

## MULTIPLE PRODUCTS - 1

### EXERCISE – 4(b)

#### I. Very Short Answer Questions:

1. If  $|\vec{p}| = 2$ ,  $|\vec{q}| = 3$  and  $(\vec{p}, \vec{q}) = \frac{\pi}{6}$  then find  $|\vec{p} \times \vec{q}|^2$

Sol:  $|\vec{p} \times \vec{q}|^2 = |\vec{p}|^2 |\vec{q}|^2 \sin^2(\vec{p}, \vec{q})$

$$= 4 \times 9 \times \frac{1}{4} = 9$$

2. If  $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$      $\vec{b} = \vec{i} - 3\vec{j} - 5\vec{k}$  then find  $|\vec{a} \times \vec{b}|$

Sol:  $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{vmatrix}$

$$= \vec{i}(5+3) - \vec{j}(-10-1) + \vec{k}(-6+1)$$

$$\vec{a} \times \vec{b} = 8\vec{i} + 11\vec{j} - 5\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{64+121+25} = \sqrt{210}$$

3. If  $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$      $\vec{b} = \vec{i} + 4\vec{j} - 2\vec{k}$  then find  $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ .

Sol:  $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$$= 2\vec{b} \times \vec{a} \quad \{ \because -\vec{a} \times \vec{b} = \vec{b} \times \vec{a} \}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \vec{i}(4-6) - \vec{j}(1+4) + \vec{k}(-3-8)$$

$$= -2\vec{i} - 5\vec{j} - 11\vec{k}$$

$$\therefore (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2\vec{b} \times \vec{a}$$

$$= -2(2\vec{i} + 5\vec{j} + 11\vec{k})$$

4. If  $4\vec{i} + \frac{2p}{3}\vec{j} + p\vec{k}$  is parallel to the vector  $\vec{i} + 2\vec{j} + 3\vec{k}$  then find p.

**Sol:** Let  $\vec{a} = 4\vec{i} + \frac{2p}{3}\vec{j} + p\vec{k}$

$$\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\because \vec{a} \parallel \vec{b} \quad \vec{a} \times \vec{b} = 0 \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & \frac{2p}{3} & p \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$$\vec{i}(2p - 2p) - \vec{j}(12 - p) + \vec{k}(8 - \frac{2p}{3}) = 0$$

$$\therefore 12 - p = 0 \text{ and } 8 - \frac{2p}{3} = 0$$

$$\therefore p = 12$$

5. Compute  $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

**Sol:**  $\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$

$$\cancel{\vec{a} \times \vec{b}} + \cancel{\vec{a} \times \vec{c}} + \cancel{\vec{b} \times \vec{c}} - \cancel{\vec{a} \times \vec{b}} - \cancel{\vec{a} \times \vec{c}} - \cancel{\vec{b} \times \vec{c}} = 0$$

6. If  $p = xi + yj + zk$  then find  $|\vec{p} \times \vec{k}|^2$

**Sol:**  $|\vec{p} \times \vec{k}|^2 = |\vec{p}|^2 |\vec{k}|^2 - (\vec{p} \cdot \vec{k})^2$        $\because |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$$(x^2 + y^2 + z^2)(1) - \{(xi + yj + zk) \cdot k\}^2$$

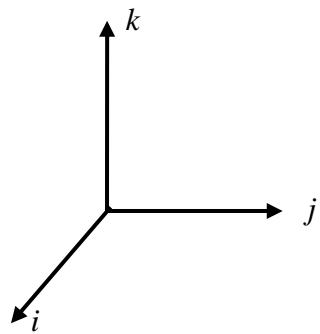
$$(x^2 + y^2 + z^2) - \{z\}^2 = x^2 + y^2$$

7. Compute  $2\vec{j} \times (3\vec{i} - 4\vec{k}) + (\vec{i} + 2\vec{j}) \times \vec{k}$

Sol:  $6\vec{j} \times \vec{i} - 8\vec{j} \times \vec{k} + \vec{i} \times \vec{k} + 2\vec{j} \times \vec{k}$

$$-6\vec{k} - 8\vec{i} - \vec{j} + 2\vec{i}$$

$$-6\vec{i} - \vec{j} - 6\vec{k}$$



8. Find unit vector perpendicular to both  $\vec{i} + \vec{j} + \vec{k}$  and  $2\vec{i} + \vec{j} + 3\vec{k}$

Sol: Let  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$        $\vec{b} = 2\vec{i} + \vec{j} + 3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \vec{i}(3-1) - \vec{j}(3-2) + \vec{k}(1-2)$$

$$\vec{a} \times \vec{b} = 2\vec{i} - \vec{j} - \vec{k}$$

Unit vector  $\perp^r$  to both  $\vec{a}$  &  $\vec{b}$  =  $\pm \frac{\vec{a} + \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \pm \frac{(2\vec{i} - \vec{j} - \vec{k})}{\sqrt{2^2 + 1^2 + 1^2}} = \pm \frac{(2\vec{i} - \vec{j} - \vec{k})}{\sqrt{6}}$$

9. If  $\theta$  is the angle between the vectors  $\vec{i} + \vec{j}$  and  $\vec{j} + \vec{k}$  then find  $\sin \theta$

Sol:  $\vec{a} = \vec{i} + \vec{j}$ ,       $\vec{b} = \vec{j} + \vec{k}$

$$\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i}(1-0) - \vec{j}(1-0) + \vec{k}(1-0)$$

$$= \vec{i} - \vec{j} + \vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{3}, \quad |\vec{a}| = \sqrt{2}, \quad |\vec{b}| = \sqrt{2}$$

$$\sin \theta = \frac{\sqrt{3}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{3}}{2}$$

10. Find the area of the parallelogram having  $\vec{a} = 2\vec{j} - \vec{k}$  and  $\vec{b} = -\vec{i} - \vec{k}$  as adjacent sides

Sol: Area of parallelogram  $|\vec{a} \times \vec{b}|$

$$|\vec{a} \times \vec{b}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i}(2-0) - \vec{j}(0-1) + \vec{k}(0+2)$$

$$|\vec{a} \times \vec{b}| = 2\vec{i} + \vec{j} + 2\vec{k}$$

$$\text{Area} = |\vec{a} \times \vec{b}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

11. Find the area of the parallelogram whose diagonals are  $3\vec{i} + \vec{j} - 2\vec{k}$  and  $\vec{i} - 3\vec{j} + 4\vec{k}$

Sol: area of parallelogram  $= \frac{1}{2} |\vec{a} \times \vec{b}|$  where  $\vec{a}, \vec{b}$  are diagonals

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \vec{i}(4-6) - \vec{j}(12+2) + \vec{k}(-9-1)$$

$$\vec{a} \times \vec{b} = -2\vec{i} - 14\vec{j} - 10\vec{k}$$

$$\text{Area} = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} \sqrt{4+196+100} = \sqrt{\frac{264}{4}}$$

$$= \sqrt{\frac{300}{4}} = \sqrt{75} = 5\sqrt{3}$$

- 12. Find the area of the triangle having  $3\vec{i} + \vec{j}$  and  $-5\vec{i} + 7\vec{j}$  as two of its sides**

**Sol:** Area =  $\frac{1}{2} |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = \vec{k}(21 + 20) = 41\vec{k}$$

$$\text{Area} = \frac{1}{2} |41\vec{k}| = \frac{41}{2}$$

- 13. Find the unit vector perpendicular to the plane determined by the vectors  
 $\vec{a} = 4\vec{i} + 3\vec{j} - \vec{k}$  and  $\vec{b} = 2\vec{i} - 6\vec{j} - 3\vec{k}$**

**Sol:**  $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix} = \vec{i}(-9 - 6) - \vec{j}(-12 + 2) + \vec{k}(-24 - 6)$

$$\vec{a} \times \vec{b} = -15\vec{i} + 10\vec{j} - 30\vec{k}$$

The unit vector  $\perp^r \vec{a} \times \vec{b} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \pm \frac{(-15\vec{i} + 10\vec{j} - 30\vec{k})}{\sqrt{225 + 100 + 900}}$$

$$= \pm \frac{(-15\vec{i} + 10\vec{j} - 30\vec{k})}{35}$$

$$= \pm \frac{(3\vec{i} - 2\vec{j} + 6\vec{k})}{7}$$

14. Find the area of the triangle whose vertices are A (1, 2, 3), B (2, 3, 1) and C (3, 1, 2)

Sol:  $\overrightarrow{AB} = (2-1)\vec{i} + (3-2)\vec{j} + (1-3)\vec{k}$

$$\vec{i} + \vec{j} - 2\vec{k}$$

$$\overrightarrow{AC} = (3-1)\vec{i} + (1-2)\vec{j} + (2-3)\vec{k}$$

$$= 2\vec{i} - \vec{j} - \vec{k}$$

$$\text{Area of triangle} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$(\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}(-1-2) - \vec{j}(1+2) + \vec{k}(-1-2)$$

$$= -3\vec{i} - 3\vec{j} - 3\vec{k}$$

$$\text{Area} = \frac{1}{2} |-3\vec{i} - 3\vec{j} - 3\vec{k}|$$

$$= \frac{1}{2} \sqrt{9+9+9} = \frac{3\sqrt{3}}{2}$$

## II. Short Answer Questions:

1. If  $\vec{a} + \vec{b} + \vec{c} = 0$  then prove that  $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Sol: Given that  $\vec{a} + \vec{b} + \vec{c} = 0 \rightarrow (1)$

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = 0$  Taking cross product with  $\vec{a}$ .

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

$$\vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

$$\vec{a} \times \vec{b} = \vec{c} \times \vec{a} \rightarrow (2)$$

Taking cross product with  $\vec{b}$  on both sides

$$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = 0$$

$$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = 0$$

$$\vec{b} \times \vec{c} = -\vec{b} \times \vec{a}$$

$$\vec{b} \times \vec{c} = \vec{a} \times \vec{b} \rightarrow (3)$$

From (2)&(3)

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

- 2.** If  $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$ ,  $\vec{b} = -\vec{i} + 2\vec{j} - 4\vec{k}$  and then find  $(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c})$

**Sol:**  $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \vec{i}(-4+2) - \vec{j}(-8-1) + \vec{k}(4+1)$

$$= -2\vec{i} + 9\vec{j} + 5\vec{k}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}(2+4) - \vec{j}(-1+4) + \vec{k}(-1-2)$$

$$= 6\vec{i} - 3\vec{j} - 3\vec{k}$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c}) = (-2\vec{i} + 9\vec{j} + 5\vec{k}) \cdot (6\vec{i} - 3\vec{j} - 3\vec{k})$$

$$= -12 - 27 - 15 = -54$$

- 3.** Find the vector area and the are of the parallelogram having

**Sol:**  $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}(4-1) - \vec{j}(2+2) + \vec{k}(-1-4)$

$$= 3\vec{i} - 4\vec{j} - 5\vec{k}$$

$$\text{Vector area of parallelogram} = 3\vec{i} - 4\vec{j} - 5\vec{k}$$

$$\text{Area of parallelogram} = |\vec{3i} - 4\vec{j} - 5\vec{k}|$$

$$= \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$

4. If  $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq 0$  then show that  $\vec{a} + \vec{c} = p\vec{b}$  where p is some scalar.

Sol:  $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

$$\vec{a} \times \vec{b} - \vec{b} \times \vec{c} = 0$$

$$\vec{a} \times \vec{b} + \vec{c} \times \vec{b} = 0 \quad \therefore \vec{c} \times \vec{b} = -\vec{b} \times \vec{c}$$

$$(\vec{a} + \vec{c}) \times \vec{b} = 0$$

$\therefore \vec{a} + \vec{c}$  And  $\vec{b}$  are parallel

$\therefore \vec{a} + \vec{c} = p\vec{b}$  Where p is a scalar

5. Let  $\vec{a}$  and  $\vec{b}$  be vectors satisfying  $|\vec{a}| = |\vec{b}| = 5$  and  $(\vec{a}, \vec{b}) = 45^\circ$ . Find the triangle having  $\vec{a} - 2\vec{b}$  and  $3\vec{a} + 2\vec{b}$  as two of its sides

Sol: Area of triangle  $= \frac{1}{2} |(\vec{a} - 2\vec{b}) \times (3\vec{a} + 2\vec{b})|$

$$= \frac{1}{2} |3\vec{a} \times \vec{a} + 2\vec{a} \times \vec{b} - 6\vec{b} \times \vec{a} - 4\vec{b} \times \vec{b}|$$

$$= 4 |\vec{a} \times \vec{b}|$$

$$= 4 |\vec{a}| |\vec{b}| \sin(\vec{a}, \vec{b})$$

$$= 4 \times 5 \times 5 \times \frac{1}{\sqrt{2}} = 50\sqrt{2}$$

6. Find the vector having magnitude  $\sqrt{6}$  units and perpendicular to both  $2\vec{i} - \vec{k}$  and  $3\vec{j} - \vec{i} - \vec{k}$

**Sol:** Let  $\vec{a} = 2\vec{i} - \vec{k}$  and  $\vec{b} = 3\vec{j} - \vec{i} - \vec{k}$  be the given vectors.

Vector  $\perp^r$  to both  $\vec{a}$  and  $\vec{b} = \vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= \vec{i}(0+3) - \vec{j}(-2-1) + \vec{k}(6-0)$$

$$= 3\vec{i} + 3\vec{j} + 6\vec{k}$$

$$\text{Unit vector } \perp^r \text{ both } \vec{a} \text{ & } \vec{b} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$= \pm \frac{(3\vec{i} + 3\vec{j} + 6\vec{k})}{\sqrt{9+9+36}}$$

$$\text{Vector } \perp^r \text{ to both } \vec{a} \text{ & } \vec{b} \text{ and of magnitude is } = \pm \sqrt{6} \frac{(3\vec{i} + 3\vec{j} + 6\vec{k})}{3\sqrt{6}}$$

$$\therefore \text{Required vector} = \pm(\vec{i} + \vec{j} + 2\vec{k})$$

7. Find a unit vector perpendicular to the plane determined by the points

P (1, -1, 2), Q (2, 0, -1), R(0, 2, 1)

**Sol:**  $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= (2-1)\vec{i} + (0+1)\vec{j} + (-1-2)\vec{k}$$

$$\overrightarrow{PQ} = \vec{i} + \vec{j} - 3\vec{k}$$

$$\overrightarrow{PR} = (\overrightarrow{OR} - \overrightarrow{OP})$$

$$= (0-1)\vec{i} + (2+1)\vec{j} + (1-2)\vec{k}$$

$$= -\vec{i} + 3\vec{j} - \vec{k}$$

$$\text{Unit vector } \perp^r \text{ to plane} = \pm \frac{(\overrightarrow{PQ} \times \overrightarrow{PR})}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= \vec{i}(-1+9) - \vec{j}(-1-3) + \vec{k}(3+1)$$

$$= 8\vec{i} + 4\vec{j} + 4\vec{k}$$

$$\text{Unit vector} = \pm \frac{(8\vec{i} + 4\vec{j} + 4\vec{k})}{\sqrt{64+16+16}}$$

$$= \pm \frac{4(2\vec{i} + \vec{j} + \vec{k})}{4\sqrt{6}}$$

$$= \pm \frac{1}{\sqrt{6}}(2\vec{i} + \vec{j} + \vec{k})$$

8. If  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$  and  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ ,  $\vec{a} \neq 0$  then show that  $\vec{b} = \vec{c}$

Sol:  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0$$

$$\vec{a} \cdot (\vec{b} - \vec{c}) = 0$$

$$\vec{a} = 0 \text{ Or } \vec{b} - \vec{c} = 0 \text{ or } \vec{a} \perp^r \vec{b} - \vec{c}$$

$$\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$$

$$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0$$

$$\vec{a} \times (\vec{b} - \vec{c}) = 0$$

$$\vec{a} = 0 \text{ Or } \vec{b} - \vec{c} = 0 \text{ or } \vec{a} \text{ parallel } (\vec{b} - \vec{c})$$

$$\vec{a} \perp^r (\vec{b} - \vec{c})$$

And  $\vec{a}$  parallel  $\vec{b} - \vec{c}$  are not possible at a linear

$$\therefore \vec{a} = 0 \text{ or } \vec{b} - \vec{c} = 0$$

$$\vec{a} = 0 \quad \vec{b} = \vec{c}$$

$$\text{But } \vec{a} \neq 0 \quad \therefore \vec{b} = \vec{c}$$

**9. Find a vector of magnitude 3 and perpendicular to both the vectors**

$$\vec{b} = 2\vec{i} - 2\vec{j} + \vec{k} \text{ and } \vec{c} = 2\vec{i} + 2\vec{j} + 3\vec{k}$$

Sol:  $\vec{b} \times \vec{c} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

$$= \vec{i}(-6 - 2) - \vec{j}(6 - 2) + \vec{k}(4 +)$$

$$= -8\vec{i} - 4\vec{j} + 8\vec{k}$$

Unit vector  $\perp^r$  to  $\vec{b}$  &  $\vec{c}$   $= \pm \frac{(\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}$

$$= \pm \frac{(-8\vec{i} - 4\vec{j} + 8\vec{k})}{\sqrt{64 + 16 + 64}}$$

$$= \pm \frac{\cancel{4}(2\vec{i} + \vec{j} - 2\vec{k})}{\cancel{4\sqrt{2}}}$$

$\therefore$  Vector of magnitude 3 is  $\pm(2\vec{i} + \vec{j} - 2\vec{k})$

**10. If  $|\vec{a}| = 13$ ,  $|\vec{b}| = 5$  and  $\vec{a} \cdot \vec{b} = 60$  find  $|\vec{a} \times \vec{b}|$**

Sol:  $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2(\vec{a}, \vec{b})$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2(\vec{a}, \vec{b})$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}, \vec{b})^2$$

$$= (13)^2 \times 5^2 - (60)^2$$

$$= 169 \times 25 - 3600$$

$$= 4225 - 3600$$

$$= 625$$

$$|\vec{a} \times \vec{b}| = \sqrt{625} = 25$$

- 11. Find the unit vector perpendicular to the plane passing through the points (1, 2, 3) (2, -1, 1) and (1, 2, -40)**

**Sol:** Let  $\begin{cases} \overrightarrow{OA} = \vec{i} + 2\vec{j} + 3\vec{k} \\ \overrightarrow{OB} = 2\vec{i} - \vec{j} + \vec{k} \\ \overrightarrow{OC} = \vec{i} + 2\vec{j} - 4\vec{k} \end{cases}$  be the given position vectors

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \vec{i} - 3\vec{j} - 2\vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= -7\vec{k}$$

$$\text{Unit vector } \perp^r \text{ to the plane} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & -2 \\ 0 & 0 & -7 \end{vmatrix}$$

$$= \vec{i}(21 - 0) - \vec{j}(-7) + \vec{k}(0 - 0)$$

$$= 21\vec{i} + 7\vec{j}$$

$$\text{Unit vector } \perp^r \text{ to plane} = \pm \frac{(\overrightarrow{AB} \times \overrightarrow{AC})}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

$$= \pm \frac{(21\vec{i} + 7\vec{j})}{\sqrt{441 + 49}}$$

$$= \pm \frac{\cancel{7}(3\vec{i} + \vec{j})}{\cancel{7}\sqrt{10}}$$

### III. Long Answer Questions:

1. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  represent the vertices A, B and C respectively of  $\triangle ABC$  then prove that  $|(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$  is twice the area of  $\triangle ABC$ .

**Sol:** Let  $\overrightarrow{OA} = \vec{a}$ ,  $\overrightarrow{OB} = \vec{b}$  and  $\overrightarrow{OC} = \vec{c}$  be the given position vectors

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{c} - \vec{a}$$

$$\text{Area of triangle } ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2} (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$= \frac{1}{2} |\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|$$

$$= \frac{1}{2} |\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}|$$

$$\text{Area of triangle } ABC = \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$2 \text{ (area of triangle } ABC) = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

2. If  $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ ,  $\vec{b} = \vec{i} + \vec{j} - \vec{k}$  and  $\vec{c} = \vec{i} - \vec{j} + \vec{k}$  then compute  $\vec{a} \times (\vec{b} \times \vec{c})$  and verify that it is perpendicular to  $\vec{a}$

**Sol:**  $\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$$= \vec{i}(1-1) - \vec{j}(1+1) + \vec{k}(-1-1)$$

$$= -2\vec{j} - 2\vec{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 0 & -2 & -2 \end{vmatrix}$$

$$= \vec{i}(-6 + 8) - \vec{j}(-4 - 0) + \vec{k}(-4 - 0)$$

$$= 2\vec{i} + 4\vec{j} - 4\vec{k}$$

$$\{(\vec{a} \times (\vec{b} \times \vec{c}))\} \cdot \vec{a} = (2\vec{i} + 4\vec{j} + 4\vec{k}) \cdot (2\vec{i} + 3\vec{j} + 4\vec{k})$$

$$= 4 + 12 - 16 = 0$$

$\therefore \vec{a} \times (\vec{b} \times \vec{c})$  is  $\perp^r \vec{a}$

3. If  $\vec{a} = 7\vec{i} - 2\vec{j} + 3\vec{k}$ ,  $\vec{b} = 2\vec{i} + 8\vec{k}$  and  $\vec{c} = \vec{i} + \vec{j} + \vec{k}$  then  $\vec{a} \times \vec{b}$ ,  $\vec{a} \times \vec{c}$  and  $\vec{a} \times (\vec{b} \times \vec{c})$  verify whether the cross product is distributive over vector addities

Sol:  $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -2 & 3 \\ 2 & 0 & 8 \end{vmatrix} = \vec{i}(-16 - 0) - \vec{j}(56 - 6) + \vec{k}(0 + 4)$

$$\vec{a} \times \vec{b} = -16\vec{i} - 50\vec{j} + 4\vec{k}$$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}(-2 - 3) - \vec{j}(7 - 3) + \vec{k}(7 + 2)$$

$$= -5\vec{i} + 4\vec{j} + 9\vec{k}$$

$$\vec{a} \times \vec{c} = -8\vec{i} + 6\vec{j} + 2\vec{k}$$

$$\vec{a} \times \vec{c} + \vec{a} \times \vec{c} = (-16\vec{i} - 50\vec{j} + 4\vec{k}) + (-5\vec{i} - 4\vec{j} + 9\vec{k})$$

$$= -21\vec{i} - 54\vec{j} + 13\vec{k}$$

$$\vec{b} + \vec{c} = 2\vec{i} + 8\vec{k} + \vec{i} + \vec{j} + \vec{k}$$

$$= 3\vec{i} + \vec{j} + 9\vec{k}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -2 & 3 \\ 3 & 1 & 9 \end{vmatrix}$$

$$= \vec{i}(-18 - 3) - \vec{j}(63 - 9) + \vec{k}(7 + 6)$$

$$= -21\vec{i} - 54\vec{j} + 13\vec{k}$$

$$\therefore \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

4. If  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ ,  $\vec{c} = \vec{j} - \vec{k}$  then find vector  $\vec{b}$  such that  $\vec{a} \times \vec{b} = \vec{c}$  and  $\vec{a} \cdot \vec{b} = 3$

**Sol:** Let  $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a} \cdot \vec{b} = 3 \Rightarrow b_1 + b_2 + b_3 = 3$$

$$\vec{a} \times \vec{b} = \vec{c}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{j} - \vec{k}$$

$$\vec{i}(b_3 - b_2) - \vec{j}(b_3 - b_1) + \vec{k}(b_2 - b_1) = \vec{j} - \vec{k}$$

$$b_3 - b_2 = 0 \quad b_1 - b_3 = 1 \quad b_2 - b_1 = -1$$

$$b_3 - b_2 \quad b_1 - b_3 = 1$$

$$b_1 - b_2 = 1$$

$$b_1 = 1 + b_2$$

Sub  $b_1, b_2, b_3$  in

$$b_1 + b_2 + b_3 = 3$$

$$1 + b_2 + b_2 + b_3 = 3 \Rightarrow b_2 = 2/3, b_3 = 2/3$$

$$b_2 - b_1 = -1$$

$$\frac{2}{3} + 1 = b_1 \Rightarrow b_1 = \frac{5}{3}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$= \frac{1}{3}(5\vec{i} + 2\vec{j} + 2\vec{k})$$

- 5. If A, B, C and D are four points then show that**

$$|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}| \text{ is four times the area of } \Delta ABC$$

**Sol:** let A be the origin of vectors

$$|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$$

$$|\overrightarrow{AB}(\overrightarrow{AD} - \overrightarrow{AC}) + (\overrightarrow{AC} - \overrightarrow{AB}) \times \overrightarrow{AD} - \overrightarrow{AC} \times (\overrightarrow{AD} - \overrightarrow{AB})|$$

$$|\overrightarrow{AB} \times \overrightarrow{AD} - \overrightarrow{AB} \times \overrightarrow{AC} + \overrightarrow{AC} \times \overrightarrow{AD} - \overrightarrow{AB} \times \overrightarrow{AD} - \overrightarrow{AC} \times \overrightarrow{AD} + \overrightarrow{AC} \times \overrightarrow{AB}|$$

$$|- \overrightarrow{AB} \times \overrightarrow{AC} - \overrightarrow{AC} \times \overrightarrow{AB}| = 2 |- \overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= 4 \times \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= 4 \text{ (area of triangle ABC)}$$

- 6. If  $A_1, A_2, A_3, \dots, A_n$  are the vertices of a regular polygon with n sides and 'O' is the centre show that  $\sum_{j=1}^{n-1} \overrightarrow{OA_j} \times \overrightarrow{OA_{j+1}} = (1-n)(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$**

**Sol:** since  $A_1, A_2, A_3, \dots, A_n$  are vertices of a regular polygon

$$OA_1, A_2 \text{ Are of } OA_2, A_3 = \dots \text{ area of triangle } OA_{n-1}, A_n$$

$$\sum_{j=1}^{n-1} \overrightarrow{OA_j} \times \overrightarrow{OA_{j+1}} = \overrightarrow{OA_1} \times \overrightarrow{OA_2} + \overrightarrow{OA_2} \times \overrightarrow{OA_3} + \overrightarrow{OA_3} \times \overrightarrow{OA_4} + \dots + \overrightarrow{OA_{n-1}} \times \overrightarrow{OA_n}$$

$$\text{Since } \overrightarrow{OA_1} \times \overrightarrow{OA_2} = \overrightarrow{OA_2} \times \overrightarrow{OA_3} = \dots$$

$$\sum_{j=1}^{n-1} \overrightarrow{OA_j} \times \overrightarrow{OA}_{j+1} = (n-1) \overrightarrow{OA_1} \times \overrightarrow{OA_2}$$

$$= (1-n) \overrightarrow{OA_2} \times \overrightarrow{OA_1}$$

7. If  $\vec{a}, \vec{b}, \vec{c}$  are three vectors of equal magnitudes and each of them is incline at an angle  $60^\circ$  to the other it  $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{6}$ , then find  $|\vec{a}|$

Sol: Given that  $(\vec{a}, \vec{b}) = 60^\circ$ ,  $(\vec{b}, \vec{c}) = 60^\circ$ ,  $(\vec{c}, \vec{a}) = 60^\circ$

$$|\vec{a}| = |\vec{b}| = |\vec{c}|$$

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{6}$$

S.O.B.s

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 2|\vec{a}||\vec{b}|\cos 60^\circ + 2|\vec{b}||\vec{c}|\cos 60^\circ + 2|\vec{a}||\vec{c}|\cos 60^\circ = 6$$

$$3|\vec{a}|^2 + |\vec{a}|^2 + |\vec{a}|^2 + |\vec{a}|^2 = 6 \quad \therefore |\vec{a}| = |\vec{b}| = |\vec{c}|$$

$$|\vec{a}|^2 = 1 \Rightarrow |\vec{a}|^2 = 1$$

8. For any two vectors  $\vec{a}$  and  $\vec{b}$  show that

$$\left\{1 + |\vec{a}|^2\right\} \left\{1 + |\vec{b}|^2\right\} = |\vec{1} - \vec{a} \cdot \vec{b}|^2 + |\vec{a} + \vec{b} + \vec{a} \times \vec{b}|^2$$

Sol: R.H.S  $|\vec{1} - \vec{a} \cdot \vec{b}|^2 + |\vec{a} + \vec{b} + \vec{a} \times \vec{b}|^2$

$$1 + (\vec{a} \cdot \vec{b})^2 - 2(\vec{a} \cdot \vec{b})|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot (\vec{a} \times \vec{b}) + 2\vec{a} \cdot (\vec{a} \times \vec{b}) + |\vec{a} \times \vec{b}|^2$$

Here  $\vec{a} \times \vec{b}$  is  $\perp^r$   $\vec{a}$  &  $\vec{b}$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0 \text{ And } \vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

$$1 + (\vec{a} \cdot \vec{b})^2 + |\vec{a}|^2 + |\vec{b}|^2 + 0 + 0 + |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$\therefore |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$\therefore \left\{ 1 + |\vec{a}|^2 \right\} + |\vec{b}|^2 \left\{ 1 + |\vec{a}|^2 \right\}$$

$$\left\{ 1 + |\vec{a}|^2 \right\} \left\{ 1 + |\vec{b}|^2 \right\} = R.H.S$$

9. If  $\vec{a}, \vec{b}, \vec{c}$  are unit vectors such that  $\vec{a}$  is perpendicular to the plane of  $\vec{b}, \vec{c}$  and the angle between  $\vec{b}$  and  $\vec{c}$  is  $\frac{\pi}{3}$  then find  $|\vec{a} + \vec{b} + \vec{c}|$ .

**Sol:**  $\vec{a}$  is perpendicular to the plane of  $\vec{b}, \vec{c}$

$$\therefore \vec{a} \cdot \vec{b} = 0 \quad \vec{a} \cdot \vec{c} = 0$$

$$(\vec{b}, \vec{c}) = \frac{\pi}{3} \quad |\vec{a}| = 1, |\vec{b}| = 1, |\vec{c}| = 1$$

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{|\vec{a} + \vec{b} + \vec{c}|^2}$$

$$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a}}$$

$$= \sqrt{1+1+1+2(0)+2|\vec{b}||\vec{c}|\cos\frac{\pi}{3}+0}$$

$$\sqrt{3+2 \times \frac{1}{2}} = 2$$

$$(\vec{a} \times \vec{d}) \cdot \vec{b} = (-1\vec{i} - 13\vec{j} + 10\vec{k}) \cdot (-\vec{i} + 3\vec{j} + 2\vec{k})$$

$$= 11 - 39 + 20 = -8$$

$$(\vec{a} \times \vec{d}) \cdot \vec{c} - (\vec{a} \times \vec{d}) \cdot \vec{b}$$

$$-88 - (-8)$$

$$-88 + 8 = -80$$