MULTIPLE PRODUCTS -2

EXERCISE - 4(c)

I. Very Short Answer Questions

1. Compute $[\vec{i}-\vec{j}\quad \vec{j}-\vec{k}\quad \vec{k}-\vec{i}]$

Sol: $[\vec{i}-\vec{j}\quad \vec{j}-\vec{k}\quad \vec{k}-\vec{i}]=\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{vmatrix}=1(1-0)+1(0-1)+0=0$

2. If $\vec{a}=\vec{i}-2\vec{j}-3\vec{k}$ **,** $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$ **,** $\vec{c}=\vec{i}+3\vec{j}-2\vec{k}$

Sol: $\vec{a}.(\vec{b}\times\vec{c})=[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}1 & -2 & -3\\ 2 & 1 & -1\\ 1 & 3 & -2\end{vmatrix}$

1 (-2 + 3) +2(-4 +1) -3 (6-1)

1 – 6 – 15 = - 20

3. If $\vec{a}=(1,-1,-6)$, $\vec{b}=(1,-3,4)$ and $\vec{c}=(2,-5,3)$ then compute the following

i) $\vec{a}.(\vec{b}\times\vec{c})$ **ii)** $\vec{a}\times(\vec{b}\times\vec{c})$ **iii)** $(\vec{a}\times\vec{b})\times\vec{c}$

Sol: $\vec{a}=\vec{i}-\vec{j}-6\vec{k}$, $\vec{b}=\vec{i}-3\vec{j}+4\vec{k}$ and $\vec{c}=2\vec{i}-5\vec{j}+3\vec{k}$

i) $\vec{a}.(\vec{b}\times\vec{c})=[\vec{a}\ \vec{b}\,\vec{c}]=\begin{vmatrix}1 & -1 & -6\\ 1 & -3 & 4\\ 2 & -5 & 3\end{vmatrix}$

1 (-9 + 20) +1(3 – 8) -6 (-5 + 6)

11 – 5 – 6 = 0

ii) $\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -3 & 4\\ 2 & -5 & 3\end{vmatrix}$

$=\vec{i}(-90+20)-\vec{j}(3-8)+\vec{k}(-5+6)$

$=11\vec{i}+5\vec{j}+\vec{k}$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & -6\\ 11 & 5 & 1\end{vmatrix}$$

$$=\vec{i}(-1+30)-\vec{j}(1+66)+\vec{k}(5+11)$$

$$=29\vec{i}-67\vec{j}+16\vec{k}$$

iii) $$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & -6\\ 1 & -3 & 4\end{vmatrix}$$

$$=\vec{i}(-4-18)-\vec{j}(4+6)+\vec{k}(-3+1)$$

$$=-22\vec{i}-10\vec{j}-2\vec{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -22 & -10 & -2\\ 2 & -5 & 3\end{vmatrix}=\vec{i}(-30-10)-\vec{j}(-66+4)+\vec{k}(110+20)$$

$$=-40\vec{i}+62\vec{j}+130\vec{k}$$

4. Simplify the following

i) $(\vec{i}-2\vec{j}+3\vec{k})\times(2\vec{i}+\vec{j}-\vec{k}).(\vec{j}+\vec{k})$

ii) $(2\vec{i}-3\vec{j}+\vec{k}).(\vec{i}-\vec{j}+2\vec{k})\times(2\vec{i}+\vec{j}+\vec{k})$

Sol: $$(\vec{i}-2\vec{j}+3\vec{k})\times(2\vec{i}+\vec{j}-\vec{k}).(\vec{j}+\vec{k})=\begin{vmatrix}1 & -2 & 3\\ 2 & 1 & -1\\ 0 & 1 & 1\end{vmatrix}$$

$$=1(1+1)+2(2-0)+3(2-0)$$

= 12

ii) $$(2\vec{i}-3\vec{j}+\vec{k}).(\vec{i}-\vec{j}+2\vec{k})\times(2\vec{i}+\vec{j}+\vec{k})=\begin{vmatrix}2 & -3 & 1\\ 1 & -1 & 2\\ 2 & 1 & 1\end{vmatrix}$$

= 2 (-1 -2) +3 (1 – 4) + 1(1 + 2)

= - 6 – 9 + 3 = -12

5. Find the volume of parallelopiped having coterminus edges $\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\vec{j}$, $\vec{i}+2\vec{j}-\vec{k}$

Sol: Let $\vec{a}=\vec{i}+\vec{j}+\vec{k}$ $\quad \vec{b}=\vec{i}-\vec{j}$ $\quad \vec{c}=\vec{i}+2\vec{j}-\vec{k}$

Volume of parallelopeped $[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 2 & -1\end{vmatrix}$

= 1(1 – 0) – 1 (-1 – 0) +1(2 + 1)

= 1 + 1 + 3 = 5

6. Find the for which the vectors $2\vec{i}-3\vec{j}+\vec{k}$, $\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{j}-t\vec{k}$ are coplanar

Sol: Let $\vec{a}=2\vec{i}-3\vec{j}+\vec{k}$, $\quad \vec{b}=\vec{i}+2\vec{j}-3\vec{k}$, $\quad \vec{c}=\vec{j}-t\vec{k}$ be the vectors

Since $\vec{a},\vec{b},\vec{c}$ are coplanar then $[\vec{a}\,\vec{b}\,\vec{c}]=0$

$$\begin{vmatrix}2 & -3 & 1\\ 1 & 2 & -3\\ 0 & 1 & -t\end{vmatrix}=0$$

2 (-2t + 3) +3 (-t -0) +1(1 – 0) = 0

-4t + 6 – 3t + 1 $\Rightarrow$ 7t = 7$\Rightarrow$ t =1

7. For non-coplanar vector $\vec{a},\vec{b}$ and $\vec{c}$ determine p for which the vector $\vec{a}+\vec{b}+\vec{c}$, $\vec{a}+p\vec{b}+2\vec{c}$ and $-\vec{a}+\vec{b}+\vec{c}$ are coplanar

Sol: since $\vec{a},\vec{b},\vec{c}$ are non coplanar $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ the vectors $\vec{a}+\vec{b}+\vec{c}$, $\vec{a}+p\vec{b}+2\vec{c}$, $-\vec{a}+\vec{b}+\vec{c}$ are coplanar

$$\begin{vmatrix}1 & 1 & 1\\ 1 & p & 2\\ -1 & 1 & 1\end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}]=0$$

$\therefore$ 1(p -2) -1 91 + 2) + 1(1 +P) = 0 $\quad \therefore[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$

P – 2 – 3 + 1 + p = 2

2p – 4 = 0

P = 2

8. Determine λ, for which the volume of the paralleloped having coterminus edges $\vec{i}+\vec{j}$, $3\vec{i}-\vec{j}$ and

Sol: Let $\vec{a}=\vec{i}+\vec{j}$ $\vec{b}=3\vec{i}-\vec{j}$ $\vec{c}=3\vec{i}+\lambda\vec{k}$ be the given cotermines edges

$$\left|[\vec{a}\,\vec{b}\,\vec{c}]\right|=16$$

$$\begin{vmatrix}1 & 1 & 0\\ 3 & -1 & 0\\ 0 & 3 & \lambda\end{vmatrix}=16$$

$$1(-\lambda-0)-1(3\lambda-0)=16$$

$$-4\lambda=16\Rightarrow\lambda=\pm4$$

9. Find the volume of the tetrahedron having the edges $\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\vec{j}$ and $\vec{i}+2\vec{j}+\vec{k}$

Sol: Let $\vec{a}=\vec{i}+\vec{j}+\vec{k}$, $\vec{b}=\vec{i}-\vec{j}$, $\vec{c}=\vec{i}+2\vec{j}+\vec{k}$

Volume of tetrahedron = $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

$$=\frac{1}{6}\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 2 & 1\end{vmatrix}=\frac{1}{6}(-1-0)-1(1-0)+1(2+1)$$

$$=\frac{1}{6}(1)$$

10. Let $\vec{a},\vec{b}$ and $\vec{c}$ be non coplanar vector and $\vec{\alpha}=\vec{a}+2\vec{b}+3\vec{c}$, $\vec{\beta}=2\vec{a}+\vec{b}-2\vec{c}$, $\vec{\gamma}=3\vec{a}-7\vec{c}$ then find $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]$

Sol: $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]=\begin{vmatrix}1 & 2 & 3\\ 2 & 1 & -2\\ 3 & 0 & -7\end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}]$

= 1 (-7 -0) -2 (-14 + 6) +3 (0 – 3)

= -7 + 16 – 9 = 0

11. **Let $\vec{a},\vec{b}$ and $\vec{c}$ be non coplanar vectors if $[2\vec{a}-\vec{b}+3\vec{c}\ \ \vec{a}+\vec{b}-2\vec{c}\ \ \vec{a}+\vec{b}-2\vec{c}]=\lambda[\vec{a}\ \vec{b}\ \vec{c}]$ then find λ**

Sol: $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix}[\vec{a}\ \vec{b}\vec{c}]=\lambda[\vec{a}\vec{b}\vec{c}]$

$\{2(-3+2)+1(-3+2)+3(1-1)\}[\vec{a}\vec{b}\vec{c}]=\lambda[\vec{a}\vec{b}\vec{c}]$

$=3[\vec{a}\vec{b}\vec{c}]=\lambda[\vec{a}\vec{b}\vec{c}]$

$\lambda=-3$

12. **Let $\vec{a},\vec{b}$ and $\vec{c}$ be non coplanar vectors if $[\vec{a}+2\vec{b}\ \ 2\vec{b}+\vec{c}\ \ 5\vec{c}+\vec{a}]\ \lambda[\vec{a}\ \vec{b}\ \vec{c}]$ then find λ**

Sol: Given that $[\vec{a}+2\vec{b}\ \ 2\vec{b}+\vec{c}\ \ 5\vec{c}+\vec{a}]=\lambda[\vec{a}\vec{b}\vec{c}]$

$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 5 \end{vmatrix}[\vec{a}\vec{b}\vec{c}]=\lambda[\vec{a}\ \vec{b}\ \vec{c}]$

$\{1(10-0)-2(0-1)+1(0-2)\}\ [\vec{a}\vec{b}\vec{c}]=\lambda\ [\vec{a}\vec{b}\vec{c}]$

$10+2-2=\lambda \Rightarrow \lambda=10$

13. **If $\vec{a},\vec{b},\vec{c}$ are non coplanar vector then find the value of $\dfrac{(\vec{a}+2\vec{b}-\vec{c}).\,[(\vec{a}-\vec{b})\times(\vec{a}-\vec{b}-\vec{c})]}{[\vec{a}\vec{b}\vec{c}]}$**

Sol: $\dfrac{(\vec{a}+2\vec{b}-\vec{c}).\,[(\vec{a}-\vec{b})\times(\vec{a}-\vec{b}-\vec{c})]}{[\vec{a}\vec{b}\vec{c}]}$

$=\dfrac{\begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{vmatrix}[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]}$

= -1 – 2 = - 3

14. **$\vec{a},\vec{b},\vec{c}$ Are mutually $\perp^r$ unit vectors then find the value of $[\vec{a}\ \vec{b}\ \vec{c}]^2$**

Sol: Since $\vec{a},\ \vec{b},\ \vec{c}$ are mutually perpendicular unit vector let $\vec{a}=\vec{i}$, $\vec{b}=\vec{j}$, $\vec{c}=\vec{k}$

$[\vec{a}\vec{b}\vec{c}]^2=[\vec{i}\ \vec{j}\ \vec{k}]^2$

$=\{\vec{i}.\ \vec{j}\times\vec{k}\}^2$

$= \{\vec{i}.\ \vec{j}\}^2 \qquad \{\because\ \vec{j}\times\vec{k} = i\}$

$= 1 \qquad \because\ i.i = 1$

$\because [\vec{a}\,\vec{b}\ \vec{c}]^2 = 1$

15. $\vec{a},\vec{b},\ \vec{c}$ **are nonzero vectors and** $\vec{a}$ **is** $\perp^r$ **both** $\vec{b}$ **and** $\vec{c}$**. If** $|\vec{a}| = 2$**,** $|\vec{b}| = 3$**,** $|\vec{c}| = 4$ **and** $(\vec{b},\vec{c}) = \dfrac{2\pi}{3}$ **find** $|[\vec{a}\ \vec{b}\vec{c}]|$

Sol: $\vec{a}$ is $\perp^r$ $\vec{b}\ \&\ \vec{c} \qquad \therefore \vec{a}.\vec{b} = \vec{a}.\vec{c} = 0$

$\vec{b}\times\vec{c}$ is $\perp^r$ $\vec{b}\ \&\ \vec{c}$

$\therefore \vec{a}$ Parallel $\vec{b}\times\vec{c} \qquad \therefore \vec{a}.(\vec{b}\times\vec{c}) = \pm|\vec{a}||\vec{b}\times\vec{c}|\ [\because \theta = 0\ or\ 180^0]$

$|[\vec{a}\,\vec{b}\,\vec{c}]| = |\vec{a}\,\vec{b}\times\vec{c}|$

$= |\vec{a}||\vec{b}\times\vec{c}|$

$= ||\vec{a}||\vec{b}||\vec{c}||\ \sin(\vec{b}\times\vec{c})$

$= 2\times 3\times 4\times\dfrac{\sqrt{3}}{2} = 12\sqrt{3}$

16. **If** $\vec{a},\vec{b},\vec{c}$ **are unit coplanar vector then find** $[2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}]$

Sol: Given that $\vec{a},\vec{b},\vec{c}$ are coplanar $[\vec{a},\vec{b},\vec{c}] = 0$

$$[2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}] = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}] = 0$$

II. Short Answer Questions:

1. **If** $[\vec{b}\,\vec{c}\ \vec{d}]+[\vec{c}\ \vec{a}\ \vec{d}]+[\vec{a}\,\vec{b}\,\vec{d}] = [\vec{a}\ \vec{b}\ \vec{c}]$ **then show that the points with position vectors** $\vec{a},\vec{b},\vec{c}$ **and** $\vec{d}$ **are coplanar**

Sol: Let $\overline{OA} = \vec{a}$, $\overline{OB} = \vec{b}$, $\overline{OC} = \vec{c}$ and $\overline{OD} = \vec{d}$ be the given position vectors

$\overline{AB} = \vec{b}-\vec{a}, \qquad \overline{AC} = \vec{c}-\vec{a}, \qquad \overline{AD} = \vec{d}-\vec{a}$

If the points are coplanar then

$[\overline{AB}\ \overline{AC}\ \overline{AD}] = 0$

$[\vec{b}-\vec{a}\ \ \vec{c}-\vec{a}\ \vec{d}-\vec{a}]=0$

$(\vec{b}-\vec{a}).\ (\vec{c}-\vec{a})\times(\vec{d}-\vec{a})=0$

$(\vec{b}-\vec{a}).\{\vec{c}\times\vec{d}-\vec{c}\times\vec{a}-\vec{a}\times\vec{d}+\vec{a}\times\vec{a}\}=0$

$[\vec{b}\ \ \vec{c}\ \vec{d}]-[\vec{b}\ \vec{c}\ \vec{a}]-[\vec{b}\ \vec{a}\ \vec{d}]-[\vec{a}\ \vec{c}\ \vec{d}]-[\vec{a}\ \vec{c}\ \vec{a}]-[\vec{a}\ \vec{a}\ \vec{d}]=0 \qquad \{\because \vec{a}\times\vec{b}=0\}$

$[\vec{b}\ \vec{c}\,\vec{d}]-[\vec{b}\,\vec{c}\,\vec{a}]+[\vec{a}\ \vec{b}\,\vec{d}]+[\vec{c}\,\vec{a}\,\vec{d}]=0$

$\therefore [\vec{b}\ \vec{c}\,\vec{d}]+[\vec{a}\,\vec{b}\,\vec{d}]+[\vec{c}\ \vec{a}\,\vec{d}]=[\vec{a}\,\vec{b}\,\vec{c}]$

$\therefore If\ [\vec{b}\ \vec{c}\,\vec{d}]+[\vec{a}\,\vec{b}\,\vec{d}]+[\vec{c}\ \vec{a}\,\vec{d}]=[\vec{a}\,\vec{b}\,\vec{c}]$

Then $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar

2. If $\vec{a},\vec{b}$ and $\vec{c}$ are non coplar vector then prove that the four points with position vectors $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar

Sol: let $\overline{OA}=2\vec{a}+3\vec{b}-\vec{c}$, $\overline{OB}=\vec{a}-2\vec{b}+3\vec{c}$

$\overline{OC}=3\vec{a}+4\vec{b}-2\vec{c}$, $\overline{OD}=\vec{a}-6\vec{b}+6\vec{c}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\vec{a}-5\vec{b}+4\vec{c}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\vec{a}+\vec{b}-\vec{c}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=-\vec{a}-9\vec{b}+7\vec{c}$

$$[\overrightarrow{AB}\overrightarrow{AC}\overrightarrow{AD}]=\begin{vmatrix}-1 & -5 & 4\\ 1 & 1 & -1\\ -1 & -9 & 7\end{vmatrix}[\vec{a}\vec{b}\vec{c}]$$

{-1(7-9) +5(7-1) +4(-9+1)} $[\vec{a}\vec{b}\vec{c}]$

{2+30-32} $[\vec{a}\vec{b}\vec{c}]$

3. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non zero and non collinear vectors and $\theta\neq 0, \pi$ is the angle between the vectors $\vec{b}$ and $\vec{c}$ if $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$ then find $\sin\theta$.

Sol: $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

$(\vec{c}.\vec{a})\vec{b}-(\vec{c}.\vec{b})\vec{a}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

$(\vec{c}.\vec{a})\vec{b}-(\vec{c}.\vec{b})\vec{a}-\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}=0$

$(\vec{c}.\vec{a})\vec{b}-\{(\vec{c}.\vec{b})+\frac{1}{3}|\vec{b}||\vec{c}|\}\vec{a}=0$

sin $\vec{a},\vec{b}$ are non collinear

$\vec{c}.\vec{a}=0$ and $\vec{c}.\vec{b}+\frac{1}{3}|\vec{c}||\vec{b}|=0$

$|\vec{c}||\vec{b}|\cos\theta+\frac{1}{3}|\vec{c}||\vec{b}|=0$

$|\vec{b}||\vec{c}|\{\cos\theta+\frac{1}{3}\}=0$

$\because \vec{b},\vec{c}$ are non zero vectors

$\vec{b}\neq 0 \qquad \vec{c}\neq 0$

$\therefore \cos\theta+\frac{1}{3}=0 \Rightarrow \cos\theta=-\frac{1}{3}$

$\sin\theta=\sqrt{1-cos^2\theta}=\sqrt{1-\frac{1}{9}}$

$\sin\theta=\frac{2\sqrt{2}}{3}$

4. Find the volume of the tetrahedron whose vertices are (1, 2, 1) (3, 2, 5) (2, -1, 0) and (-1, 0, 1)

Sol: Let $\overline{OA}=\vec{i}+2\vec{j}+\vec{k} \qquad \overline{OB}=3\vec{i}+2\vec{j}+5\vec{k}$

$\overline{OC}=2\vec{i}-\vec{j} \qquad \overline{OD}=-\vec{i}+\vec{k}$

$\overline{AB}=\overline{OB}-\overline{OA}=2\vec{i}+0+4\vec{k} \qquad \overline{AC}=\vec{i}-3\vec{j}-\vec{k}$

$\overline{AD}=-2\vec{i}-2\vec{j}$

The volume of tetrahedron = $\frac{1}{6}[\overline{AB}\,\overline{AC}\,\overline{AD}]$

$$=\frac{1}{6}\begin{vmatrix}2 & 0 & 4\\ 1 & -3 & -1\\ -2 & -2 & 0\end{vmatrix}$$

$=\frac{1}{6}|2(0-2)+4(-2-6)|$

$=\frac{1}{6}|-4-32|=6$

$\therefore$ Volume = 6

5. **Show that** $(\vec{a}+\vec{b}).(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})=2[\vec{a}\ \vec{b}\ \vec{c}]$

Sol: $(\vec{a}+\vec{b}).(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})=\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{vmatrix}[\vec{a}\ \vec{b}\ \vec{c}]$

$=\{1(1-0)-1(0-1)\}[\vec{a}\ \vec{b}\ \vec{c}]$

$=2[\vec{a}\ \vec{b}\ \vec{c}]$

6. **Show that the equation of the plane passing through the points with position vectors** $3\vec{i}-5\vec{j}-\vec{k}$ **,** $-\vec{i}+5\vec{j}+7\vec{k}$ **and parallel to the vectors** $3\vec{i}-\vec{j}+7\vec{k}$ **is 3x + 2y – z = 0**

Sol: Let $\overline{OA}=3\vec{i}-5\vec{j}-\vec{k}\,;\overline{OB}=-\vec{i}+5\vec{j}+7\vec{k}$ be the given points.

Let $\overline{OP}=x\vec{i}+y\vec{j}+z\vec{k}$ be any point on the plane

$\overline{AP}=(x-3)\vec{i}+(y+5)\vec{j}+(z+1)\vec{k}$

$\overline{AB}=\overline{OB}-\overline{OA}$

$=-4\vec{i}+10\vec{j}+8\vec{k}$

The plane is parallel to $3\vec{i}-\vec{j}+7\vec{k}$

$\therefore$ Let $\vec{p}=3\vec{i}-\vec{j}+7\vec{k}$

$\overline{AP},\overline{AB},\vec{p}$ are co planer.

$\therefore[\overline{AP},\overline{AB},\vec{p}]=0$

$\begin{vmatrix}x-3 & y+5 & z+1\\ -4 & 10 & 8\\ 3 & -1 & 7\end{vmatrix}=0$

(X-3)(70+8)-(y+5)(-28-24)+(z+1)(4-30)=0

78(x-3) +52(y+5)-26(z+1) =0

3(x-3)+2(y+5)-(z+10=0

3x-9+2y+10-z-1=0

3x+2y-z=0

7. Prove that $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})]=(\vec{a}.\vec{a})(\vec{b}\times\vec{a})$

Sol: $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})$

$\vec{a}\times[(\vec{a}.\vec{b})\vec{a}-(\vec{a}.\vec{a})\vec{b}]$

$(\vec{a}\times\vec{a})(\vec{a}.\vec{b})-(\vec{a}.\vec{a})(\vec{a}\times\vec{b})$

$0+(\vec{a}.\vec{a})(\vec{b}\times\vec{a})$

$=(\vec{a}.\vec{a})(\vec{b}\times\vec{a})$

8. If $\vec{a},\vec{b},\vec{c}$ **and** $\vec{d}$ **are coplanar vectors then show that** $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$

Sol: $\vec{a}\times\vec{b}$ is $\perp^r$ $\vec{a}$ and $\vec{b}$

$\vec{c}\times\vec{d}$ Is $\perp^r$ $\vec{c}$ and $\vec{d}$

Since $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar

$\vec{a}\times\vec{b},\vec{c}\times\vec{d}$ are $\perp^r$ to same plane

$\therefore \vec{a}\times\vec{b}\ parallel\ \vec{c}\times\vec{d}$

Hence $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$

9. Show that $[(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})].\vec{d}=[\vec{a}.\vec{d}]\,[\vec{a}\ \vec{b}\,\vec{c}]$

Sol: $[(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})].\vec{d}$

$[(\vec{a}\times\vec{b}.\vec{c})\ \vec{a}-(\vec{a}\times\vec{b}.\vec{a}).\vec{c}].\vec{d}$

$\left\{[\vec{a}\,\vec{b}\,\vec{c}]\ \vec{a}-(0)\vec{c}\right\}.\vec{d}$ $\quad\because \vec{a}\times\vec{b}.\vec{c}=[\vec{a}\,\vec{b}\ \vec{c}]$

$\vec{a}\times\vec{b}.\vec{c}=0$

$\therefore[\vec{a}\vec{b}\,\vec{c}\,]=\vec{a}.\vec{d}$

10. Show that $\vec{a}.[(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})]=0$

Sol: $\vec{a}.[(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})]$

$$=\begin{vmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 1 & 1\end{vmatrix}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\{1(1-1)-0(0-1)+0(0-1)\}\ [\vec{a}\ \vec{b}\ \vec{c}]=0$$

11. Find λ in order that the four points A (3, 2, 1), $B(4,\lambda,5)$, C (4, 2, -2), D (6, 5, -1) be coplanar

Sol: Let $\overline{OA}=3\vec{i}+2\vec{j}+\vec{k}$ $\quad\overline{OB}=4\vec{i}+\lambda\vec{j}+5\vec{k}$

$\overline{OC}=4\vec{i}+2\vec{j}-2\vec{k}$ $\quad\overline{OD}=6\vec{i}+5\vec{j}-\vec{k}$

$\overline{AB}=\overline{OB}-\overline{OA}=\vec{i}+(\lambda-2)\vec{j}+4\vec{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=\vec{i}-3\vec{k}$

$\overline{AD}=\overline{OD}-\overline{OA}=3\vec{i}+3\vec{j}-2\vec{k}$

Since the points are coplanar

$[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}]=0$

$$\begin{vmatrix}1 & \lambda-2 & 4\\1 & 0 & -3\\3 & 3 & -2\end{vmatrix}=0$$

$1(0+9)-(\lambda-2)(-2+9)+4(3-0)=0$

$9-7(\lambda-2)+12=0$

$21-7\lambda+14=0$

$\lambda=5$

12. Prove that the four points $4\vec{i}+5\vec{j}+\vec{k}$, $-(\vec{j}+\vec{k})$, $3\vec{i}+9\vec{j}+4\vec{k}$ and $-4\vec{i}+4\vec{j}+4\vec{k}$ are coplanar.

Sol: Let $\overline{OA}=4\vec{i}+5\vec{j}+\vec{k}$ $\quad\overline{OB}=-\vec{j}-\vec{k}$

$\overline{OC}=3\vec{i}+9\vec{j}+4\vec{k}$ $\quad\overline{OD}=-4\vec{i}+4\vec{j}+4\vec{k}$

Be the given points

$\overline{AB}=\overline{OB}-\overline{OA}=-4\vec{i}-6\vec{j}-2\vec{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=-\vec{i}+4\vec{j}+3\vec{k}$

$\overline{AD}=\overline{OD}-\overline{OA}=-8\vec{i}-\vec{j}+3\vec{k}$

$$[\overline{AB}\ \overline{AC}\ \overline{AD}]=\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix}$$

= -4 (12 + 3) +6 (-3 + 24) -2 (1 + 32)

= - 60 + 126 – 66 =0

$[\overline{AB}, \overline{AC}, \overline{AD}]=0$

Hence A, B, C, D are coplanar

13. If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar then show that the vector $\vec{a}-\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are coplanar

Sol: Let $\vec{p}=\vec{a}-\vec{b}$ $\quad\vec{q}=\vec{b}+\vec{c}$

$\vec{r}=\vec{c}+\vec{a}$ be the given vector since $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$

$$[\vec{p}\ \vec{q}\ \vec{r}]=\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{vmatrix}[\vec{a}\ \vec{b}\ \vec{c}]$$

$\{1(1-0)+1(0-1)\}[\vec{a}\ \vec{b}\ \vec{c}]=0$

$\therefore \vec{p}, \vec{q}, \vec{r}$ are coplanar

14. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B and C respectively then prove that $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ is $\perp^r$ to plane of ΔABC

Sol: Given that $\overrightarrow{OA}=\vec{a}$ $\quad \overrightarrow{OB}=\vec{b}$ and $\overrightarrow{OC}=\vec{c}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\vec{c}-\vec{a}$

Vector $\perp^r$ the plane containing A, B, C is $\overrightarrow{AB}\times\overrightarrow{AC}$

$=(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$

$=\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}$

$=\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+0$

$\therefore \vec{a}\times\vec{a}=0$

$\therefore \vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$ is $\perp^r$ the plane of triangle ABC

III. Long Answer Questions:

1. Show that $\{\vec{a}\times(\vec{c}\times\vec{b})\}\times\vec{c}=(\vec{a}.\vec{c})(\vec{b}\times\vec{c})$

Sol: $\{\vec{a}\times(\vec{b}\times\vec{c})\}\times\vec{c}$

$\{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}\}\times\vec{c}$

$(\vec{a}.\vec{c})(\vec{b}\times\vec{c})-(\vec{a}.\vec{b})(\vec{c}\times\vec{c})$

$(\vec{a}.\vec{c})(\vec{b}\times\vec{c}) \qquad \therefore \vec{c}\times\vec{c}=0$

2. If A = (1, -2, -1), B= (4, 0, -3), C= (1, ,2 -1) and D = (2, -4, -5) find the distance between AB and CD.

Sol: Equation of the line passing through the points A (1, -2, -1), B(4, 0, -3) is

$\vec{r}=(1-t)\vec{a}+t\vec{b}$

$=\vec{a}+t(\vec{b}-\vec{a})$

$\vec{r}=(\vec{i}-2\vec{j}-\vec{k})+t(3\vec{i}+2\vec{j}-2\vec{k})\rightarrow(2)$

This is of the form

$\vec{r} = \vec{a} + t\vec{b}$

$\therefore \vec{a} = \vec{i} - 2\vec{j} - \vec{k}$ $\quad\quad$ $\vec{b} = 3\vec{i} + 2\vec{j} - 2\vec{k}$

Equation of the line passing through the points C (1, 2, -1) and D = (2, -4, -5) is
$\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + s(\vec{i} - 6\vec{j} - 4\vec{k}) \rightarrow (2)$

This is of the form

$\vec{r} = \vec{c} + s\vec{d}$

$\therefore \vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ $\quad\quad$ $\vec{d} = \vec{i} - 6\vec{j} - 4\vec{k}$

Distance between (1) & (2) $= \dfrac{\left|[\vec{a} - \vec{c}\ \vec{b}\,\vec{d}]\right|}{\left|\vec{b} \times \vec{d}\right|}$

$$[\vec{a} - \vec{c} \quad \vec{b} - \vec{d}] = \begin{vmatrix} 0 & -4 & 0 \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix}$$

=0(-8-12)+4(-12+2)+0(-18-2)

=-40

$$\vec{b} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix}$$

= $\vec{i}$ (-8-12)- $\vec{j}$ (-12+2)+ $\vec{k}$ (-18-2)

$= -20\vec{i} + 10\vec{j} - 20\vec{k}$

$(\vec{b} \times \vec{d}) = \sqrt{400 + 100 + 400} = 30$

Distance= $\dfrac{\left|(\vec{a} - \vec{c} \quad \vec{b} - \vec{d})\right|}{(\vec{b} \times \vec{d})} = \dfrac{40}{30} = \dfrac{4}{3}$

3. **If** $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$ $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ **then find** $\vec{a} \times (\vec{b} \times \vec{c})$ *and* $\left|(\vec{a} \times \vec{b}) \times \vec{c}\right|$

Sol: $(\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}(-1-2) - \vec{j}(-2-1) + \vec{k}(4-1)$

$= -3\vec{i} + 3\vec{j} + 3\vec{k}$

$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$

$= \vec{i}(-6-3) - \vec{j}(3+3) + \vec{k}(3-6)$

$= -9\vec{i} - 6\vec{j} - 3\vec{k}$

$(\vec{a} \times \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

$= \vec{i}(-2-1) - \vec{j}(1-2) + \vec{k}(1+4)$

$= -3\vec{i} + \vec{j} + 5\vec{k}$

$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & +5 \\ 1 & 2 & -1 \end{vmatrix}$

$= \vec{i}(-1-10) - \vec{j}(3-5) + \vec{k}(-6-1)$

$= -11\vec{i} + 2\vec{j} - 7\vec{k}$

$\left|(\vec{a} \times \vec{b}) \times \vec{c}\right| = \sqrt{12+4+49} = \sqrt{174}$

4. **If** $\vec{a}=\vec{i}-2\vec{j}-3\vec{k}$**,** $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$ **and** $\vec{c}=\vec{i}+3\vec{j}-2\vec{k}$ **verify that** $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

Sol: $\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & -1\\ 1 & 3 & -2\end{vmatrix}$

$=\vec{i}(-2+3)-\vec{j}(-4+1)+\vec{k}(6-1)$

$=\vec{i}+3\vec{j}+5\vec{k}$

$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & -3\\ 1 & 3 & 5\end{vmatrix}$

$=\vec{i}(-10+9)-\vec{j}(5+3)+\vec{k}(3+2)$

$=-\vec{i}-8\vec{j}+5\vec{k}$

$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & -3\\ 2 & 1 & -1\end{vmatrix}$

$=\vec{i}(2+3)-\vec{j}(-1+6)+\vec{k}(1+4)$

$=5\vec{i}-5\vec{j}+5\vec{k}$

$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 5 & -5 & 5\\ 1 & 3 & -2\end{vmatrix}$

$=\vec{i}(10-15)\vec{j}(-10-5)+\vec{k}(15+5)$

$=-5\vec{i}+15\vec{j}+20\vec{k}$

$\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

5. **If** $\vec{a}=2\vec{i}+\vec{j}-3\vec{k}$ **,** $\vec{b}=\vec{i}-2\vec{j}+\vec{k}$ **,** $\vec{c}=-\vec{i}+\vec{j}-4\vec{k}$ **,** $\vec{d}=\vec{i}+\vec{j}+\vec{k}$ **then compute** $\left|(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\right|$

Sol: $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & -3\\ 1 & -2 & 1\end{vmatrix}$

$=\vec{i}(1-6)-\vec{j}(2+3)+\vec{k}(-4-1)$

$=-5\vec{i}-5\vec{j}-5\vec{k}$

$\vec{c}\times\vec{d}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & -4\\ 1 & 1 & 1\end{vmatrix}$

$=\vec{i}(1+6)-\vec{j}(-1+4)+\vec{k}(-1-1)$

$=5\vec{i}-3\vec{j}-2\vec{k}$

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -5 & -5 & -5\\ 5 & -3 & -2\end{vmatrix}$

$=\vec{i}(10-15)-\vec{j}(10+25)+\vec{k}(-15+25)$

$=-5\vec{i}-35\vec{j}+140\vec{k}$

$\left|(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\right|=\sqrt{25+1225+1600}=\sqrt{2850}$

$=\sqrt{25\times 114}$

$=5\sqrt{114}$

6. **If** $A=(1,a,a^2)$**,** $B=(1,b,b^2)$**.** $C=(1,c,c^2)$ **are non coplanar vector and** $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$ **then show that abc + 1 = 0**

Sol: $\vec{A},\vec{B},\vec{C}$ are non co planar then

$[\vec{A},\vec{B},\vec{C}]\neq 0$

$$\therefore \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\neq 0 \quad \text{and} \quad \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1=b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$$

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}+\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}=0$$

$$-\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}+abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0 \text{ (by inter changing } c_2 \text{ and } c_3)$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}+abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\{1+abc\}=0$$

But $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\neq 0$

$\therefore 1+abc=0$

7. **If** $\vec{a},\vec{b},\vec{c}$ **are non zero vectors then** $\left|(\vec{a}\times\vec{b}).\vec{c}\right|=|\vec{a}||\vec{b}||\vec{c}| \Leftrightarrow \vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$

Sol: Let α be the angle between $\vec{a}\times\vec{b}$, and $\vec{c}$ β be the angle between $\vec{a},\vec{b}$

$\left|(\vec{a}\times\vec{b}).\vec{c}\right|=|\vec{a}||\vec{b}||\vec{c}|$

$|\vec{a}\times\vec{b}|\,|\vec{c}|\cos\alpha = |\vec{a}||\vec{b}|\,|\vec{c}| \qquad \because |\vec{c}|\neq 0$

$|\vec{a}\times\vec{b}|\sin\beta\cos\alpha = |\vec{a}||\vec{b}|$

$\sin\beta\cos\alpha = 1 \qquad \therefore |\vec{a}|\neq 0,\ |\vec{b}|\neq 0$

$\therefore \sin\beta = 1 \qquad \cos\alpha = 1$

$\beta = 90^0 \qquad \alpha = 0^0$

Here $\vec{a}\times\vec{b}$ parallel $\vec{c}$

And $\vec{a}\perp^r \vec{b}$

$\therefore \vec{c}\, is \perp^r \vec{a}$ and $\vec{c}\perp^r \vec{b}$

$\therefore \vec{a}.\vec{b} = 0,\ \vec{b}.\vec{c} = 0,\ \vec{c}.\vec{a} = 0$

$\therefore \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

8. If $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} + 2\vec{k}$ then find $1|a\times b)\times c|$ and $|a\times(b\times c)|$

Sol: $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$= \vec{i}(2-1) - \vec{j}(4-1) + \vec{k}(2-1)$

$= \vec{i} - 3\vec{j} + \vec{k}$

$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & -3 & 1 \end{vmatrix}$

$= \vec{i}(-2+9) - \vec{j}(1-3) + \vec{k}(-3+2)$

$= 7\vec{i} + 2\vec{j} - \vec{k}$

$|\vec{a}\times(\vec{b}\times\vec{c})| = |7\vec{i} + 2\vec{j} - \vec{k}|$

$= \sqrt{49+4+1} = \sqrt{54}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 3\\ 2 & 1 & 1\end{vmatrix}$$

$$=\vec{i}(-2-3)-\vec{j}(1-6)+\vec{k}(1+4)$$

$$=-5\vec{i}+5\vec{j}+5\vec{k}$$

$$\left|(\vec{a}\times\vec{b})\times\vec{c}\right|=\sqrt{5^2+15^2+(-10)^2}=\sqrt{350}\quad\Rightarrow\quad 5\sqrt{14}$$

9. Let $\vec{a},\vec{b},\vec{c}$ be non coplanar vectors $\overrightarrow{a^1}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$; $\overrightarrow{b^1}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$; $\overrightarrow{c^1}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$ then show that $(\vec{a}+\vec{b}).\overrightarrow{a^1}+(\vec{b}+\vec{c}).\overrightarrow{b^1}+(\vec{c}+\vec{a}).\overrightarrow{c^1}=3$ and also $[\vec{a}\vec{b}\vec{c}][\overrightarrow{a^1}\,\overrightarrow{b^1}\,\overrightarrow{c^1}]=1$

Sol: $(\vec{a}+\vec{b}).\overrightarrow{a^1}(\vec{b}+\vec{c}).\overrightarrow{b^1}(\vec{c}+\vec{a}).\overrightarrow{c^1}$

$$(\vec{a}+\vec{b})\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}+(\vec{b}+\vec{c})\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}+(\vec{c}+\vec{a})\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

$$\frac{\vec{a}.(\vec{b}\times\vec{c})}{[\vec{a}\vec{b}\vec{c}]}+\frac{\vec{b}.(\vec{b}\times\vec{c})}{[\vec{a}\vec{b}\vec{c}]}+\frac{\vec{b}.(\vec{c}\times\vec{a})}{[\vec{a}\vec{b}\vec{c}]}+\frac{\vec{c}.(\vec{c}\times\vec{a})}{[\vec{a}\vec{b}\vec{c}]}+\frac{\vec{c}.(\vec{a}\times\vec{b})}{[\vec{a}\vec{b}\vec{c}]}+\frac{\vec{a}.(\vec{a}\times\vec{b})}{[\vec{a}\vec{b}\vec{c}]}$$

$$\frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]}+0+\frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]}+0+\frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]}+0$$

1+ 1 + 1 =3 { ∵ If two vector are identical in product then the value = 0}

$$[\vec{a}\vec{b}\vec{c}]\left[\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}\;\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}\;\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}\right]$$

$$\frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]^3}\left[\vec{b}\times\vec{c}\;\;\vec{c}\times\vec{a}\;\;\vec{a}\times\vec{b}\right]$$

$$\frac{1}{[\vec{a}\vec{b}\vec{c}]^2}\;(\vec{b}\times\vec{c}).\left\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\right\}$$

$$\frac{1}{[\vec{a}\vec{b}\vec{c}]^2}\;(\vec{b}\times\vec{c}).\left\{(\vec{c}\times\vec{a}.\vec{b})\vec{a}-(\vec{c}\times\vec{a}.\vec{b})\vec{b}\right\}$$

$$=\frac{1}{[\vec{a}\vec{b}\vec{c}]^2}(\vec{b}\times\vec{c}).\{[\vec{a}\vec{b}\vec{c}]\vec{a}\}=\frac{[\vec{a}\vec{b}\vec{c}]^2}{[\vec{a}\vec{b}\vec{c}]^2}$$

10. If $|\vec{a}|=1$, $|\vec{b}|=1$, $|\vec{c}|=2$ and $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=0$ then find the angle between $\vec{a}$ and $\vec{c}$

Sol: $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=0$

$$\left|\vec{a}\times(\vec{a}\times\vec{c})\right|=\left|-\vec{b}\right|$$

$$\left|(\vec{a}.\vec{a})\vec{c}-(\vec{a}.\vec{c})\vec{a}\right|=\left|\vec{b}\right|$$

$$\left|\,|\vec{a}|^2\,\vec{c}-|\vec{a}||\vec{c}|\cos\theta\vec{a}\right|=\left|\vec{b}\right|$$

$$\left|\vec{c}-2\vec{a}\cos\theta\right|=\left|\vec{b}\right|$$

S.O.B.S

$$|\vec{c}|^2+4|\vec{a}|^2-4\vec{c}.\vec{a}\cos\theta=|\vec{b}|^2$$

$$4+4\cos^2\theta-4|\vec{c}||\vec{a}|\cos^2\theta=|\vec{b}|^2$$

$$4+4\cos^2\theta-4\times2\times1\cos^2\theta=1$$

$$3=4\cos^2\theta\Rightarrow\cos^2\theta=\frac{3}{4}$$

$$\cos\theta=\pm\frac{\sqrt{3}}{2}$$

$$\theta=30^0 \text{ or } \theta=150^0$$

11. Let $\vec{a}=\vec{i}-\vec{k}$, $\vec{b}=x\vec{i}+\vec{k}+(1-x)\vec{k}$, $\vec{c}=y\vec{i}+x\vec{k}+(1+x+y)\vec{k}$ then prove that $[\vec{a}\,\vec{b}\,\vec{c}]$ is independent of both x and y

Sol: $[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}1 & 0 & -1\\ x & 1 & 1-x\\ y & x & 1+x-y\end{vmatrix}$

$$=1\left\{(1+\cancel{x}-y)-\cancel{x}+x^2\right\}-0(x+x^2-xy-y+xy)-1(x^2-y)$$

$$=1-\cancel{y}+\cancel{x^2}-0-\cancel{x^2}+\cancel{y^2}=1$$

This is independent of x and y

12. Let $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$, $\vec{c}=\vec{i}+3\vec{k}$ if $\vec{a}$ is a unit vector then find the maximum value of $[\vec{a}\vec{b}\vec{c}]$

Sol: Given $|\vec{a}|=1$

$\vec{b}=2\vec{i}+\vec{j}-\vec{k}\Rightarrow|\vec{b}|=\sqrt{6}$

$\vec{c}=\vec{i}+3\vec{k}\Rightarrow|\vec{c}|=\sqrt{10}$

$[\vec{a}\,\vec{b}\,\vec{c}]=\vec{a}.(\vec{b}\times\vec{c})$

$=|\vec{a}|.|\vec{b}\times\vec{c}|\cos\theta$

Since $[\vec{a}\vec{b}\vec{c}]$ is maximum then

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & -1\\ 1 & 0 & 3\end{vmatrix}$$

$=\vec{i}(3-0)-\vec{j}(6+1)+\vec{k}(0-1)$

$=3\vec{i}-7\vec{j}-\vec{k}$

$|\vec{b}\times\vec{c}|=\sqrt{9+49+1}=\sqrt{59}$

$\therefore$ Maximum value of $[\vec{a}\vec{b}\vec{c}]=|\vec{a}||\vec{b}\times\vec{c}|$

$=1\times\sqrt{59}$

12. Let $\vec{a}=\vec{i}-\vec{j}$, $\vec{b}=\vec{j}-\vec{k}$, $\vec{c}=\vec{k}-\vec{i}$ find unit vector $\vec{d}$ such that $\vec{a}.\vec{d}=0[\vec{b}\vec{c}\vec{d}]=0$

Sol: Let $\vec{d}=d_1\vec{i}+d_2\vec{j}+d_3\vec{k}$

$\vec{a}.\vec{d}=0\Rightarrow d_1-d_2=0$

$d_1-d_2=0\rightarrow(1)$

$[\vec{b}\vec{c}\vec{d}]=0$

$$\begin{vmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ d_1 & d_2 & d_3\end{vmatrix}=0$$

$$0(0-d_2)-1(-d_3-d_1)-1(-d_2-0)+0+d_3+d_1+d_2=0 \to (2)$$

Since $\vec{d}$ is the unit vector

$$\left|\vec{d}\right|=\sqrt{d_1^2+d_2^2+d_3^2}=1$$

$$d_1^2+d_2^2+d_3^2=1 \to (3)$$

Sub (2) in (1)

$$d_3+d_1+d_1=0$$

$$d_3=-2d_1$$

Sub d_3, d_2 in (3)

$$d_1^2+d_1^2+4d_1^2=1$$

$$d_1=\pm\frac{1}{\sqrt{6}}$$

$$\therefore \vec{d}=d_1\vec{i}+d_2\vec{j}+d_3\vec{k}$$

$$=\pm\frac{1}{\sqrt{6}}\vec{i}\pm\frac{1}{\sqrt{6}}\vec{j}\pm\frac{1}{\sqrt{6}}\vec{k}$$

$$=\pm\frac{1}{\sqrt{6}}(\vec{i}+\vec{j}+\vec{k}).$$