

5. **Let** $\vec{a}=\vec{i}+2\vec{j}+3\vec{k}$, $\vec{b}=-\vec{i}+2\vec{j}+\vec{k}$. If $\vec{c}=3\vec{i}+\vec{j}+t\vec{k}$ is perpendicular to $\vec{a}+\vec{b}$ then find t.

Sol: given $\vec{c}$ is perpendicular to $\vec{a}+\vec{b}$

$\vec{c}.(\vec{a}+\vec{b})=0$

$\vec{c}.\vec{a}=-\vec{b}.\vec{c}$

$(3\vec{i}+\vec{j}+t\vec{k}).(\vec{i}+2\vec{j}+3\vec{k})=-\left\{(-\vec{i}+2\vec{j}+\vec{k}).(3\vec{i}+\vec{j}+t\vec{k})\right\}$

3 + 2 + 3t = - (-3 + 2 +t)

5 + 3t = 1 –t $\Rightarrow 4t=-4$

t = -1

6. **If the volume of a parallelepiped with coterminous edges** $\vec{i}+x\vec{j}-x^2\vec{k}$ **,** $\vec{i}+\vec{j}-\vec{k}$ **, and** $\vec{i}-\vec{j}+\vec{k}$ **is 4 cub is units then find x**

Sol: Let $\vec{a}=\vec{i}+xj-x^2\vec{k}$ $\quad \vec{b}=\vec{i}+\vec{j}-\vec{k}$ $\quad \vec{c}=\vec{i}-\vec{j}+\vec{k}$ be the given vectors given volume = 4

$[\vec{a}\ \vec{b}\ \vec{c}]=4$

$$\begin{vmatrix}1 & x & -x^2\\ 1 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}=4$$

$1(1-1)-x(1+1)-x^2(-1-1)=4$

$2x^2-2x-4=0$

$x^2-x-2=0$

(x – 2) (x + 1) = 0 $\quad \therefore x=2\ or\ -1$

7. **If** $\vec{a},\vec{b},\vec{c}$ **are unit vectors such that** $\vec{a}+\vec{b}+\vec{c}=0$ **the least value of** $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

Sol: $\vec{a}+\vec{b}+\vec{c}=\vec{o}$

Squaring on both sides

$(\vec{a}+\vec{b}+\vec{c})^2=(\vec{o})^2$

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}.\vec{b}+2\vec{b}.\vec{c}+2\vec{c}.\vec{a}=0$$

$$1+1+1+2+\{\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\}=0$$

$$\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=-\frac{3}{2}$$

8. If $\vec{a}$ is a unit vector and $\vec{x}$ is any vector such that $(\vec{x}-\vec{a}).(\vec{x}+\vec{b})=15$ then find $|\vec{x}|$

Sol: Given that $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15$

$$|\vec{x}|^2-|\vec{a}|^2=15$$

But $|\vec{a}|=1$

$$\therefore |\vec{x}|^2-1=15 \Rightarrow |\vec{x}|^2=16$$

$$|\vec{x}|=4$$

9. If $\vec{i}\times(\vec{r}\times\vec{i})+\vec{j}\times(\vec{r}\times\vec{j})+\vec{k}\times(\vec{r}\times\vec{k})=\vec{a}\times\vec{b}$ then find $\vec{r}$ in term of $\vec{a}$ and $\vec{b}$

Sol: $\vec{i}\times(\vec{r}\times\vec{i})+\vec{j}\times(\vec{r}\times\vec{j})+\vec{k}\times(\vec{r}\times\vec{k})=\vec{a}\times\vec{b}$

$$(\vec{i}.\vec{i})\vec{r}-(\vec{i}.\vec{r})\vec{i}+(\vec{j}.\vec{j})+\vec{r}-(\vec{j}.\vec{r})+\vec{j}+(\vec{k}.\vec{k})\vec{r}-(\vec{k}.\vec{r})\vec{k}=\vec{a}\times\vec{b}$$

$$3\vec{r}-\{(\vec{r}.\vec{i})\vec{i}+(\vec{r}.\vec{j})\vec{j}+(\vec{r}.\vec{k})\vec{j}+(\vec{k}.\vec{k})\vec{k}\}=\vec{a}\times\vec{b}$$

But $(\vec{r}.\vec{i})\vec{i}+(\vec{r}.\vec{j})\vec{j}+(\vec{r}.\vec{k})\vec{k}\}=\vec{r}$

$$\therefore 3\vec{r}-\vec{r}=\vec{a}\times\vec{b}$$

$$2\vec{r}=\vec{a}\times\vec{b}\Rightarrow\vec{r}=\frac{1}{2}(\vec{a}\times\vec{b})$$

10. $\vec{a},\vec{b},\vec{c}$ **are non-zero vectors and no two of them are collinear. If** $\vec{a}+2\vec{b}$ **is collinear with** $\vec{c}$ **and** $\vec{b}+3\vec{c}$ **is collinear with** $\vec{a}$ **then find the value of** $\vec{a}+2\vec{b}+6\vec{c}$**.**

Sol: Given that $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$

$\therefore \vec{a}+2\vec{b}=k\vec{c} \rightarrow (1)$

$\vec{b}+3\vec{c}$ is collinear with $\vec{a}$

$\therefore \vec{b}+3\vec{c}=m\vec{a} \rightarrow (2)$

From (2) we have

$\vec{b}=m\vec{a}-3\vec{c}$

Sub $\vec{b}$ in (1) we have

$\vec{a}+2(m\vec{a}-3\vec{c})=k\vec{c}$

$\vec{a}+2m\vec{a}-6\vec{c}=k\vec{c}$

$\Rightarrow (2m+1)\vec{a}-(6+k)\vec{c}=0$

Since $\vec{a}$, $\vec{c}$ are not collinear

2m+1=0 6+k=0

M=$-\frac{1}{2}$ k=-6

$\therefore \vec{a}+2\vec{b}=-6\vec{c}$

$\therefore \vec{a}+2\vec{b}+6\vec{c}=0$

II. Short Answer Questions :

1. **If** $\vec{a}=\vec{i}+2\vec{j}-3\vec{k}$ **and** $\vec{b}=3\vec{i}-\vec{j}+2\vec{k}$ **then find the cosine of the angle between** $2\vec{a}+\vec{b}$ **and** $\vec{a}+2\vec{b}$

Sol: $\vec{a}=\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{b}=3\vec{i}-\vec{j}+2\vec{k}$

$2\vec{a}=2\vec{i}+4\vec{j}-6\vec{k}$ and $2\vec{b}=6\vec{i}-2\vec{j}+4\vec{k}$

$2\vec{a}+\vec{b}=(2\vec{i}+4\vec{j}-6\vec{k})+(3\vec{i}-\vec{j}+2\vec{k})$

$=5\vec{i}+3\vec{j}-4\vec{k}$

$\vec{a}+2\vec{b}=(\vec{i}+2\vec{j}-\vec{k})+(6\vec{i}-2\vec{j}+4\vec{k})$

$=7\vec{i}+\vec{k}$

$$\cos\theta=\frac{(2\vec{a}+\vec{b}).(\vec{a}+2\vec{b})}{\left|2\vec{a}+\vec{b}\right|\left|\vec{a}+2\vec{b}\right|}$$

$$=\frac{(5\vec{i}+3\vec{j}-4\vec{k}).(7\vec{i}+\vec{k})}{\left|5\vec{i}+3\vec{j}-4\vec{k}\right|\left|\vec{i}+\vec{k}\right|}$$

$$=\frac{35-4}{\sqrt{25+9+16}\ \sqrt{49+1}}=\frac{31}{50}$$

2. Find the angles made by $\vec{a}=2\vec{i}+3\vec{j}+6\vec{k}$ with X, Y, Z axes

Sol: $\vec{a}=2\vec{i}+3\vec{j}+6\vec{k}$

$$\left.\begin{array}{l}\textit{unit vector}\\ \textit{in the direction of } \vec{a}\end{array}\right\}=\frac{\vec{a}}{|\vec{a}|}=\frac{2\vec{i}+3\vec{j}+6\vec{k}}{\left|2\vec{i}+3\vec{j}+6\vec{k}\right|}=\frac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{2^2+3^2+6^2}}$$

$$=\left(\frac{2}{7}\right)\vec{i}+\left(\frac{3}{7}\right)\vec{j}+\left(\frac{6}{7}\right)\vec{k}$$

Here $\frac{2}{7},\frac{3}{7},\frac{6}{7}$ are the direction cosines of line

$\therefore \cos\alpha=\frac{2}{7},\ \cos\beta=\frac{3}{7},\ \cos\gamma=\frac{6}{7}$

$\alpha=\cos^{-1}\left(\frac{2}{7}\right),\ \beta=\cos^{-1}\left(\frac{3}{7}\right),\ \gamma=\cos^{-1}\left(\frac{6}{7}\right)$

3. **Let** $\vec{a}=\vec{i}+2\vec{j}+3\vec{k}$ **,** $\vec{b}=-\vec{i}+2\vec{j}+3\vec{k}$ **,** $\vec{c}=3\vec{i}+\vec{j}$ **and** $\vec{d}$ **is a non zero vector perpendicular to** $\vec{a}\,\&\,\vec{b}$ **if then angle between** $\vec{c}$ **and** $\vec{d}$ **is** θ **then find** $\cos\theta$

Sol: Since $\vec{d}$ is $\perp^r$ both $\vec{a}\,\&\,\vec{b}$

Let $\vec{a}\times\vec{b}$ is $\perp^r$ both $\vec{a}\,\&\,\vec{b}$

$\vec{d}$ parallel $\vec{a}\times\vec{b}$

$\therefore \vec{d}=p(\vec{a}\times\vec{b})$ where $p\in R$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 3\\ -1 & 2 & 3\end{vmatrix}=\vec{i}(6-6)-\vec{j}(3+3)+\vec{k}(2+2)$$

$\vec{a}\times\vec{b}=-6\vec{j}+4\vec{k}$

$\therefore \vec{d}=p(-6\vec{j}+4\vec{k})$

$$\cos\theta=\frac{\vec{c}.\vec{d}}{|\vec{c}||\vec{d}|}=\frac{p(3\vec{i}+\vec{j}).(-6\vec{j}+4\vec{k})}{|3\vec{i}+\vec{j}||p(-6\vec{i}+4\vec{j}|}$$

$$=\frac{p(-6)}{(\sqrt{q+1})\,R\sqrt{36+16}}$$

$$=\frac{-6}{\sqrt{10}\sqrt{52}}=\frac{-6}{\sqrt{10}\,2\sqrt{13}}$$

$$\therefore \cos\theta=\frac{-3}{\sqrt{130}}$$

4. **If** $\vec{a}=\vec{i}-2\vec{j}+3\vec{k}$ **,** $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$ **and** $\vec{c}=\vec{i}+3\vec{j}-2\vec{k}$ **verify that** $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

Sol: $$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & -1\\ 1 & 3 & -2\end{vmatrix}$$

$$=\vec{i}(-2+3)-\vec{j}(-4+1)+\vec{k}(6-1)$$

$= \vec{i} + 3\vec{j} + 5\vec{k}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

$= \vec{i}(-10-9) - \vec{j}(5-3) + \vec{k}(3+2)$

$= -19\vec{i} - 2\vec{j} + 5\vec{k}$

$\vec{a}.\vec{c} = 1-6-6 = -11$

$\vec{a}.\vec{b} = 2-2-3 = -3$

$R.H.S.(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= -11(2\vec{i} + \vec{j} - \vec{k}) + 3(\vec{i} + 3\vec{j} - 2\vec{k})$

$= -22\vec{i} - 11\vec{j} + 11\vec{k} + 3\vec{i} + 9\vec{j} - 6\vec{k}$

$= -19\vec{i} - 2\vec{j} + 5\vec{k}$

$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

5. If A = (1, -1, 0), B = (1, 2, 1), C = (2, 3, 0) and D= (0, 1, t) be four points if the projection of AB on CD is equal to that of CD on AB then find t

Sol: $\overrightarrow{AB} = 3\vec{j} + \vec{k}$, $\overrightarrow{CD} = -2\vec{i} - 2\vec{j} + t\vec{k}$

Projection of AB an CD = projection of CD on AB

$$\frac{\overrightarrow{AB}.\overrightarrow{CD}}{\left|\overrightarrow{CD}\right|} = \frac{\overrightarrow{CD}.\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

$\therefore \left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$

$\left|3\vec{i} + \vec{k}\right| = \left|-2\vec{i} - 2\vec{j} + t\vec{k}\right|$

$\sqrt{3^2 + 1^2} = \left|-2\vec{i} - 2\vec{j} + t\vec{k}\right|$

$\sqrt{3^2+1^2}=\sqrt{4+4+t^2}$

$10=8+t^2 \Rightarrow t^2=2$

$t=\pm\sqrt{2}$

6. **Find the perpendicular distance from the origin to the plane passing through the points (1, 0, 5) (1, -5, -1) and (-3, 5, 0)**

Sol: Let $\vec{a}=\vec{i}+5\vec{j}$, $\vec{b}=\vec{i}-5\vec{j}-\vec{k}$

$\vec{c}=-3\vec{i}+5\vec{j}$ be the given points. The equation of plane passing through the points $\vec{a},\vec{b},\vec{c}$ is

$[\vec{r}-\vec{a}\ \vec{b}-\vec{a}\ \ \vec{c}-\vec{a}]=0$

$(\vec{r}-\vec{a}).(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=0$

$\left\{\vec{r}.(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right\}-\vec{a}.(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=0$

$\vec{b}-\vec{a}=-5\vec{j}-6\vec{k}$ $\vec{c}-\vec{a}=-4\vec{i}+5\vec{j}-5\vec{k}$

$$(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & -5 & -6\\ -4 & 5 & -5\end{vmatrix}$$

$\vec{i}(25+30)-\vec{j}(-24)+\vec{k}(0-20)$

$55\vec{i}+24\vec{j}-20\vec{k}$

$\vec{a}.\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=(\vec{i}+5\vec{k}).(55\vec{i}+24\vec{j}-20\vec{k})$

= 55 - 100 = - 45

$\therefore$ equation of plane is

$$\frac{\vec{r}.(55\vec{i}+24\vec{j}-20\vec{k})}{|55\vec{i}+24\vec{j}-20\vec{k}|}=\frac{-45}{|55\vec{i}+24\vec{j}-20\vec{k}|}$$

$\therefore \perp^r$ distance from the origin to the plane

$$= \frac{|-45|}{\sqrt{(55)^2 + (24)^2 + (-20)^2}}$$

$$= \frac{45}{\sqrt{4001}}$$

7. If $\vec{a} = 3\vec{i} - 4\vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 6\vec{k}$ are diagonals of a parallelogram then show that it is a rhombus

Sol: Given $\vec{a} = 3\vec{i} - 4\vec{j} - \vec{k} \quad \vec{b} = 2\vec{i} + 3\vec{j} - 6\vec{k}$

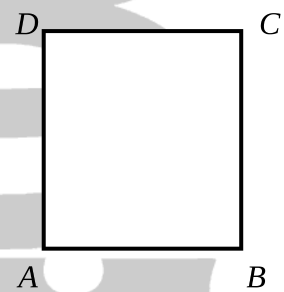
$\vec{a}.\vec{b} = (3\vec{i} - 4\vec{j} - \vec{k}).(2\vec{i} + 3\vec{j} - 6\vec{k})$

= 6 – 12 + 6 = 0

$\therefore \vec{a} \perp^r \vec{b}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \rightarrow (1)$

$\Rightarrow \overrightarrow{BD} = -\overrightarrow{AB} + \overrightarrow{BC} \quad \therefore \overrightarrow{AD} = \overrightarrow{BC}$



$\therefore \overrightarrow{BD} = \overrightarrow{BC} - \overrightarrow{AB} \rightarrow (2)$

(1) + (2) We have

$\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$

$\Rightarrow \overrightarrow{BC} = \frac{\vec{a} + \vec{b}}{2} = \frac{5\vec{i} - \vec{j} - 7\vec{k}}{2} \Rightarrow |\overrightarrow{BC}| = \frac{1}{2}\sqrt{25 + 1 + 49} = \frac{5\sqrt{2}}{2}$

$\overrightarrow{AC} - \overrightarrow{BD} = 2\overrightarrow{AB}$

$\overrightarrow{AB} = \frac{1}{2}(\vec{i} - 7\vec{j} + 5\vec{k})$

$|\overrightarrow{AB}| = \frac{1}{2}\sqrt{1 + 49 + 25} = \frac{5\sqrt{3}}{2}$

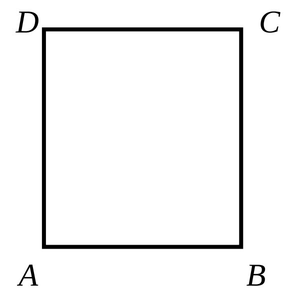
And $\overrightarrow{AB}.\overrightarrow{BO} \neq 0$ i.e. adjacent sides are equal

$\overrightarrow{AB}.\overrightarrow{BC} \neq 0$ i.e. adjacent sides are not perpendicular

$\overrightarrow{AC}.\overrightarrow{BD} = 0$ i.e. diagonals are perpendicular

Hence ABCD is a rhombus

8. Find the acute angle between the diagonals of to parallelogram whose adjacent sides are $\overline{AB} = 3\vec{i} - 2\vec{j} + 2\vec{k}$ $\overline{BC} = \vec{i} - 2\vec{k}$



Sol: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= 3\vec{i} - 2\vec{j} + 3\vec{k} + \vec{i} - 2\vec{k}$

$= 4\vec{i} - 2\vec{j}$

$\overrightarrow{BO} = \overrightarrow{BA} + \overrightarrow{AD}$

$= -\overrightarrow{AB} + \overrightarrow{BC}$ $\quad\quad \therefore \overrightarrow{AD} + \overrightarrow{BO}$

$\therefore \overrightarrow{BC} - \overrightarrow{AB}$

$= \vec{i} - 2\vec{k} - 3\vec{i} + 2\vec{j} - 2\vec{k}$

$= -2\vec{i} + 2\vec{j} - 4\vec{k}$

$$\cos\theta = \frac{\overrightarrow{AC}.\overrightarrow{BD}}{\left|4\vec{i} - 2\vec{j}\right|\left|2\vec{i} + 2\vec{j} - 4\vec{k}\right|}$$

$$= \frac{-8-4}{\sqrt{16+4}\sqrt{4+4+16}} = \frac{4}{2\sqrt{5}2\sqrt{6}}$$

$\sin\theta$ is acute

$$\cos\theta = \frac{3}{\sqrt{3}\sqrt{10}} = \sqrt{\frac{3}{10}}$$

$$\theta = \cos^{-1}\sqrt{\frac{3}{10}}$$

9. If the position vector of vertices of triangle are $0, \vec{a}$ and $\vec{b}$ and triangle is its area then prove that $4\Delta^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})$

Sol: Let $\overline{OA} = \vec{a}$ $\overline{OB} = \vec{b}$ be the given position vector of A and B

$\therefore$ area of triangle OAB = $\Delta = \frac{1}{2}|\overline{OA} \times \overline{OB}|$

$2\Delta = |\vec{a} \times \vec{b}|$

$4\Delta^2 = |\vec{a} \times \vec{b}|^2$

$= |\vec{a}|^2 |\vec{b}|^2 \sin^2(\vec{a}, \vec{b})$

$= |\vec{a}|^2 |\vec{b}|^2 \{1 - \cos^2(\vec{a}, \vec{b})\}$

$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2(\vec{a}, \vec{b})$

$4\Delta^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2$

10. In the coordinate plane $A(1, y_0)$ is a point on the curve $y = 2\sqrt{x}$. The tangent at A to the curve meets x-axis in B. If O is then find $\overline{AB}.\overline{OA}$

Sol: Given curve is $y = 2\sqrt{x}$

$A(1, y_0)$ is a point on the curve

$\therefore y_0 = 2\sqrt{1} \Rightarrow y_0 = 2$

$\frac{dy}{dx} = \frac{2}{2\sqrt{x}}$

Slope of tangent $= \left(\frac{dy}{dx}\right)_{(1, y_0)} = \frac{1}{\sqrt{1}} = 1$

Equation of tangent of A is

$(y - y_0) = 1(x - 1)$

$y - y_0 = x - 1$

$x - y = 1 - y_0$

This cuts x-axis at B

Put y=0

$\therefore x = 1 - y_0$

$\therefore B = (1 - y_0, 0)$

$\overrightarrow{AB}.\overrightarrow{OA} = (-y_0\vec{i} - y\vec{j}).(\vec{i} + y_0\vec{j})$

$= -y_0 - {y_0}^2$

= -2 – 4 = - 6

11. For any two vectors $\vec{a}$ and $\vec{b}$ prove that $(\vec{a}\times\vec{b})^2 + (\vec{a}.\vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

Sol: $(\vec{a}\times\vec{b})^2 + (\vec{a}.\vec{b})^2$

$|\vec{a}|^2 |\vec{b}|^2 \sin^2(\vec{a},\vec{b}) + (\vec{a}.\vec{b})^2$

$|\vec{a}|^2 |\vec{b}|^2 \{1 - \cos^2(\vec{a},\vec{b})\} + (\vec{a}.\vec{b})^2$

$|\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2(\vec{a},\vec{b}) + (\vec{a}.\vec{b})^2$

$|\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2 + (\vec{a}.\vec{b})^2$

$= |\vec{a}|^2 |\vec{b}|^2 = R.H.S$

12. **If** $\vec{a}=\frac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k})$**,** $\vec{b}=\frac{1}{7}(3\vec{i}-6\vec{j}+2\vec{k})$**,** $\vec{c}=\frac{1}{7}(6\vec{i}+2\vec{j}-3\vec{k})$ **are such that** $\vec{a}\times\vec{b}=\lambda\vec{c}$ **then find** λ

Sol: $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 6\\ 3 & -6 & 2\end{vmatrix}$

$=\frac{1}{49}\{\vec{i}(6+36)-\vec{j}(4-18)+\vec{k}(-12-9)$

$=\frac{1}{49}\{42\vec{i}+14\vec{j}-21\vec{k}\}$

$\frac{7}{49}\{6\vec{i}+2\vec{j}-3\vec{k}\}=\frac{1}{7}\{6\vec{i}+2\vec{j}-3\vec{k}\}$

$\vec{a}\times\vec{b}=\vec{c}$

$\therefore\lambda=1$

III. Long Answer Questions:

1. **If the vectors** $(\cos x)\vec{i}+(\sin x)\vec{j}$ **and** $x\vec{i}+(\sin x)\vec{j}$ **are collinear vector then find the range of x**

Sol: Let $\vec{a}=(\cos x)\vec{i}+(\sin x)\vec{j}$

$\vec{b}=x\vec{i}+(\sin x)\vec{j}$

Since $\vec{a},\vec{b}$ are collinear

$\vec{a}\times\vec{b}=\vec{o}$

$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos x & \sin x & 0\\ x & \sin x & 0\end{vmatrix}=\vec{0}$

$\vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(\cos x\,\sin x-x\sin x)=\vec{0}$

$\therefore\cos x\sin x-x\sin x=0$

$\sin x\{\cos x - x\} = 0$

Sin x = 0 or cos x = x

$x = \{n\pi / n \in z\}$

$\therefore$ Rnage of $x = \{n\pi / n \in z\} u(x_o)$ where x_0 shatisfites the equation

$\cos x_0 = x_0$ and $\frac{\pi}{6} < x_0 < \frac{\pi}{3}$

2. **A (1, -1, 2), B (1, 0, 2), C(2, -2, 3), D(4, 2, 1) are the vertices of a tetrahedral then find the distance of the plane ΔABC from is**

Sol: Cartesian equation of the plane passing through A(1, -1, 2), B(1, 0, 2), C(2, -2, 3) is

$$\begin{vmatrix} x-1 & y+1 & z-2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 0 \Rightarrow (x-1)(1) - (y+1)0 + (z-2)(0-1)$$

$$\Rightarrow x - 1 - z + 2 = 0 \rightarrow x - z + 1 = 0$$

$\therefore \perp^r$ distance from D to the plane

$$= \frac{|4-1+1|}{\sqrt{1^2+1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

3. **If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{b}| = |\vec{c}|$ then show that**

$[(\vec{a}+\vec{b}) \times (\vec{a}+\vec{c})] \times (\vec{b}\times\vec{c}).(\vec{b}\times\vec{c}) = 0$

Sol: $[(\vec{a}+\vec{b}) \times (\vec{a}+\vec{c})] \times \left[(\vec{b}\times\vec{c})\right].(\vec{b}\times\vec{c})$

$[\vec{a}+\vec{a}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}] \times \left[(\vec{b}\times\vec{c})\right].(\vec{b}+\vec{c})$

$[\vec{0}+(\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{b}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{b}\times\vec{c})].(\vec{b})$

$[((\vec{a}\times\vec{c}.\vec{c})\vec{b}-(\vec{a}\times\vec{c}.\vec{b})\vec{c}+(\vec{b}\times\vec{a}.\vec{c})\vec{b}-(\vec{b}\times\vec{a}.\vec{b})\vec{c}+\vec{o}].(\vec{b}+\vec{c})$

$[0+[\vec{a}\vec{b}\vec{c}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}].(.\vec{b}+\vec{c})$

$[\vec{a}\vec{b}\vec{c}](\vec{c}.\vec{b})+[\vec{a}\,\vec{b}\,\vec{c}](\vec{c}.\vec{c}) = [\vec{a}\vec{b}\vec{c}]\vec{b}.\vec{b}-[\vec{a}\,\vec{b}\,\vec{c}](\vec{b}-\vec{c}) \qquad \therefore \vec{b}-\vec{c} = \vec{c}-\vec{b}$

$[\vec{a}\,\vec{b}\,\vec{c}]|\vec{c}|^2 - [\vec{a}\,\vec{b}\,\vec{c}]|\vec{b}|^2 = 0 \qquad \therefore |\vec{b}| = |\vec{c}|$

4. In the space $A(\vec{a})$ is a point and $\vec{b}$ is a vector. If $p(\vec{r})$ is any prove that such that $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ show that locus of P is a straight line

Sol: Given $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$

$\vec{r}\times\vec{b} - \vec{a}\times\vec{b} = 0$

$(\vec{r}-\vec{a})\times\vec{b} = 0$

$\vec{r}-\vec{a}$ parallel $\vec{b}$

$\therefore \vec{r}-\vec{a} = t\vec{b}$ where $t \in R$

$\vec{r} = \vec{a} + t\vec{b}$

This represent a straight line passing through the point $\vec{a}$ and parallel $\vec{b}$

$\therefore \vec{r}$ lies on a straight lines

Hence locus of P is a straight line

5. $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular unit vector and $\vec{d}$ is a unit vector equally inclined to each of $\vec{a},\vec{b}$ and $\vec{o}$ and are angle 60^0 and find $|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2$

Sol: Given $\vec{a},\vec{b},\vec{c}$ are three mutually $\perp^r$ vectors

$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0 \qquad |\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{d}| = 1$

$(\vec{d},\vec{a}) = (\vec{d},\vec{b}) = (\vec{d},\vec{c}) = 60^0 \qquad |\vec{d}| = 1$

$|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2 = |\vec{a}+\vec{b}|^2 + |\vec{c}+\vec{d}|^2 + 2(\vec{a}.\vec{b}).(\vec{c}+\vec{d})$

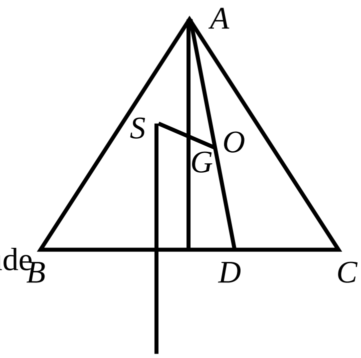
$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} + |\vec{c}|^2 + |\vec{d}|^2 + 2\vec{c}.\vec{d} + 2\vec{a}\div\vec{c} + 2\vec{a}.\vec{d} + 2\vec{b}.\vec{c} + 2\vec{b}.\vec{d}$

$= 1+1+0+1+1+2|\vec{c}||\vec{d}|\cos 60^0 + 0 + 02|\vec{a}||\vec{d}|\cos 60^0 + 2|\vec{b}||\vec{d}|\cos 60^0|$

= 7

Long Answer Questions :-

1. **In a triangle ABC prove that**



i) $\overline{SA}+\overline{SB}+\overline{SC}=\overline{SO}$ **here 'S' is the circum**

ii) $\overline{OA}+\overline{OB}+\overline{OC}=2\overline{SO}$ **Centre O is orthocenter**

Sol : In triangle ABC $\overline{SX}$ is the $\perp^r$ bisector AX is the median AD is the altitude

S is the circum centre 'G' is the B centroid O is the orthocenter.

SG = GO = 1:2 since SX and AD are parallel

$\overline{AO}=2\overline{SX}$

Let S be the origin of vector

$\overline{SX}=\dfrac{\overline{SB}+\overline{SC}}{2}\Rightarrow \overline{SB}+\overline{SC}=2\overline{SX}$

$\Rightarrow \overline{SB}+\overline{SC}=\overline{AO}\ \{\because 2\overline{SX}=\overline{AB}\}$

Adding $\overline{SA}$ on both sides

$\overline{SA}+\overline{SB}+\overline{SC}=\overline{SA}+\overline{AO}$

$\overline{SA}+\overline{SB}+\overline{SC}=\overline{SO}$

ii) In triangle OSA

$\overline{OS}+\overline{SA}=\overline{OA}$

In triangle OSB $\overline{OS}+\overline{SB}=\overline{OB}$

In triangle OSC $\overline{OS}+\overline{SC}=\overline{OC}$

Adding we have

$\overline{OA}+\overline{OB}+\overline{OC}=(\overline{OS}+\overline{SA})+(\overline{OS}+\overline{SB})+(\overline{OS}+\overline{SB})$

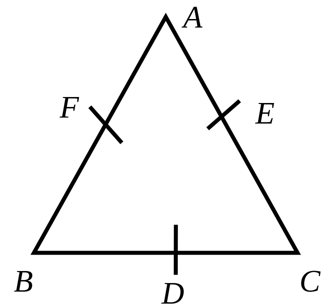
$\overline{OA}+\overline{OB}+\overline{OC}=3\overline{OS}+(\overline{SA}+\overline{SB}+\overline{SC}\ )$

$=3\overline{OS}-\overline{OS}$

$= 2\overrightarrow{OS}$

2. In triangle ABC, D, E, F are the mid points of the sides BC, CA and AB respectively then show that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\overrightarrow{O}$

Sol : Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OC}=\vec{c}$ be the position vectors of A, B, C respectively



$\overrightarrow{OD}=\frac{\vec{b}+\vec{c}}{2}$, $\overrightarrow{OE}=\frac{\vec{a}+\vec{c}}{2}$, $\overrightarrow{OF}=\frac{\vec{a}+\vec{b}}{2}$

$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\overrightarrow{OD}-\overrightarrow{OA}+\overrightarrow{OE}-\overrightarrow{OB}+\overrightarrow{OF}-\overrightarrow{OC}$

$=\frac{\vec{b}+\vec{c}}{2}-\vec{a}+\frac{\vec{a}+\vec{c}}{2}-\vec{b}+\frac{\vec{a}+\vec{b}}{2}-\vec{c}$

$=\frac{\vec{b}+\vec{c}+\vec{a}+\vec{c}+\vec{a}+\vec{b}}{2}-\vec{a}-\vec{b}-\vec{c}$

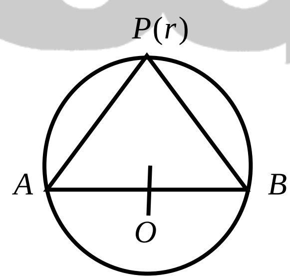
$=\vec{a}+\vec{b}+\vec{c}-\vec{a}-\vec{b}-\vec{c}$

$=\vec{0}$

3. By vector method show that angle in semi circle is a right angle

Sol: Let APB be a semi circle with centre O.

OA = OB = OP



Let $\overrightarrow{OA}=-\vec{a}$, $\overrightarrow{OB}=\vec{a}$ and $\overrightarrow{OP}=\vec{r}$

$|\overrightarrow{OP}|=|\overrightarrow{OA}|\Rightarrow|\vec{r}|=|\vec{a}|$

$\overrightarrow{AP}=\vec{r}+\vec{a}$ $\quad$ $\overrightarrow{BP}=\vec{r}-\vec{a}$

$\overrightarrow{AP}.\overrightarrow{BP}=(\vec{r}+\vec{a}).(\vec{r}-\vec{a})$

$=(\vec{r})^2-(\vec{a})^2$

$=|\vec{r}|^2-|\vec{a}|^2$

$=|\vec{a}|^2-|\vec{a}|^2=0$

$\therefore \overrightarrow{AP}.\overrightarrow{BP} = 0 \quad \therefore AP \perp^r BP$

Hence $\angle APB = 90^0$

4. In ΔABC prove that i) $a^2 = b^2 + c^2 - 2bc\cos A$ ii) a = b cos c + c cos B

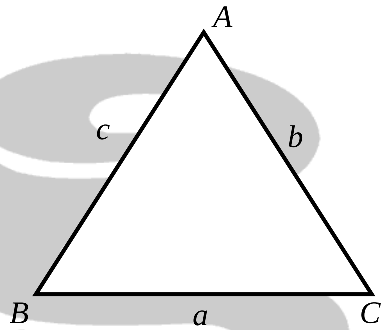
Sol: i) In a triangle ABC

$\left|\overrightarrow{BC}\right| = a \qquad \left|\overrightarrow{CA}\right| = b, \qquad \left|\overrightarrow{AB}\right| = c$

$A = (\overrightarrow{AB}, \overrightarrow{AC}) \quad B = (\overrightarrow{BC}, \overrightarrow{BA}) \quad C = (\overrightarrow{CB}, \overrightarrow{CA})$

We know that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$\overrightarrow{AB} + \overrightarrow{CA} = -\overrightarrow{BC}$



S.O.B.S

$(\overrightarrow{AB} + \overrightarrow{CA})^2 = (-\overrightarrow{BC})^2$

$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 + 2\overrightarrow{AB}.\overrightarrow{CA} = \left|\overrightarrow{BC}\right|^2$

$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 - 2\overrightarrow{AB}.\overrightarrow{CA} = \left|\overrightarrow{BC}\right|^2$

$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 - 2\left|\overrightarrow{AB}\right|\left|\overrightarrow{CA}\right|\cos(\overrightarrow{AB}, \overrightarrow{AC}) = \left|\overrightarrow{BC}\right|^2$

$c^2 + b^2 - 2cb\cos A = a^2$

$\therefore a^2 = b^2 + c^2 - 2bc\cos A$

ii) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$\overrightarrow{BC} = -\overrightarrow{AB} - \overrightarrow{CA}$

$\overrightarrow{BC}.\overrightarrow{BC} = (-\overrightarrow{AB} - \overrightarrow{CA}).\overrightarrow{BC}$

$\left|\overrightarrow{BC}\right|^2 = \overrightarrow{AB}.\ \overrightarrow{BC} - \overrightarrow{CA}.\overrightarrow{BC}$

$\left|\overrightarrow{BC}\right|^2 = \overrightarrow{BA}.\ \overrightarrow{BC} + \overrightarrow{CB}.\overrightarrow{CA}$

$$\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos B + \left|\overrightarrow{CB}\right|\left|\overrightarrow{CA}\right|\cos C$$

$$a^2 = ca\cos B + ab\cos C$$

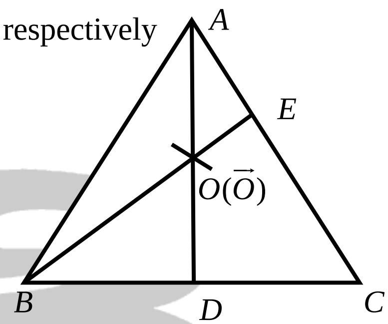
A = b cos c + c cos B

5. By using vector method prove that altitudes of a triangle are concurrent

Sol: In a triangle ABC let the altitudes AD and BE meet at O.

Let O be the origin of vectors

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ be the Prove that of A, B, C respectively



$\overrightarrow{AD} \perp^r \overrightarrow{BC}$ $\therefore \overrightarrow{OA} \perp^r BC$ $\Rightarrow \overrightarrow{OA}.\overrightarrow{BC} = 0$

$\therefore \vec{a}.(\vec{c} - \vec{b}) = 0 \Rightarrow \vec{a}.\vec{c} = \vec{a}.\vec{b} \rightarrow (1)$

$\overrightarrow{BE} \perp^r AC \Rightarrow \overrightarrow{OB} \perp^r AC$

$\therefore \overrightarrow{OB}.\overrightarrow{AC} = 0$ $\vec{b}.(\vec{c} - \vec{a}) = 0$

$\vec{b}.\vec{c} = \vec{b}.\vec{a} \rightarrow (2)$

From (1) and (2)

$\vec{a}.\vec{c} = \vec{b}.\vec{c}$

$\vec{a}.\vec{c} - \vec{b}.\vec{c} = 0$

$(\vec{a} - \vec{b}).\vec{c} = 0 \rightarrow \vec{c}.(\vec{b} - \vec{a}) = 0$

$\overrightarrow{OC}.\overrightarrow{AB} = 0$

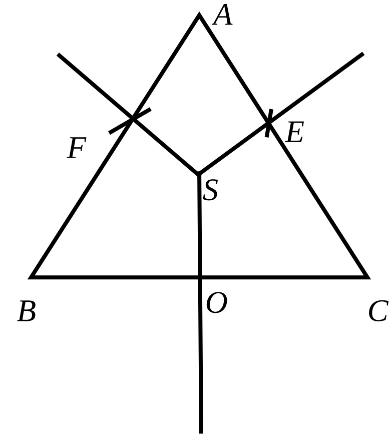
$\overrightarrow{OC} \perp^r AB$

$\therefore$ Altitude through c also passes through O

$\therefore$ Altitudes are concurrent.

6. Show that perpendicular bisector are concurrent

Sol: Let SO, SE be the $\perp^r$ bisectors of BC and AC respectively



Let S be the origin of vectors

Let $\overline{SA}=\vec{a}$ $\overline{SB}=\vec{b}$ $\overline{SC}=\vec{c}$

$$\overline{SD}=\frac{\overline{SB}+\overline{SC}}{2}\Rightarrow \overline{SD}=\frac{\vec{b}+\vec{c}}{2}$$

$$\overline{SD}\perp^r \overline{BC}\Rightarrow \overline{SD}.\overline{BC}=0$$

$$\frac{(\vec{b}+\vec{c})}{2}.(\vec{c}-\vec{b})=0$$

$$|\vec{c}|^2-|\vec{b}|^2=0\Rightarrow |\vec{c}|^2=|\vec{b}|^2\rightarrow(1)$$

$$\overline{SE}=\frac{\overline{SA}+\overline{SC}}{2}=\frac{\vec{a}+\vec{c}}{2}$$

$$\overline{SE}\perp^r \overline{AC}\Rightarrow \overline{SE}.\overline{AC}=0$$

$$\left(\frac{\vec{a}+\vec{c}}{2}\right).(\vec{c}-\vec{a})=0$$

$$|\vec{c}|^2-|\vec{a}|^2=0\Rightarrow |\vec{c}|^2=|\vec{a}|^2\rightarrow(2)$$

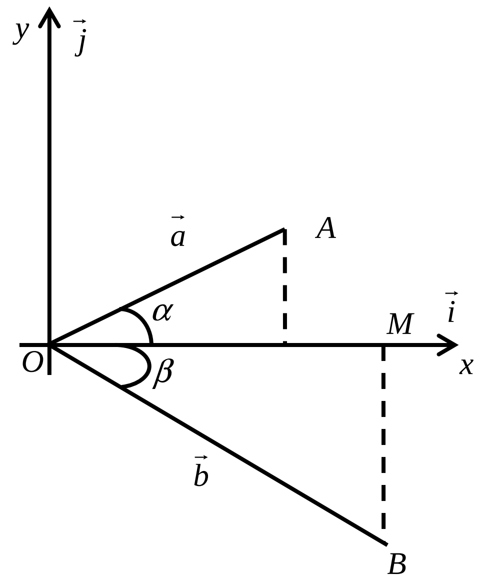
From (1) and (2) $|\vec{a}|^2=|\vec{b}|^2$

$$|\vec{a}|^2-|\vec{b}|^2=0$$

$$(\vec{a}-\vec{b}).\frac{(\vec{a}+\vec{b})}{2}=0$$

$$\overline{BA}.\overline{SF}=0\Rightarrow \overline{BA}\perp^r \overline{SF}$$

$\therefore \perp^r$ bisector of AB passes through S hence $\perp^r$ bisectors are concurrent



7. **Prove that i)** $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

ii) $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Sol: Let $\vec{i},\vec{j}$ be two unit vectors along $\overrightarrow{ox}$ i.e. x-axis $\overrightarrow{oy}$ i.e. y-axis respectively

Let $\overline{OA}=\vec{a}$, $\overline{OB}=\vec{b}$ such that $(\vec{a},\vec{i})=\alpha$, $(\vec{a},\vec{j})=\beta$ and

$|\overline{OA}|=|\vec{a}|=1$ and $|\overline{OB}|=|\vec{b}|=1$

L be the projection of A on x-axis M be the projection B on x-axis

$A=(\cos\alpha,\sin\alpha,0)$ $\qquad$ $B=(\cos\beta,\sin\beta,0)$

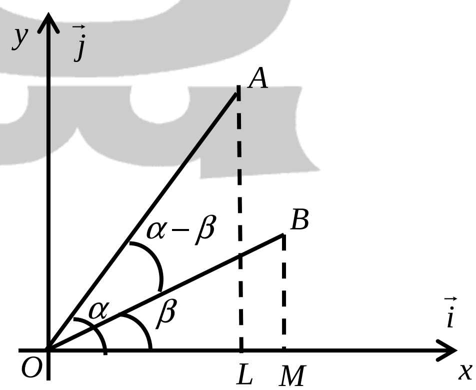
$\overline{OA}=(\cos\alpha)\vec{i}+(\sin)\vec{j}$ $\qquad$ $\overline{OB}=(\cos\beta)\vec{i}-(\sin\beta)\vec{j}$

$(\overline{OA},\overline{OB})=\alpha+\beta$

$$\cos(\alpha+\beta)=\frac{\overline{OA}.\overline{OB}}{|\overline{OA}||\overline{OB}|}$$

$$\cos(\alpha+\beta)=\frac{\{(\cos\alpha)\vec{i}+(\sin\alpha)\vec{j}\}.\{(\cos\beta)\vec{i}-(\sin\beta)\vec{j}\}}{1\times1}$$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\varpi$



ii) Let $\vec{i},\vec{j}$ be two unit vectors along $\overrightarrow{ox}$ i.e x-axis $\overrightarrow{oy}$ i.e. y-axis respectively

Let $\overline{OA}=\vec{a}$, $\overline{OB}=\vec{b}$ be two unit vectors such that

$(\vec{a},\vec{i})=\alpha$, $(\vec{a},\vec{j})=\beta$

$A=(\cos\alpha,\sin\alpha,0)$, $\quad B=(\cos\beta,\sin\beta,0)$

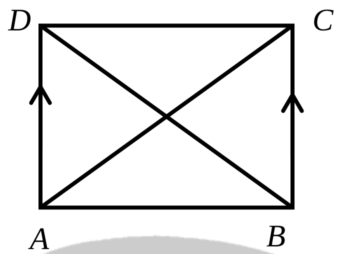
$\overline{OA}=(\cos\alpha)\vec{i}+(\sin\alpha)\vec{j}$ $\qquad$ $\overline{OB}=(\cos\beta)\vec{i}+(\sin\beta)\vec{j}$

$(\overrightarrow{OA}, \overrightarrow{OB}) = \alpha - \beta$

$$\cos(\alpha - \beta) = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}$$

8. By vector method show that in a parallelogram the sum of the squares of the diagonals is twice the sum of squares of its adjacent sides

Sol: $(\overrightarrow{AC})^2 + (\overrightarrow{BD})^2 = (\overrightarrow{AB} + \overrightarrow{BC})^2 + (\overrightarrow{BA} + \overrightarrow{AD})^2$



But $\overrightarrow{BA} = -\overrightarrow{AB}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

$= (\overrightarrow{AB} + \overrightarrow{BC})^2 + (-\overrightarrow{AB} + \overrightarrow{BC})^2$

$= (\overrightarrow{AB} + \overrightarrow{BC})^2 + (\overrightarrow{BC} - \overrightarrow{AB})^2$

$= |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + 2\overrightarrow{AB}.\overrightarrow{BC} + |\overrightarrow{BC}|^2 + |\overrightarrow{AB}|^2 - 2\overrightarrow{BC}.\overrightarrow{AB}$

$= 2\left\{|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2\right\}$

9. Show that in a parallelogram if diagonals are equal then prove that it is a rectangle

Sol: Given that $|\overrightarrow{AC}| = |\overrightarrow{BD}|$

$|\overrightarrow{AB} + \overrightarrow{BC}| = |\overrightarrow{BA} + \overrightarrow{AD}|$

$|\overrightarrow{AB} + \overrightarrow{BC}| = |-\overrightarrow{AB} + \overrightarrow{BC}| \qquad \because \overrightarrow{BA} = -\overrightarrow{AB} \quad \overrightarrow{AD} = \overrightarrow{BC}$

$|\overrightarrow{AB} + \overrightarrow{BC}| = |\overrightarrow{BC} - \overrightarrow{AB}|$

S.O.B.S

$|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + 2\overrightarrow{AB}.\overrightarrow{BC} = |\overrightarrow{BC}|^2 + |\overrightarrow{AB}|^2 - 2\overrightarrow{BC}.\overrightarrow{AB}$

$4\overrightarrow{AB}.\overrightarrow{BC} = 0$

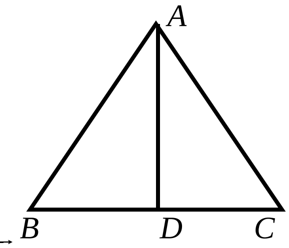
$\overrightarrow{AB}.\overrightarrow{BC}$

$\overrightarrow{AB} \perp^r \overrightarrow{BC}$

$\therefore$ ABCD is a rectangle

10. If D is the mid point of the side BC of triangle ABC, then show by vectors method that $AB^2 + BC^2 = 2(AD^2 + BD^2)$

Sol: D is the mid point of BC $\overrightarrow{BD} = \overrightarrow{DC}$



$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$ $\qquad$ $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

$(\overrightarrow{AB})^2 = (\overrightarrow{AD} - \overrightarrow{BD})^2$ $\qquad$ $\{\because \overrightarrow{DB} = -\overrightarrow{BD}\}$

$(\overrightarrow{AC})^2 = (\overrightarrow{AD} + \overrightarrow{DC})^2$

$(\overrightarrow{AB})^2 + (\overrightarrow{AC})^2 = (\overrightarrow{AD})^2 + (\overrightarrow{BD})^2 - 2\overrightarrow{AD}.\overrightarrow{BD} + \overrightarrow{AD}^2 + \overrightarrow{DC}^2 + 2\overrightarrow{AD}.\overrightarrow{DC}$

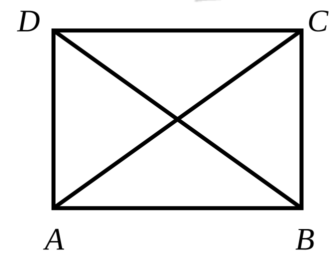
$= 2\left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BD}\right|^2 - 2\cancel{\overrightarrow{AD}.\overrightarrow{BD}} + \left|\overrightarrow{BD}\right|^2 + 2\cancel{\overrightarrow{AD}.\overrightarrow{BD}}$ $\qquad$ $\{\because \overrightarrow{BD} = \overrightarrow{DC}\}$

$= 2\left|\overrightarrow{AD}\right|^2 + 2\left|\overrightarrow{BD}\right|^2$

$\therefore AB^2 + AC^2 = 2\left[\overrightarrow{AD}^2 + \overrightarrow{BD}^2\right]$

11. Show by vector method the diagonals of a rhombus are at right angles

Sol: In a rhombus ABCD $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ $\qquad$ $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$\overrightarrow{BD} = -\overrightarrow{AB} + \overrightarrow{BC}$ $\qquad$ $\because \overrightarrow{BA} = -\overrightarrow{AB}$ $\quad$ $\overrightarrow{AD} = \overrightarrow{BC}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ $\qquad$ $\overrightarrow{BD} = \overrightarrow{BC} - \overrightarrow{AB}$

$\overrightarrow{BD}.\overrightarrow{BD} = (\overrightarrow{BC} + \overrightarrow{AB}).(\overrightarrow{BC} - \overrightarrow{AB})$

$= \left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2$

$= \cancel{\left|\overrightarrow{BC}\right|^2} - \left|\overrightarrow{BC}\right|^2 = 0$

$\overrightarrow{AC}.\overrightarrow{BD} = 0 \Rightarrow \overrightarrow{AC} \perp^r \overrightarrow{BD}$

Hence diagonals are at right angles Cross Products

1. **If $\vec{a},\vec{b}$ are two vectors then their cross product denoted by $\vec{a}\times\vec{b}$ defined as** $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\vec{n}\sin(\vec{a},\vec{b})$ **where $\vec{n}$ is the unit vector $\perp^r$ to both $\vec{a}$ and $\vec{b}$**

2. $\vec{a}\times\vec{b}$ is a vector

3. **If** $\vec{a}\times\vec{b} = 0$ then either of $\vec{a},\vec{b}$ is a null vector (or) $\vec{a}$ parallel $\vec{b}$

4. **If** $\vec{a},\vec{b}$ are not parallel vector then $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

5. **i)** $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ ii) $\vec{a}\times\vec{b} = \vec{o}$

6. $\vec{a}\times(-\vec{b}) = (-\vec{a})\times\vec{b} = -(\vec{a}\times\vec{b})$

7. $-\vec{a}\times-\vec{b} = \vec{a}\times\vec{b}$

8. $l\,\vec{a}\times m\,\vec{b} = lm(\vec{a}\times\vec{b})$

9. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ {left distributive}

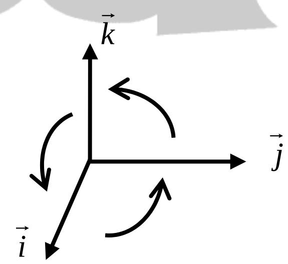
10. $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$ {right distributive}

11. If $\vec{i},\vec{j},\vec{k}$ are orthonormal unit vector then

i) $\vec{i}\times\vec{i} = \vec{o}$, $\vec{j}\times\vec{j} = \vec{o}$, $\vec{k}\times\vec{k} = \vec{o}$

ii) $\vec{i}\times\vec{j} = \vec{k}$ $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$

$\vec{j}\times\vec{i} = -\vec{k}$ $\vec{k}\times\vec{j} = -\vec{i}$ $\vec{i}\times\vec{k} = -\vec{j}$



12. If $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ $\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ then $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

13. If $\vec{a},\vec{b}$ are two vectors then $\sin(\vec{a},\vec{b}) = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

14. If $\vec{a},\vec{b}$ are two non parallel vectors then unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

15. If $\vec{a},\vec{b}$ are two vectors then $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

Sol: Let $(\vec{a},\vec{b})^2=\theta\ (\vec{a}\times\vec{b})=|\vec{a}||\vec{b}|\vec{n}\sin\theta$

$$(\vec{a},\vec{b})^2=|\vec{a}|^2|\vec{b}|^2|\vec{n}|^2\sin^2\theta$$

$$(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2\{1-\cos^2\theta\}$$

$$=|\vec{a}|^2|\vec{b}|^2-|\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2$$

$$(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$$

16. Vector Area

If Δ is the area of a figure and and $\vec{n}$ is units the perpendicular to the lamina then $\Delta\vec{n}$ is called vector area .

17. The vector area of a triangle ABC is $\frac{1}{2}\overline{AB}\times\overline{AC}=\frac{1}{2}\overline{BC}\times\overline{BA},\ =\frac{1}{2}\overline{CA}\times\overline{CB}$

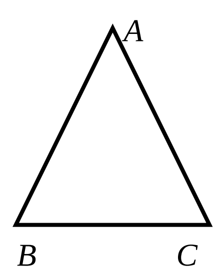
Sol: In a triangle $\overrightarrow{ABC}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ are the vectors represented by the sides AB, BC, CA

$$A=(\overrightarrow{AB},\overrightarrow{AC})\quad B=(\overrightarrow{BA},\overrightarrow{BC})\quad C=(\overrightarrow{CB},\overrightarrow{CA})$$

Let $\vec{n}$ be the unit vector $\perp^r$ $\overrightarrow{AB},\overrightarrow{AC}$ and $\overrightarrow{AB},\overrightarrow{AC},\vec{n}$ form right handed system area of triangle ABC

$$\Delta=\frac{1}{2}AB.AC\sin A$$

$$\Delta=\frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin A$$



$$\Delta\vec{n} = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\vec{n}\sin A$$

$$\Delta\vec{n} = \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}$$

$$\Delta\vec{n} = \frac{1}{2}\overrightarrow{BC}\times\overrightarrow{BA} = \frac{1}{2}\overrightarrow{CA}\times\overrightarrow{CB}$$

18. If $\vec{a},\vec{b},\vec{c}$ are the prove that of the vertices of the triangle ABC then vector area $= \frac{1}{2}\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\}$

Sol: $\overrightarrow{OA} = \vec{a}$ $\overrightarrow{OB} = \vec{b}$ $\overrightarrow{OC} = \vec{c}$ be the given vertices

Vector area $= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}$

$$= \frac{1}{2}\left\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right\}$$

$$= \frac{1}{2}\left\{\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}\right\}$$

$$= \frac{1}{2}\left\{\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right\}$$

19. Using vector method show that in triangle ABC $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Sol: In a triangle ABC $A = (\overrightarrow{AB},\overrightarrow{AC})$, $B = (\overrightarrow{BA},\overrightarrow{BC})$, $C = (\overrightarrow{CA},\overrightarrow{CB})$

$\left|\overrightarrow{AB}\right| = c$, $\left|\overrightarrow{BC}\right| = a$, $\left|\overrightarrow{CA}\right| = b$

We know that area of area of triangle ABC

$$= \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{BA}\right| = \frac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{CB}\right|$$

$$= \left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin A = \left|\overrightarrow{BC}\right|\left|\overrightarrow{BA}\right|\sin B = \left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|\sin c$$

$$\Rightarrow cb\sin A = ac\ \sin B = ab\ \sin C$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

20. If ABCD is a parallelogram and $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$ then the vector area of ABCD is $\vec{a} \times \vec{b}$

21. If ABCD is a parallelogram and $\overrightarrow{AC} = \vec{a}$, $\overrightarrow{BD} = \vec{b}$ then the vector area of ABCD is $\frac{1}{2}(\vec{a} \times \vec{b})$