SCALAR TRIPLE PRODUCT

Topics Covered

1. **Scalar triple product**
2. **Vector equation of plane in different forms**
3. **Skew lines, shortest distance between skew lines**
4. **Vector triple product**
5. **Scalar product of four vectors**
6. **Vector product of four vectors**

Def:- 1. If $\vec{a},\vec{b},\vec{c}$ are the three vectors, then the real numbers $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product denoted by $[\vec{a}\ \vec{b}\ \ \vec{c}]$. This is read as 'box' $\vec{a},\ \vec{b},\ \ \vec{c}$

2. If V is the volume of the parallelepiped with coterminous edges $\vec{a},\ \vec{b},\ \ \vec{c}$ then $V=\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

3. If $\vec{a},\vec{b},\ \vec{c}$ form the right handed system of vectors then $V=[\vec{a}\ \vec{b}\ \vec{c}]$

4. If $\vec{a},\ \vec{b},\ \vec{c}$ form left handed system of vectors then $-V=[\vec{a},\ \vec{b},\ \vec{c}]$

<u>5. Properties of scalar triple product ;-</u>

i) The scalar triple product is independent of the position of dot and cross.

i.e. $\vec{a}\times\vec{b}.\vec{c}=\vec{a}.\vec{b}\times\vec{c}$

ii) The value of the scalar triple product is unaltered so long as the cyclic order remains unchanged

$[\vec{a}\vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\,\vec{a}\,\vec{b}]$

iii) The value of a scalar triple product is zero if two of its vectors are equal

$[\vec{a}\,\vec{a}\ \vec{b}]=0\ [\vec{b}\ \vec{b}\ \vec{c}]=0$

iv) If $\vec{a},\vec{b},\vec{c}$ are coplanar then $[\vec{a}\vec{b}\vec{c}]=0$

v) If $\vec{a},\vec{b},\vec{c}$ form right handed system then $[\vec{a}\,\vec{b}\,\vec{c}]>0$

vi) If $\vec{a},\vec{b},\vec{c}$ form left handed system then $[\vec{a}\ \vec{b}\ \vec{c}]<0$

vii) The value of the triple product changes its sign when two vectors are interchanged

$[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]$

viii) If l,m, n are three scalars $\vec{a},\ \vec{b},\ \vec{c}$ are three vectors then $[l\vec{a}\,m\vec{b}\ n\vec{c}]=lmn[\vec{a}\ \vec{b}\ \vec{c}]$

Theorem1 : Three non zero non parallel vectors $\vec{a}\,\vec{b}\,\vec{c}$ nare coplanar iff $[\vec{a}\;\vec{b}\,\vec{c}]=0$

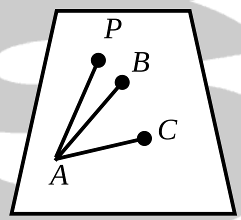
Theorem2 : If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, $\vec{c}=c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$ then $[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Theorem 3: If $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$, $\vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$, $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$ where $\vec{l},\vec{m},\vec{n}$ form a right handed system of non coplanar vectors, then $[\vec{a}\vec{b}\vec{c}]=\begin{vmatrix} \vec{m}\times\vec{n} & \vec{n}\times\vec{l} & \vec{l}\times\vec{m} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Theorem 4: The vector equation of plane passing through the points A, B, C having position vectors $\vec{a},\vec{b},\vec{c}$ **is** $[\vec{r}-\vec{a}\;\;\vec{b}-\vec{a}\;\;\vec{c}-\vec{a}]=0$ **(or)** $\vec{r}.\{(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})\}=[\vec{a}\;\vec{b}\;\vec{c}]$

Sol: Let $\overline{OP}=\vec{r}$ be any point on the plane $\overline{OA}=\vec{a}$, $\overline{OB}=\vec{b}$, $\overline{OC}=\vec{c}$ are the given points

$\overline{AP},\overline{AB},\overline{AC}$ are coplanar



$[\overline{AP}\;\overline{AB}\;\overline{AC}]=0$

$[\vec{r}-\vec{a}\;\;\vec{b}-\vec{a}\;\;\vec{c}-\vec{a}]=0$

$(\vec{r}-\vec{a}).\,(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=0$

$(\vec{r}-\vec{a}).\,\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\}=0$

$\vec{r}.\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\}-\vec{a}.\vec{b}\times\vec{c}-\vec{a}.\vec{c}\times\vec{a}-\vec{a}.\vec{a}\times\vec{b}=0$

$\vec{r}.\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\}=[\vec{a}\,\vec{b}\,\vec{c}]\left\{\because \vec{a}.\vec{c}\times\vec{a}=0\;\; \vec{a}.\vec{a}\times\vec{b}=0\right\}$

Theorem 5: The vectors equation of plane passing through the points A, B with position vectors $\vec{a},\vec{b}$ and parallel to the vector $\vec{c}$ is $[\vec{r}-\vec{a}\;\;\vec{b}-\vec{a}\;\;\vec{c}]=0$ (or) $[\vec{r}\;\;\vec{b}\;\;\vec{c}+[\vec{r}\,\vec{c}\;\;\vec{a}]=[\vec{a}\;\;\vec{b}\;\;\vec{c}]$

Theorem 6; The vector equation of the plane passing through the point A with position vector $\vec{a}$ and parallel to $\vec{b},\vec{c}$ is $[\vec{r}-\vec{a}\;\;\vec{b}\;\;\vec{c}]=0$ i.e. $[\vec{r}\;\;\vec{b}\;\;\vec{c}]=[\vec{a}\;\;\vec{b}\;\;\vec{c}]$

__**Skew lines :-**__ Two lines are said to be skew lines if there exist no plane passing through them i.e. the lines lie on two difference planes

Def:- l_1 and l_2 are two skew lines. If P is a point on l_1 and Q is a point on l_2 such that $\overleftrightarrow{PQ} \perp^r l_1$ and $\overleftrightarrow{PQ} \perp^r l_2$ then PQ is called shortest distance and $\overleftrightarrow{PQ}$ is called shortest distance line between the lines l_1 and l_2.

The shortest distance between the skew lines $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is $\dfrac{\left|[\vec{a} - \vec{c}\ \vec{b}\ \vec{d}]\right|}{\left|\vec{b} \times \vec{d}\right|}$

VECTOR TRIPLE PRODUCT :-

Theorem: If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then

i) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ ii) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{b}$

Proof: i) Let $\vec{a} = a_1\vec{l} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{l} + b_2\vec{j} + b_3\vec{k}$, $\vec{c} = c_1\vec{l} + c_2\vec{j} + c_3\vec{k}$ be three vectors $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{l}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \vec{l}\{c_3(a_3b_1 - a_1b_3) - c_2(a_1b_2 - a_2b_1)\} - \vec{j}\{c_3(a_2b_3 - a_3b_2 - c_1(a_1b_2 - a_2b_1)\} + \vec{k}\{c_2(a_2b_3 - a_3b_2) - c_1(a_3b_1 - a_1b_3)\}$$

$$(\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a} = (a_1c_1 + a_2c_2 + a_3c_3)\{b_1\vec{l} + b_2\vec{j} + b_3\vec{k})$$

$$(\cancel{a_1b_1c_1} + a_2b_1c_2 + a_3b_1c_3 - \cancel{a_1b_1c_2} - a_1b_3c_3)\vec{l} + (a_1b_2c_1 + a_2b_3c_2 + a_3b_3c_3 - a_2b_1c_1 - \cancel{a_2b_2c_2} - a_2b_3c_3)\vec{j}$$
$$+ (a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_3b_1c_1 - a_3b_2c_2 - a_3b_3c_3)\vec{k}$$

$$\Rightarrow \{c_3(a_3b_1 - a_1b_3) - c_2(a_1b_2 - a_2b_1)\}\vec{l} + \vec{j}\{c_3(a_2b_3 - a_3b_2) - c_1(a_1b_2 - a_2b_1) + \vec{k}\{c_2(a_2b_3 - a_3b_2) - c_1(a_3b_1 - a_1b_3)\}$$

Hence proved

Proof ii ; $\vec{b} \times \vec{c} = \begin{vmatrix} i & j & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \vec{l}(b_2c_3 - b_3c_2) - \vec{j}(b_1c_3 - b_3c_1) + \vec{k}(b_1c_2 - b_2c_1)$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\vec{i} & \vec{j} & k\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1\end{vmatrix}$$

$$=\vec{l}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}-\vec{j}\{a_1(b_1c_2-b_2c_1)-a_3(b_1c_2-b_2c_1)\}+\vec{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$

R.H.S. $(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$$(a_1c_1+a_2c_2+a_3c_3)\{b_1\vec{l}+b_2\vec{j}+b_3\vec{k}\}-(a_1b_1+a_2b_2+a_3b_3)\{c_1\vec{l}+c_2\vec{j}+c_3\vec{k}\}$$

$$\Rightarrow\vec{l}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}-\vec{j}\{a_1cb_1c_2-b_2c_1)-a_3(b_1c_2-b_2c_1)\}+\vec{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$

Product of four vectors :-

Theorem: If $\vec{a},\vec{b},\vec{c},\vec{d}$ are four vectors then $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$

Proof: $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\{(\vec{a}\times\vec{b})\times\vec{c}\}.\vec{d}$ $\{\because$ dot and cross are inter changeable$\}$

$$\{(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}\}.\vec{d}=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{b}.\vec{c})(\vec{a}.\vec{d})$$

$$=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$$

Theorem: If $\vec{a},\vec{b},\vec{c},\vec{d}$ are four vectors then $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}\,]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}\,]\vec{d}=[\vec{a}\,\vec{c}\,\vec{d}\,]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}\,]\vec{a}$

Proof:- $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{a}\times\vec{b}.\vec{d})\vec{c}-(\vec{a}\times\vec{b}.\vec{c})\vec{d}$

$$=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{c}\times\vec{d}\,.\vec{a}]\vec{b}-(\vec{c}\times\vec{d}\,.\vec{b}]\vec{a}$$

$$=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}-(\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

EXERCISE 4(a)

I. Very Short Answer Questions:

1. Find the angle between the vectors $\vec{i}+2\vec{j}+3\vec{k}$ **and** $3\vec{i}-\vec{j}+2\vec{k}$

Sol: Let $\vec{a}=\vec{i}+2\vec{j}+3\vec{k}$ $\quad \vec{b}=3\vec{i}-\vec{j}+2\vec{k}$

Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$

$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$=\frac{3-2+6}{\sqrt{1^2+2^2+3^2}\sqrt{3^2+1^2+2^2}}$$

$$=\frac{7}{12}=\frac{1}{2}$$

$$\theta=60^0$$

2. If the vectors $2\vec{i}+\lambda\vec{j}-\vec{k}$ **and** $4\vec{i}-2\vec{j}+2\vec{k}$ **are perpendicular to each other then find** λ

Sol: Let $\vec{a}=2\vec{i}+\lambda\vec{j}-\vec{k}$ $\quad \vec{b}=4\vec{i}-2\vec{j}+2\vec{k}$ be the given vectors

Given that $\vec{a}\perp^{r}\vec{b}$ $\qquad \therefore \vec{a}.\vec{b}=0$

$(2\vec{i}+\lambda\vec{j}-\vec{k}).(4\vec{i}-2\vec{j}+2\vec{k})=0$

$8-2\lambda-2=0\Rightarrow 6=2\lambda\Rightarrow\lambda=3$

3. For what values of λ **the vectors** $\vec{i}-\lambda\vec{j}+2\vec{k}$ **and** $8\vec{i}+6\vec{j}-\vec{k}$ **are at right angles**

Sol: Let $\vec{a}=\vec{i}-\lambda\vec{j}+2\vec{k}$ $\quad \vec{b}=8\vec{i}+6\vec{j}-\vec{k}$ be the given vectors

$\vec{a}$, $\vec{b}$ are perpendicular

$\therefore \vec{a}.\vec{b}=0\Rightarrow(\vec{i}-\lambda\vec{j}+2\vec{k}).8\vec{i}+6\vec{j}-\vec{k})=0$

$=8-6\lambda-2=0\Rightarrow 6=6\lambda\Rightarrow\lambda=1$

4. $\vec{a}=2\vec{i}-\vec{j}+\vec{k}$, $\vec{b}=\vec{i}-3\vec{j}-5\vec{k}$. **Find the vector $\vec{c}$ such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the sides of a triangle**

Sol: since $\vec{a},\vec{b},\vec{c}$ are the sides of a triangle $\vec{a}+\vec{b}+\vec{c}=\vec{O}$

$\therefore \vec{c}=-\vec{a}-\vec{b}$

$=-2\vec{i}+\vec{j}-\vec{k}-\vec{i}+3\vec{j}+5\vec{k}$

$\therefore \vec{c}=-3\vec{i}+\vec{j}+4\vec{k}$

5. **Find the angle between the planes** $\vec{r}.(2\vec{i}-\vec{j}+2\vec{k})=3$ **and** $\vec{r}.(3\vec{i}+6\vec{j}+\vec{k})=4$

Sol: Angle between the planes be θ

$$\cos\theta=\frac{\overrightarrow{m_1}.\overrightarrow{m_2}}{|\overrightarrow{m_1}||\overrightarrow{m_2}|}$$

$$=\frac{(2\vec{i}-\vec{j}+2\vec{k}).(3\vec{i}+6\vec{j}+\vec{k})}{\sqrt{2^2+1^2+2^2}\ \sqrt{3^2+6^2+1^2}}$$

$$=\frac{\not{6}-\not{6}+2}{\sqrt{9}\ \sqrt{6}}$$

$$\theta=\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

6. **Find the radius of the sphere whose equation is** $\vec{r}^{\,2}=2\vec{r}.(4\vec{i}-2\vec{j}+2\vec{k})$

Sol: $\vec{r}^{\,2}-2\vec{r}.(4\vec{i}-2\vec{j}+2\vec{k})=0$

This is of the form $\vec{r}^{\,2}-2\vec{r}.\vec{c}=a^2-c^2$

Here $\vec{c}=4\vec{i}-2\vec{j}+2\vec{k}$

And $a^2-c^2=0$

$a^2=|\vec{c}|^2 \Rightarrow a^2=|4\vec{i}-2\vec{j}+2\vec{k}|^2$

$a^2=\left\{\sqrt{16++4}\right\}^2$

$a=\sqrt{24}$

$\therefore$ Radius of sphere = $\sqrt{24}$

7. If $(\vec{r}-2\vec{j}-4\vec{k}).(\vec{r}+2\vec{j}+2\vec{k})=0$ is the equation of sphere then ifnd its centre

Sol: $\{\vec{r}-(2\vec{i}-\vec{j}+4\vec{k})\}.\{\vec{r}-(-2\vec{i}+2\vec{j}+2\vec{k})\}=0$

This is of the form $(\vec{r}-\vec{a}).(\vec{r}-\vec{b})=0$

$\vec{a}=2\vec{i}-\vec{j}+4\vec{k}$, $\vec{b}=-2\vec{i}+2\vec{j}-2\vec{k}$

Centre $=\dfrac{\vec{a}+\vec{b}}{2}=\dfrac{2\vec{i}+\vec{j}+4\vec{k}-2\vec{i}+2\vec{j}-2\vec{k}}{2}$

$=\dfrac{\vec{j}+2\vec{k}}{2}$

Centre $=\left(0,\dfrac{1}{2},1\right)$

8. Let $\vec{e_1}$ and $\vec{e_2}$ be the unit vector containing angle θ. If $\frac{1}{2}|\vec{e_1}-\vec{e_2}|=\sin\lambda\theta$ then find λ

Sol: Given that $|\vec{e_1}|=1$ $\quad |\vec{e_2}|=1$

$\frac{1}{2}|\vec{e_1}-\vec{e_2}|=\sqrt{\frac{1}{4}(\vec{e_1}-\vec{e_2})^2}$

$=\sqrt{\frac{1}{4}\{1+1-2|\vec{e_1}||\vec{e_2}|\cos\theta\}}$

$\therefore \sin\lambda\theta=\sqrt{\frac{1}{4}\{2-2\cos\theta\}}$

$=\sqrt{\frac{1}{4}4\sin^2\frac{\theta}{2}}$

$\sin\lambda\theta=\sin\frac{\theta}{2}\Rightarrow\lambda=\frac{1}{2}$

9. If $\vec{a}=(4,3,5)$ is the centre of the sphere which passes through the point (-1, -1, 2) then find the cartesian equation of the sphere.

Sol: Centre of sphere = (4, 3, 5)

Point of sphere = (-1, -1, 2)

Radius = $\sqrt{(4+1)^2+(3+1)^2+(5-2)^2}$

$\sqrt{25+16+9}=5\sqrt{2}$

$\therefore$ Equation of sphere is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

$(x-4)^2+(y-3)^2+(z-5)^2=(5\sqrt{2})^2$

$\Rightarrow (x-4)^2+(y-3)^2+(z-5)^2=50$

10. Let $\vec{a}=\vec{i}+\vec{j}+\vec{k}$, $\vec{b}=2\vec{i}+3\vec{j}+\vec{k}$ find

i) The projection vector of $\vec{b}$ on $\vec{a}$ and its magnitude

ii) The vector components of $\vec{b}$ in the direction of $\vec{a}$ and $\perp^r$ to $\vec{a}$

Sol: i) The projection vector of $\vec{b}$ on $\vec{a}$ is

$$\frac{(\vec{b}.\vec{a})\vec{a}}{|\vec{a}|^2}=\frac{(2+3+1)(\vec{i}+\vec{j}+\vec{k})}{(\sqrt{1+1+1})^2}$$

$=2\vec{i}+2\vec{j}+2\vec{k}$

$\therefore$ Magnitude $=\sqrt{2^2+2^2+2^2}=2\sqrt{3}$

ii) The projection of vector of $\vec{b}$ on $\vec{a}$ = $2\vec{i}+2\vec{j}+2\vec{k}$ the projection vector of

$$\vec{b}\perp^2\vec{a}=\vec{b}-\frac{(\vec{b}.\vec{a})\vec{a}}{|\vec{a}|^2}$$

$(2\vec{i}+3\vec{j}+\vec{k}\vec{j})-(2\vec{i}+2\vec{j}+2\vec{k})=\vec{j}-\vec{k}$

11. Find the equations of the plane through the point (3, -2, 1) and perpendicular to the vector (4, 7, -4)

Sol: Vector equation of plane passing through the point $\vec{a}$ and perpendicular to $\vec{m}$ is

$(\vec{r}-\vec{a}).\vec{m}=0$

Here $\vec{a}=3\vec{i}-2\vec{j}+\vec{k}$ $\quad \vec{m}=4\vec{i}+7\vec{j}-\vec{k}$

$\therefore$ Equations is $(\vec{r}-(3\vec{i}-2\vec{j}+\vec{k})\}.(\vec{i}+7\vec{j}-4\vec{k})=0$

$\vec{r}.(4\vec{i}+7\vec{j}-4\vec{k})-(3\vec{i}-2\vec{j}+\vec{k}).(4\vec{i}+7\vec{j}-4\vec{k})=0$

$\vec{r}.(4\vec{i}+7\vec{j}-4\vec{k})-\{12-14-4\}=0$

$\vec{r}.(4\vec{i}+7\vec{j}-4\vec{k})=-6$

12. If $\vec{a}=2\vec{i}+2\vec{j}-3\vec{k}$, $\vec{b}=3\vec{i}-\vec{j}+2\vec{k}$ then find the angle between $2\vec{a}+\vec{b}$ and $\vec{a}+2\vec{b}$

Sol: $2\vec{a}=4\vec{i}+4\vec{j}-6\vec{k}$

$\vec{b}=3\vec{i}-\vec{j}+2\vec{k}$

$2\vec{a}+\vec{b}=7\vec{i}+3\vec{j}-4\vec{k}$

$\vec{a}=2\vec{i}+2\vec{j}-3\vec{k}$

$2\vec{b}=6\vec{i}-2\vec{j}+4\vec{k}$

$\vec{a}+2\vec{b}=8\vec{i}+\vec{k}$

Let θ be the angle between the vectors

$$\cos\theta=\frac{(7\vec{i}+3\vec{j}-4\vec{k}).(8\vec{i}+\vec{k})}{\sqrt{7^2+3^2+4^2}\ \sqrt{8^2+12}}$$

$$=\frac{56-4}{\sqrt{74}\sqrt{65}}\Rightarrow\theta=\cos^{-1}\left(\frac{52}{\sqrt{74\times 65}}\right)$$

II. Short Answer Questions:

1) If α,β and γ be the angles made by the vector $3\vec{I}-6\vec{j}+2\vec{k}$ with positive directions of the co ordinate axis then find $\cos\alpha, \cos\beta and \cos\gamma$.

Sol: Let $\vec{a}=3\vec{I}-6\vec{j}+2\vec{k}$

Unit vector in the direction of $\vec{a}=\frac{\vec{a}}{|\vec{a}|}$

$$=\frac{3\vec{I}-6\vec{j}+2\vec{k}}{\sqrt{3^2+6^2+2^2}}$$

$$=(\frac{3}{7})\vec{I}-(\frac{6}{7})\vec{j}+(\frac{2}{7})\vec{k}$$

$$\cos\alpha=\frac{3}{7}\quad \cos\beta=-\frac{6}{7}\quad \cos\gamma=\frac{2}{7}$$

2. Find the angles made by the straight line passing through the points (1, -3, 2) and (3, -5, 1) with the coordinate axes

Sol: Let A = (1, -3, 2) B = (3, -5, 1) be the given points

Dr's of AB = 3 – 1, -5 + 3, 1 -2

2, -2, -1

DC's of AB $=\frac{2}{\sqrt{2^2+2^2+1^2}}, \frac{-2}{\sqrt{2^2+2^2+1^2}}, \frac{-1}{\sqrt{2^2+2^2+1^2}}$

$\cos\alpha=\frac{2}{3}, \cos\beta=-\frac{2}{3}, \cos\gamma=\frac{-1}{3}$

3. Find unit vector parallel to the XOY – plane and perpendicular to the vector $4\vec{i}-3\vec{j}+\vec{k}$

Sol: Let $a\vec{i}+b\vec{j}$ be the vector in XOY plane since we want unit vector

$\sqrt{a^2+b^2}=1$ $\quad a^2+b^2=1\rightarrow(1)$

The vector $a\vec{i}+b\vec{j}$ is $\perp^r$ to $4\vec{i}-3\vec{j}+\vec{k}$

$\therefore(a\vec{i}+b\vec{j}).(4\vec{i}-3\vec{k}+\vec{k})=0$

4a – 3 b = 0

4a = 3b

$\frac{a}{3}=\frac{b}{4}=k$

a = 3k b = 4k

sub a, b in (1)

$9k^2+16k^2=1\Rightarrow k^2=\frac{1}{25}$

$k=\pm\frac{1}{5}$

$\therefore$ Vector $=\pm\frac{1}{5}(3\vec{i}+4\vec{j})$

4. In triangle $\angle ACB = 90^0$ if P and Q are points of trisection of AB then prove that

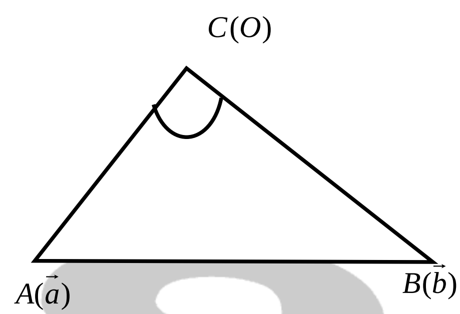
$$CP^2 + CQ^2 = \frac{5}{9}(AB)$$

Sol: Let $\vec{C}(\vec{O})$ be the origin of vectors

$\overline{CA} = \vec{a}$, $\overline{CB} = \vec{b}$

P divides AB in the ratio 1:2

$$\overline{CP} = \frac{2\vec{a} + \vec{b}}{3}$$



Q divides AB in the ratio 2:1

$$\overline{LQ} = \frac{\vec{a} + 2\vec{b}}{3}$$

$\therefore \angle ACB = 90^0 \Rightarrow \vec{a}.\vec{b} = 0$

And $\left|\overline{CA}\right|^2 + \left|\overline{CB}\right|^2 = \left|\overline{AB}\right|^2$

$$\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 = \left|\overline{AB}\right|^2$$

$$CP^2 + CQ^2 = \left|\overline{CP}\right|^2 + \left|\overline{CQ}\right|^2$$

$$= \left|\frac{2\vec{a} + \vec{b}}{3}\right|^2 + \left|\frac{\vec{a} + 2\vec{b}}{3}\right|^2$$

$$= \frac{4\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + \left|\vec{a}\right|^2 + 4\left|\vec{b}\right|^2}{9} \qquad \because \vec{a}.\vec{b} = 0$$

$$= \frac{5}{9}\left\{\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right\}$$

$$= \frac{5}{9}\left|\overline{AB}\right|^2$$

$$CP^2 + CQ^2 = \frac{5}{9}AB^2$$

5. If $\vec{a}+\vec{b}+\vec{c}=0$, $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$ then find the angle between $\vec{a}$ and $\vec{b}$

Sol: Given that $\vec{a}+\vec{b}+\vec{c}=0$

$\vec{a}+\vec{b}=-\vec{c}$

$|\vec{a}+\vec{b}|=|-\vec{c}|$

Squaring on both sides

$|\vec{a}+\vec{b}|^2=|-\vec{c}|^2$

$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}=|\vec{c}|^2$

$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}\cos\theta=|\vec{c}|^2$

Where $\theta=(\vec{a},\vec{b})$

$9+25+2\times3\times5\cos\theta=7^2$

$30\cos\theta=49-34$

$(x^2+y^2+z^2)-\{z\}^2=x^2+y^2 \quad \Rightarrow \theta=60^0$

$\therefore$ Angle between $\vec{a}$ and $\vec{b}$ $=60^0$

6. If $|\vec{a}|=2$, $|\vec{b}|=3$ and $|\vec{c}|=4$ and each $\vec{a},\vec{b},\vec{c}$ is perpendicular to sum of the other two vector then find the magnitude of $\vec{a}+\vec{b}+\vec{c}$

Sol: Given that $\vec{a}\perp^r\ \vec{b}+\vec{c}$; $\vec{c}\perp^r\ (\vec{a}+\vec{b})$ and $\vec{b}\perp^r\ (\vec{a}+\vec{c})$

$\therefore \vec{a}.(\vec{b}+\vec{c})=0\rightarrow(1)$, $\vec{b}.(\vec{c}+\vec{a})=0$, $\vec{c}.(\vec{a}+\vec{b})=0$

$\therefore$ Adding (1), (2), (3) we have

$2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$

$|\vec{a}+\vec{b}+\vec{c}|=\sqrt{|\vec{a}+\vec{b}+\vec{c}|^2}$

$=\sqrt{|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})}$

$=\sqrt{2^2+3^2+4^2+0}=\sqrt{29}$

7. Find the equation of the plane passing through the point $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$ and $\perp^r$ and between the distance of this plane from the origin

Sol: Equation of the plane is $(\vec{r}-\vec{a}).\vec{m}=0$

Here $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$ $\quad \vec{m}=3\vec{i}-2\vec{j}-2\vec{k}$

$\{\vec{r}-(2\vec{i}+3\vec{j}-\vec{k})\}.(3\vec{i}-2\vec{j}-2\vec{k})=0$

$\vec{r}.(3\vec{i}-2\vec{j}-2\vec{k})-(6-6+2)=0$

$\vec{r}.(3\vec{i}-2\vec{j}-2\vec{k})=2$

$$\vec{r}.\frac{(3\vec{i}-2\vec{j}-2\vec{k})}{|3\vec{i}-2\vec{j}-2\vec{k}|}=\frac{2}{|3\vec{i}-2\vec{j}-2\vec{k}|}$$

$$\vec{r}.\frac{(3\vec{i}-2\vec{j}-2\vec{k})}{|3\vec{i}-2\vec{j}-2\vec{k}|}=\frac{2}{\sqrt{17}}$$

This is of the form $\vec{r}.\hat{n}=p$

Where $p=\perp^r$ distance from the origin to the plane

$$=\frac{2}{\sqrt{17}}$$

8. Find the vector equation and its cartesian form of sphere with $\vec{a}=3\vec{i}+4\vec{j}-2\vec{k}$ and $\vec{b}=-2\vec{i}-\vec{j}$ as the ends points of a diameter

Sol: Vector equation of sphere $(\vec{r}-\vec{a}).(\vec{r}-\vec{b})=0$

$\therefore$ Equation is $\{\vec{r}-(3\vec{i}+4\vec{j}-2\vec{k})\}.(\vec{r}-(-2\vec{i}-\vec{j})=0$

$\overrightarrow{r^2}-\vec{r}.\left(\vec{i}+3\vec{j}-\overrightarrow{2k}\right)-10=0$

Cartesian form of sphere equation is

$(x-a_1)(x-b_1)+(y-a_2)(y-b_2)+(z-a_3)(z-b_3)=0$

Here (a_1,a_2,a_3), (b_1,b_2,b_3) are the component of given vector $\vec{a},\vec{b}$

$\therefore (x-3)(x+2)+(y-4)(y+1)+(z+y)(z-0)=0$

$x^2+y^2+z^2-x-3y+2z-10=0$

9. **If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are the position vectors of four coplanar points such that $(\vec{a}-\vec{d}).(\vec{b}-\vec{c})=(\vec{b}-\vec{d}).(\vec{c}-\vec{a})=0$ show that the point $\vec{d}$ represents the orthocenter of triangle with $\vec{a},\vec{b},\vec{c}$ as its vertices**

Sol: Let $\overline{OA}=\vec{a}$, $\overline{OB}=\vec{b}$, $\overline{OC}=\vec{c}$, $\overline{OD}=\vec{d}$ be the given position vectors

Given that $(\vec{a}-\vec{d}).(\vec{b}-\vec{c})=0$

$(\overline{OA}-\overline{OD}).(\overline{OB}-\overline{OC})=0$

$\overline{DA}.\overline{CB}=0 \Rightarrow \overline{DA}\perp^r \overline{CB}$

$(\vec{b}-\vec{d}).(\vec{c}-\vec{a})=0$

$(\overline{OB}-\overline{OD}).(\overline{OC}-\overline{OA})=0$

$\overline{DB}.\overline{AC}=0 \Rightarrow \overline{DB}\perp^r \overline{AC}$

$\therefore$ AD, BD are altitude from A and B to the sides BC, AC of triangle ABC they are meeting in D hence $\vec{d}$ is the orthocenter of triangle.

III. Long Answer Questions:

1. **Show that the points (5, -1, 1) (7, -4, 7), (1,-6,10) and (-1, -3, 4) are the vertices of a rhombus**

Sol: Let A = (5, -1, 1), B = (7, -4, 7), C = (1, -6, 10), D (-1, -3, 4) be the given vertices

$|\overline{AB}|=|2\vec{i}-3\vec{j}+6\vec{k}|=\sqrt{4+9+36}=7$

$|\overline{BC}|=|-6\vec{i}-2\vec{j}+3\vec{k}|=\sqrt{49}=7$

$|\overline{CD}|=|-2\vec{i}+3\vec{j}-6\vec{k}|=\sqrt{49}=7$

$|\overline{AD}|=|-6\vec{i}-2\vec{j}+3\vec{k}|=\sqrt{49}=7$

$\overline{BA}.\overline{BC}=(-2\vec{i}+3\vec{j}-6\vec{k}).(-6\vec{i}-2\vec{j}+3\vec{k})$

$=12-6-18\neq 0$

$\therefore \angle B\neq 90^0$

$\overline{AC}.\overline{BD}=(-4\vec{i}-5\vec{j}+9\vec{k}).(-8\vec{i}+\vec{j}-3\vec{k})$

= 32 – 5 – 27 =0

$\therefore \overline{AC} \perp^r \overline{BD}$

Hence ABCD is a rhombus

2. Let $\vec{a} = 4\vec{i} + 5\vec{j} - \vec{k}$, $\vec{b} = \vec{i} - \vec{j} + 5\vec{k}$ and $\vec{c} = 3\vec{i} + \vec{j} - \vec{k}$ find the vector which is $\perp^r$ to both $\vec{a}$ and $\vec{b}$ whose magnitude is 21 times the magnitude of $\vec{c}$.

Sol: Let $\vec{p} = p_1\vec{i} + p_2\vec{j} + p_3\vec{k}$ be the required

$\vec{p}.\vec{a} = 0 \qquad 4p_1 + 5p_2 - p_3 = 0 \rightarrow (1)$

$\vec{p}.\vec{b} = 0 \qquad \Rightarrow p_1 + 4p_2 + 5p_3 = 0 \rightarrow (2) \qquad \because \vec{p} \perp^r$ both $\vec{a}$ & $\vec{b}$

$|\vec{p}| = 21|\vec{c}|$

$\sqrt{p_1^2 + p_2^2 + p_3^2} = 21\sqrt{9+1+1} \Rightarrow p_1^2 + p_2^2 + p_3^2 = 41 \times 11 \rightarrow (3)$

By solving (1) & (2)

$p_1 \quad p_2 \quad p_3$

5 -1 4 5

-4 5 1 -4

$$\frac{p_1}{25-4} = \frac{p_2}{-1-20} = \frac{p_3}{-16-5}$$

$p_1 = 21k, \qquad p_2 = -21k, \qquad p_3 = -21k$

Sub p_1, p_2, p_3 we have

$$41 \times 3k^2 = 441 \times 11 \Rightarrow k = \frac{\sqrt{11}}{\sqrt{3}}$$

$$\therefore p_1 = 21 \times \frac{\sqrt{11}}{\sqrt{3}} = 7\sqrt{33}, \; p_2 = -7\sqrt{33}, \; p_3 = -7\sqrt{33}$$

$\therefore$ Vector $= \pm 7\sqrt{33}(\vec{i} - \vec{j} - \vec{k})$

3. If A (-6, 1, 6), B(6, -2, 3), C(-2, -3, -1) and D(-5, -9, -7) are four points. Show that the points A, B and D lie on a sphere with centre C. What s the radius of the sphere

Sol: $|\overline{CA}| = |4\vec{i} - 4\vec{j} - 7\vec{k}| = \sqrt{16+16+49} = 9$

$|\overline{CB}| = |8\vec{i} + \vec{j} + 4\vec{k}| = \sqrt{64+1+16} = 9$

$$\left|\overline{CD}\right| = \left|-3\vec{i} - 6\vec{j} - 6\vec{k}\right| = \sqrt{9+36+36} = 9$$

The point C is equi distant from A, B, D hence A, B, D lie on a sphere with centre 'C'.

Radius of sphere = 9

4. **G is the centroid of triangle ABC and a, b, are the lengths of the sides BC, CA and AB respectively prove that** $a^2+b^2+c^2 = 3(OA^2+OB^2+OC^2) - 9(OG)^2$

Sol: Let $\left|\overline{BC}\right| = a$, $\left|\overline{CA}\right| = b$, $\left|\overline{AB}\right| = c$

$$a^2+b^2+c^2 = \left|\overline{BC}\right|^2 + \left|\overline{CA}\right|^2 + \left|\overline{AB}\right|^2$$

$$= \left|\overline{OC}-\overline{OB}\right|^2 + \left|\overline{OA}-\overline{OC}\right|^2 + \left|\overline{OB}-\overline{OA}\right|^2$$

$$= \left|\overline{OC}\right|^2 + \left|\overline{OB}\right|^2 - 2(\overline{OC}) + (\overline{OB}) + \left|\overline{OA}\right|^2 + \left|\overline{OC}\right|^2 - 2(\overline{OA})(\overline{OC}) + (\overline{OB})^2 + \left|\overline{OC}\right|^2 - 2(\overline{OB}.\overline{OC})$$

$$a^2+b^2+c^2 = 2\left\{(\overline{OA})^2 + \left|\overline{OB}\right|^2 + \left|\overline{OC}\right|^2\right\} - 2\{\overline{OA}.\overline{OB} + \overline{OB}.\overline{OC} + \overline{OC}.\overline{OD}\} \rightarrow (1)$$

We know that

$$\overline{OG} = \frac{\overline{OA}+\overline{OB}+\overline{OC}}{3}$$

$$9(\overline{OG})^2 = \left|\overline{AB}\right|^2 + \left|\overline{OB}\right|^2 + \left|\overline{OC}\right|^2 + 2(\overline{OA}.\overline{OB} + \overline{OB}.\overline{OC} + \overline{OC}.\overline{OA})$$

$$\therefore 2\{\overline{OA}.\overline{OB} + \overline{OB}.\overline{OC} + \overline{OC}.\overline{OA}\} = 9(OG)^2 - \left\{\left|\overline{OA}\right|^2 + \left|\overline{OB}\right|^2 + \left|\overline{OC}\right|^2\right\}$$

Sub this in (1) we have

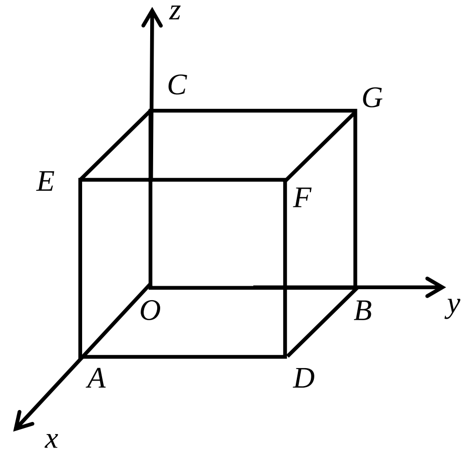
$$a^2+b^2+c^2 = 2\left\{\left|\overline{OA}\right|^2 + \left|\overline{OB}\right|^2 + \left|\overline{OC}\right|^2\right\} - \left\{9(OG)^2 - \left|OA\right|^2 + \left|\overline{OB}\right|^2 + \left|\overline{OC}\right|^2\right\}$$

$= 3\left\{(OA)^2 + OB^2 + OC^2\right\} - 9(OG)^2$ Hence prove

5. **A line makes angles** $\theta_1, \theta_2, \theta_3 and \theta_4$ **with the diagonals of a cube show that** $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 = \frac{4}{3}$**.**

Sol: Let $\overline{OA}, \overline{OB}, \overline{OC}$ be the edges of cube along x-axis y-axis z-axis let OADB GCEF be the unit cube

O = (0, 0, 0), A= (1, 0, 0) (lies on x-axis)



D = (1, 1, 0) (lies on x y plane)

B = (0, 1, 0) (lies on G = (0, 1, 1,) (lies on z-plane)

C= (0, 0, 1) (lies on (z-axis)

E = (1, 0, 1) (lies on zx plane)

F = (1, 1, 1) (lies on space)

The diagonals of cube

$\overrightarrow{OF}, \overrightarrow{AG}, \overrightarrow{DC}, \overrightarrow{BE}$

$\overrightarrow{OF} = \vec{i} + \vec{j} + \vec{k}$

$\overrightarrow{AG} = \overrightarrow{OG} - \overrightarrow{OA} = (-\vec{i} + \vec{j} + \vec{k})$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = -\vec{i} - \vec{j} + \vec{k} \Rightarrow \overrightarrow{CD} = \vec{i} + \vec{j} - \vec{k}$

$\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = \vec{i} - \vec{j} + \vec{k}$

Let $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ be the vector along the given lie

$\cos\theta_1 = \dfrac{a_1 + a_2 + a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{3}}$ $\quad \cos\theta_2 = \dfrac{-a_1 + a_2 + a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{3}}$

$\cos\theta_3 = \dfrac{a_1 + a_2 - a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{3}}$ $\quad \cos\theta_4 = \dfrac{a_1 - a_2 + a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{3}}$

Here $\theta_1, \theta_2, \theta_3, \theta_4$ are the angle before the line, diagonals $\overrightarrow{OF}, \overrightarrow{AG}, \overrightarrow{CD}$ and $\overrightarrow{BE}$ respectively

$\therefore \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4$

$= \dfrac{(a_1 + a_2 + a_3)^2 + (-a_1 + a_2 + a_3)^2 + (a_1 + a_2 - a_3)^2 + (a_1 - a_2 + a_3)^2}{3(a_1^2 + a_2^2 + a_3^2)}$

$= a_1^2 + a_2^2 + a_3^2 + 2\cancel{a_1}a_2 + 2\cancel{a_2}a_3 + 2\cancel{a_3}a_1 + a_1^2 + a_2^2 + a_3^2 - 2\cancel{a_1}a_2 + 2\cancel{a_2}a_3$

$-2\cancel{a_1}a_3 + a_1^2 + a_2^2 + a_3^2 + 2\cancel{a_1}a_2 - 2\cancel{a_1}a_3 + a_1^2 + a_2^2 + a_3^2$

$-2\cancel{a_1}a_3 - 2\cancel{a_2}a_3 + 2\cancel{a_1}a_3$

$3(a_1^2 + a_2^2 + a_3^2)$

$$= \frac{4(a_1^2 + a_2^2 + a_3^2)}{3(a_1^2 + a_2^2 + a_3^2)} = \frac{4}{3}$$

$$\therefore \cos^2\theta_1 + \cos^2\theta_1 + \cos^2\theta_3 + \cos^2\theta_4 = \frac{4}{3}$$