Product of vectors

SCALAR PRODUCT

Definitions and Key Points :

Def: Let $\vec{a},\vec{b}$ be two vectors dot product (or) scalar product (or) direct product (or) inner product denoted by $\vec{a}.\vec{b}$ .Which is defined as $|\vec{a}||\vec{b}|\cos\theta$ where $\theta=(\vec{a},\vec{b})$.

* The product $\vec{a}.\vec{b}$ is zero when $|\vec{a}|=0$ (or) $|\vec{b}|=0$ (or) $\theta=90^0$.

*** Sign of the scalar product** : Let $\vec{a},\vec{b}$ are two non-zero vectors

(i) If θ is acute then $\vec{a}.\vec{b}>0$ (i.e $0<\theta<90^0$).

(ii) If θ is obtuse then $\vec{a}.\vec{b}<0$ (i.e $90^0<\theta<180^0$).

(iii) If $\theta=90^0$ then $\vec{a}.\vec{b}=o$.

(iv) If $\theta=0^0$ then $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|$.

(v) If $\theta=180^0$ then $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|$.

Note:-

(1) The dot product of two vectors is always scalar.

(2) $\vec{a}.\vec{b}=\vec{b}.\vec{a}$ i.e dot product of two vectors is commutative.

(3) If $\vec{a}.\vec{b}$ are two vectors then $\vec{a}.(-\vec{b})=(-\vec{a}).\vec{b}=-(\vec{a}.\vec{b})$.

(4) $(-\vec{a}).(-\vec{b})=\vec{a}.\vec{b}$.

(5) If l,m are two scalars and $\vec{a}.\vec{b}$ are two vectors then $(l\vec{a}).(m\vec{b})=lm(\vec{a}.\vec{b})$.

(6) If $\vec{a}$ and $\vec{b}$ are two vectors then $\vec{a}.\vec{b}=\pm|\vec{a}||\vec{b}|$.

(7) If $\vec{a}$ is a vector then $\vec{a}.\vec{a}=|\vec{a}|^2$.

(8) If $\vec{a}$ is a vector $\vec{a}.\vec{a}$ is denoted by $(\vec{a})^2$ hence $(\vec{a})^2=|\vec{a}|^2$.

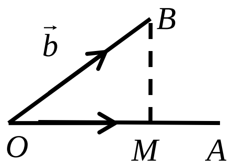
* Component and orthogonal projection

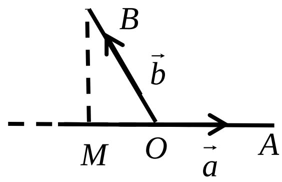
Def: Let $\vec{a}=\overrightarrow{OA}$ $\vec{b}=\overrightarrow{OB}$ be two non zero vectors let the plane passing through B and perpendicular to $\vec{a}$ intersect $\overrightarrow{OA}$ ln M.

(i) If $(\vec{a},\vec{b})$ is acute then OM is called component of $\vec{b}$ on $\vec{a}$.

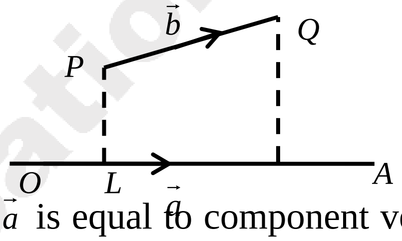
(ii) If $(\vec{a},\vec{b})$ is obtuse then –(OM) is called the component of $\vec{b}$ on $\vec{a}$.

(iii) The vector $\overline{OM}$ is called component vector of $\vec{b}$ on $\vec{a}$.





Def: Let $\vec{a}=\overrightarrow{OA}$; $\vec{b}=\overrightarrow{PQ}$ be two vectors let the planes passing through P, Q and perpendicular to $\vec{a}$ intersect $\overline{OA}$ in L, M respectively then $\overline{LM}$ is called orthogonal projection of $\vec{b}$ on $\vec{a}$



Note : i) The orthogonal projection of a vector $\vec{b}$ on $\vec{a}$ is equal to component vector of $\vec{b}$ on $\vec{a}$.

ii) Component of a vector $\vec{b}$ on $\vec{a}$ is also called projection of $\vec{b}$ on $\vec{a}$

iii) If A< B, C, D are four points in the space then the component of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ is same as the projection of $\overrightarrow{AB}$ on the ray $\overrightarrow{CD}$.

* If $\vec{a},\vec{b}$ be two vectors $(\vec{a}\neq\vec{o})$ then

i) The component of $\vec{b}$ on $\vec{a}$ is $\frac{\vec{b}.\vec{a}}{|\vec{a}|}$

iii) The orthogonal projection of $\vec{b}$ on $\vec{a}$ is $\frac{(\vec{b}.\vec{a})\vec{a}}{|\vec{a}|^2}$.

***** If $\vec{i},\vec{j},\vec{k}$ form a right handed system of Ortho normal triad then

i) $\vec{i}.\vec{j}=\vec{j}.\vec{j}=\vec{k}.\vec{k}=1$

ii) $\vec{i}.\vec{j}=\vec{j}.\vec{i}=0$; $\vec{j}.\vec{k}=\vec{k}.\vec{j}=0$; $\vec{k}.\vec{i}=\vec{i}.\vec{k}=0$

* If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$; $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$.

* If $\vec{a},\vec{b},\vec{c}$ are three vectors then

i) $(\vec{a}+\vec{b})^2=(\vec{a})^2+(\vec{b})^2+2\vec{a}.\vec{b}$

ii) $(\vec{a}-\vec{b})^2=(\vec{a})^2+(\vec{b})^2-2\vec{a}.\vec{b}$

iii) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=(\vec{a})^2-(\vec{b})^2$

iv) $(\vec{a}+\vec{b})^2=(\vec{a}-\vec{b})^2=2\left\{(\vec{a})^2+(\vec{b})^2\right\}$

v) $(\vec{a}+\vec{b})^2-(\vec{a}-\vec{b})^2=4\vec{a}.\vec{b}$

vi) $(\vec{a}+\vec{b}+\vec{c})^2=(\vec{a})^2+(\vec{b})^2+(\vec{c})^2+2\vec{a}.\vec{b}+2\vec{b}.\vec{c}+2\vec{c}.\vec{a}$.

* If $\vec{r}$ is vector then $\vec{r}=(\vec{r}.\vec{i})i+(\vec{r}+\vec{j})\vec{j}+(\vec{r}.\vec{k})\vec{k}$.

Angle between the planes :- The angle between the planes is defined as the angle between the normals to the planes drawn from any point in the space.

SPHERE * The vector equation of a sphere with centre C having position vector $\vec{c}$ and radius a is $(\vec{r}-\vec{c})^2=a^2$ i.e. $\vec{r}^2-2\vec{r}.\vec{c}+c^2=a^2$

* The vector equation of a sphere with $A(\vec{a})$ and $B(\vec{b})$ as the end points of a diameter is $(\vec{r}-\vec{a}).(\vec{r}-\vec{b})=0$ (or) $(\vec{r})^2-\vec{r}.(\vec{a}+\vec{b})+\vec{a}.\vec{b}=0$

Work done by a force :- If a force $\vec{F}$ acting on a particle displaces it from a position A to the position B then work done W by this force $\vec{F}$ is $\vec{F}.\overrightarrow{AB}$

* The vector equation of the plane which is at a distance of p from the origin along the unit vector $\vec{n}$ is $\vec{r}.\vec{n}=p$.

* The vector equation of the plane passing through the origin and perpendicular to the vector m is **r.m =0**

* The Cartesian equation of the plane which is at a distance of p from the origin along the unit vector $\mathbf{n}=\mathbf{l}\mathbf{i}+\mathbf{m}\mathbf{j}+\mathbf{n}\mathbf{k}$ of the plane is $\mathbf{n}=\mathbf{l}x+\mathbf{m}y+\mathbf{n}z$

* The vector equation of the plane passing through the point a having position vector $\vec{a}$ and perpendicular to the vector $\vec{m}$ is $(\vec{r}-\vec{a}).\vec{m}=0$.

* The vector equation of the plane passing through the point a having position vector $\vec{a}$ and parallel to the plane **r.m=q** is $(\vec{r}-\vec{a}).\vec{m}=0$.

CROSS(VECTOR) PRODUCT OF VECTORS

* Let $\vec{a},\vec{b}$ be two vectors. The cross product or vector product or skew product of vectors $\vec{a},\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and is defined as follows

i) If $\vec{a}=0$ or $\vec{b}=0$ or $\vec{a},\vec{b}$ are parallel then $\vec{a}\times\vec{b}=0$

ii) If $\vec{a}\neq 0$, $\vec{b}\neq 0$, $\vec{a},\vec{b}$ are not parallel then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|(\sin\theta)\vec{n}$ where $\vec{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$ so that $\vec{a},\vec{b},\vec{n}$ form a right handed system.

Note :- i) $\vec{a}\times\vec{b}$ is a vector

ii) If $\vec{a},\vec{b}$ are not parallel then $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

iii) If $\vec{a},\vec{b}$ are not parallel then $\vec{a},\vec{b},\vec{a}\times\vec{b}$ form a right handed system .

iv) If $\vec{a},\vec{b}$ are not parallel then $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin(\vec{a},\vec{b})$ and hence $|\vec{a}\times\vec{b}|\leq|\vec{a}||\vec{b}|$

v) For any vector $\vec{a}$ $\vec{a}\times\vec{b}=\vec{o}$

2. If $\vec{a},\vec{b}$ are two vectors $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$ this is called "anti commutative law"

3. If $\vec{a},\vec{b}$ are two vectors then $\vec{a}\times(-\vec{b})=(-\vec{a})\times\vec{b}=-(\vec{a}\times\vec{b})$

4. If $\vec{a},\vec{b}$ are two vectors then $(-\vec{a})\times(-\vec{b})=\vec{a}\times\vec{b}$

5. If $\vec{a},\vec{b}$ are two vectors l,m are two scalars then $(l\vec{a})\times(m\vec{b})=lm(\vec{a}\times\vec{b})$

6. If $\vec{a},\vec{b},\vec{c}$ are three vectos, then

i) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ ii) $(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$

7. If $\vec{i},\vec{j},\vec{k}$ from a right handed system of orthonormal triad then

i) $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{o}$ ii) $\vec{i}\times\vec{j}=\vec{k}=-\vec{j}\times\vec{i}$; $\vec{j}\times\vec{k}=\vec{i}=-\vec{k}\times\vec{j}$; $\vec{k}\times\vec{i}=\vec{j}=-\vec{i}\times\vec{k}$

* If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$.

* If $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$, $\vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ where $\vec{l},\vec{m},\vec{n}$ form a right system of non coplanar vectors then $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{m}\times\vec{n} & \vec{n}\times\vec{l} & \vec{l}\times\vec{m}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

* If $\vec{a},\vec{b}$ are two vectors then $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=a^2b^2$.

*** VECTOR AREA :-**

If A is the area of the region bounded by a plane curve and $\vec{n}$ is the unit vector perpendicular to the plane of the curve such that the direction of curve drawn can be considered anti clock wise then $A\vec{n}$ is called vector area of the plane region bounded by the curve.

* The vector area of triangle ABC is $\frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC} = \frac{1}{2}\overrightarrow{BC}\times\overrightarrow{BA} = \frac{1}{2}\overrightarrow{CA}\times\overrightarrow{CB}$

* If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the vertices of a triangle then the vector area of the triangle is $\frac{1}{2}(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$

* If ABCD is a parallelogram and $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{BC}=\vec{b}$ then the vector area of ABCD is $\vec{a}\times\vec{b}$.

* If ABCD is a parallelogram and $\overrightarrow{AC}=\vec{a}$, $\overrightarrow{BC}=\vec{b}$ then vector area of parallelogram ABCD is $\frac{1}{2}(\vec{a}\times\vec{b})$

* The vector equation of a line passing through the point A with position vector $\vec{a}$ and perpendicular to the vectors $\vec{b}\times\vec{c}$ is $\vec{r}=\vec{a}+t(\vec{b}\times\vec{c})$.

* The vector equation of a line passing through the point A with position vector $\vec{a}$ and perpendicular to the vectors $\vec{b}\times\vec{c}$ is $\vec{r}=\vec{a}+t(\vec{b}\times\vec{c})$.

SCALAR TRIPLE PRODUCT

* If $\vec{a},\vec{b},\vec{c}$ are the three vectors, then the real numbers $(\vec{a}\times\vec{b}).\vec{c}$ is called scalar triple product denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$. This is read as 'box' $\vec{a}, \vec{b}, \vec{c}$

2. If V is the volume of the parallelepiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ then $V=\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

3. If $\vec{a},\vec{b}, \vec{c}$ form the right handed system of vectors then $V=[\vec{a}\ \vec{b}\ \vec{c}]$

4. If $\vec{a}, \vec{b}, \vec{c}$ form left handed system of vectors then $-V=[\vec{a}, \vec{b}, \vec{c}]$

Note: i) The scalar triple product is independent of the position of dot and cross.

i.e. $\vec{a}\times\vec{b}.\vec{c}=\vec{a}.\vec{b}\times\vec{c}$

ii) The value of the scalar triple product is unaltered so long as the cyclic order remains unchanged

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$$

iii) The value of a scalar triple product is zero if two of its vectors are equal

$$[\vec{a}\,\vec{a}\,\vec{b}] = 0 \; [\vec{b}\,\vec{b}\,\vec{c}] = 0$$

iv) If $\vec{a},\vec{b},\vec{c}$ are coplanar then $[\vec{a}\vec{b}\vec{c}] = 0$

v) If $\vec{a},\vec{b},\vec{c}$ form right handed system then $[\vec{a}\,\vec{b}\,\vec{c}] > 0$

vi) If $\vec{a},\vec{b},\vec{c}$ form left handed system then $[\vec{a}\;\vec{b}\;\vec{c}] < 0$

vii)The value of the triple product changes its sign when two vectors are interchanged

$$[\vec{a}\;\vec{b}\;\vec{c}] = -[\vec{a}\;\vec{c}\;\vec{b}]$$

viii) If l,m, n are three scalars $\vec{a},\ \vec{b},\ \vec{c}$ are three vectors then $[l\vec{a}\, m\vec{b}\ n\vec{c}] = lmn[\vec{a}\;\vec{b}\;\vec{c}]$

* Three non zero non parallel vectors $\vec{a}\;\vec{b}\vec{c}$ nare coplanar iff $[\vec{a}\;\vec{b}\vec{c}] = 0$

* If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

* If $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$, $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$, $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$ where $\vec{l},\vec{m},\vec{n}$ form a right handed system of non coplanar vectors, then $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} \vec{m}\times\vec{n} & \vec{n}\times\vec{l} & \vec{l}\times\vec{m} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

* The vectors equation of plane passing through the points A, B with position vectors $\vec{a},\vec{b}$ and parallel to the vector $\vec{c}$ is $[\vec{r}-\vec{a}\;\vec{b}-\vec{a}\;\vec{c}] = 0$ (or) $[\vec{r}\;\vec{b}\;\vec{c} + [\vec{r}\,\vec{c}\;\vec{a}] = [\vec{a}\;\vec{b}\;\vec{c}]$

* The vector equation of the plane passing through the point A with position vector $\vec{a}$ and parallel to $\vec{b},\vec{c}$ is $[\vec{r}-\vec{a}\;\vec{b}\;\;\vec{c}] = 0$ i.e. $[\vec{r}\;\vec{b}\;\;\vec{c}] = [\vec{a}\;\vec{b}\;\vec{c}]$

Skew lines :- Two lines are said to be skew lines if there exist no plane passing through them i.e. the lines lie on two difference planes

Def:- l_1 and l_2 are two skew lines. If P is a point on l_1 and Q is a point on l_2 such that $\overline{PQ} \perp^r l_1$ and $\overline{PQ} \perp^r l_2$ then PQ is called shortest distance and $\overline{PQ}$ is called shortest distance line between the lines l_1 and l_2.

The shortest distance between the skew lines $\vec{r}=\vec{a}+t\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is $\dfrac{|[\vec{a}-\vec{c}\ \vec{b}\ \vec{d}]|}{|\vec{b}\times\vec{d}|}$

VECTOR TRIPLE PRODUCT

Cross Product of Three vectors : For any three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ then cross product or vector product of these vectors are given as $\bar{a}\times(\bar{b}\times\bar{c})$, $(\bar{a}\times\bar{b})\times\bar{c}$ or $(\bar{b}\times\bar{c})\times\bar{a}$ etc.

i. $\bar{a}\times(\bar{b}\times\bar{c})$ is vector quantity and $|\bar{a}\times(\bar{b}\times\bar{c})| = |(\bar{b}\times\bar{c})\times\bar{a}|$

ii. In general $\bar{a}\times(\bar{b}\times\bar{c}) \neq (\bar{a}\times\bar{b})\times.\bar{c}$

iii. $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\times\bar{c}$ if $\bar{a}$ and $\bar{c}$ are collinear

iv. $\bar{a}\times(\bar{b}\times\bar{c}) = -(\bar{b}\times\bar{c})\times\bar{a}$

v. $(\bar{a}\times\bar{b})\times\bar{c} = -\bar{c}\times(\bar{a}\times\bar{b}) =$

$(\bar{a}\ .\ \bar{c})\bar{b} - (\bar{a}\ .\ \bar{b})\bar{c} = \bar{a}\times(\bar{b}\times\bar{c})$

vi. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non zero vectors and $\bar{a}\times(\bar{b}\times\bar{c}) = \bar{O}$ then $\bar{b}$ and $\bar{c}$ are parallel (or collinear) vectors.

vii. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non zero and non parallel vectors then $\bar{a}\times(\bar{b}\times\bar{c})$, $\bar{b}\times(\bar{c}\times\bar{a})$ and $\bar{c}\times(\bar{a}\times\bar{b})$ are non collinear vectors.

viii. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are any three vectors then $\bar{a}(\bar{b}\times\bar{c}) + \bar{b}\times(\bar{c}\times\bar{a}) + \bar{c}\times(\bar{a}\times\bar{b}) = \bar{O}$

ix. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are any three vectors then $\bar{a}(\bar{b}\times\bar{c}) + \bar{b}\times(\bar{c}\times\bar{a}) + \bar{c}\times(\bar{a}\times\bar{b})$ are coplanar. [since sum of these vectors is zero]

x. $\bar{a}(\bar{b}\times\bar{c})$ is vector lies in the plane of $\bar{b}$ and $\bar{c}$ or parallel to the plane of $\bar{b}$ and $\bar{c}$.

PRODUCT OF FOUR VECTORS

* **Dot product of four vectors** : The dot product of four vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ is given as $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d})(\bar{b}.\bar{c}) = \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix}$

* **Cross product of four vectors** : If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are any four vectors then $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) =$
$[\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$
$= [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

* $[\bar{a}\ \bar{b}\ \bar{c}]\,[\bar{l}\ \bar{m}\ \bar{n}] = \begin{vmatrix} \bar{a}.\bar{l} & \bar{b}.\bar{l} & \bar{c}.\bar{l} \\ \bar{a}.\bar{m} & \bar{b}.\bar{m} & \bar{c}.\bar{m} \\ \bar{a}.\bar{n} & \bar{b}.\bar{n} & \bar{c}.\bar{n} \end{vmatrix}$

* The vectorial equation of the plane passing through the point $\bar{a}$ and parallel to the vectors $\bar{b}$, $\bar{c}$ is $[\bar{r}\bar{b}\bar{c}] = [\bar{a}\bar{b}\bar{c}]$.

* The vectorial equation of the plane passing through the points $\bar{a}, \bar{b}$ and parallel to the vector $\bar{c}$ is $[\bar{r}\bar{b}\bar{c}] + [\bar{r}\bar{c}\bar{a}] = [\bar{a}\bar{b}\bar{c}]$.

* The vectorial equation of the plane passing through the points $\bar{a}, \bar{b}, \bar{c}$ is $[\bar{r}\bar{b}\bar{c}] + [\bar{r}\bar{c}\bar{a}] + [\bar{r}\bar{a}\bar{b}] = [\bar{a}\bar{b}\bar{c}]$.

* If the points with the position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar, then the condition is $[\bar{a}\bar{b}\bar{d}] + [\bar{b}\bar{c}\bar{d}] + [\bar{c}\bar{a}\bar{d}] = [\bar{a}\bar{b}\bar{c}]$

* Length of the perpendicular from the origin to the plane passing through the points $\bar{a}, \bar{b}, \bar{c}$ is $\dfrac{|[\bar{a}\bar{b}\bar{c}]|}{|\bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{a}\times\bar{b}|}$.

* Length of the perpendicular from the point $\bar{c}$ on to the line joining the points $\bar{a}, \bar{b}$ is $\dfrac{|(\bar{a}-\bar{c})\times(\bar{c}-\bar{b})|}{|\bar{a}-\bar{b}|}$.

* P, Q, R are non collinear points. Then distance of P to the plane OQR is $\left|\dfrac{\overline{OP}.(\overline{OQ}\times\overline{OR})}{|\overline{OQ}\times\overline{OR}|}\right|$

* Perpendicular distance from $P(\bar{\alpha})$ to the plane passing through $A(\bar{a})$ and parallel to the vectors $\bar{b}$ and $\bar{c}$ is $\left|\dfrac{[\bar{\alpha}-\bar{a}\ \ \bar{b}\ \ \bar{c}]}{|\bar{b}\times\bar{c}|}\right|$

* Length of the perpendicular from the point $\bar{c}$ to the line $\bar{r}=\bar{a}+t\bar{b}$ is $\frac{|(\bar{c}-\bar{a})\times\bar{b}|}{|\bar{b}|}$.

PROBLEMS

VSAQ'S

1. Find the angle between the vectors $\bar{i}+2\bar{j}+3\bar{k}$ and $3\bar{i}-\bar{j}+2\bar{k}$**.**

Sol. Let $\bar{a}=\bar{i}+2\bar{j}+3\bar{k}$ and $\bar{b}=3\bar{i}-\bar{j}+2\bar{k}$

Let θ be the angle between the vectors.

Then $\cos\theta=\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

$$\cos\theta=\frac{(\bar{i}+2\bar{j}+3\bar{k})\cdot(3\bar{i}-\bar{j}+2\bar{k})}{\sqrt{\bar{i}+2\bar{j}+3\bar{k}}\sqrt{3\bar{i}-\bar{j}+2\bar{k}}}$$

$$=\frac{3-2+6}{\sqrt{14}\sqrt{14}}=\frac{7}{14}=\frac{1}{2}$$

$$\cos\theta=\frac{1}{2}\Rightarrow\cos\theta=\cos 60°$$

$$\therefore\theta=60°$$

2. **If the vectors** $2\bar{i}+\lambda\bar{j}-\bar{k}$ and $4\bar{i}-2\bar{j}+2\bar{k}$ **are perpendicular to each other, then find** λ.

Sol. Let $\bar{a}=2\bar{i}+\lambda\bar{j}-\bar{k}$ and $\bar{b}=4\bar{i}-2\bar{j}+2\bar{k}$

By hypothesis, $\bar{a},\bar{b}$ are perpendicular then $\bar{a}\cdot\bar{b}=0$

$$\Rightarrow(2\bar{i}+\lambda\bar{j}-\bar{k})\cdot(4\bar{i}-2\bar{j}+2\bar{k})=0$$

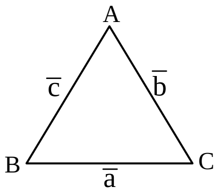
$$\Rightarrow 8-2\lambda-2=0$$

$$\Rightarrow 6-2\lambda=0$$

$$\Rightarrow\lambda=3$$

3. $\bar{a} = 2\bar{i} - \bar{j} + \bar{k}, \bar{b} = \bar{i} - 3\bar{j} - 5\bar{k}$. **Find the vector c such that** $\bar{a}, \bar{b}$ **and** $\bar{c}$ **form the sides of triangle.**

Sol.



We know that $\overline{AB} + \overline{BC} + \overline{CA} = 0$

$\bar{c} + \bar{a} + \bar{b} = 0$

$\bar{c} = -\bar{a} - \bar{b}$

$\bar{c} = -2\bar{i} + \bar{j} - \bar{k} - \bar{i} + 3\bar{j} + 5\bar{k}$

$\bar{c} = -3\bar{i} + 4\bar{j} + 4\bar{k}$

4. Find the angle between the planes $\bar{r} \cdot (2\bar{i} - \bar{j} + 2\bar{k}) = 3$ **and** $\bar{r} \cdot (3\bar{i} + 6\bar{j} + \bar{k}) = 4$.

Sol. Given $\bar{r} \cdot (2\bar{i} - \bar{j} + 2\bar{k}) = 3$

$\bar{r} \cdot (3\bar{i} + 6\bar{j} + \bar{k}) = 4$

Given equation $\bar{r} \cdot \bar{n}_1 = p,\ \bar{r} \cdot \bar{n}_2 = q$

Let θ be the angle between the planes.

Then $\cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

$$= \frac{(2\bar{i} - \bar{j} + 2\bar{k}) \cdot (3\bar{i} + 6\bar{j} + \bar{k})}{\sqrt{2\bar{i} - \bar{j} + 2\bar{k}}\sqrt{3\bar{i} + 6\bar{j} + \bar{k}}}$$

$$= \frac{6 - 6 + 2}{\sqrt{9}\sqrt{46}} = \frac{2}{3\sqrt{46}}$$

$$\cos\theta = \frac{2}{3\sqrt{46}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{3\sqrt{46}}\right)$$

5. Let $\bar{e}_1$ and $\bar{e}_2$ be unit vectors containing angle θ. If $\frac{1}{2}|\bar{e}_1 - \bar{e}_2| = \sin\lambda\theta$, then find λ.

Sol. $\frac{1}{2}|\bar{e}_1 - \bar{e}_2| = \sin\lambda\theta$

Squaring on both sides

$$\Rightarrow \frac{1}{4}(\bar{e}_1 - \bar{e}_2)^2 = \sin^2\lambda\theta$$

$$\Rightarrow \frac{1}{4}\left[(\bar{e}_1)^2 + (\bar{e}_2)^2 - 2\bar{e}_1\bar{e}_2\right] = \sin^2\lambda\theta$$

$$\Rightarrow \frac{1}{4}\left[\bar{e}_1^2 + \bar{e}_2^2 - 2|\bar{e}_1||\bar{e}_2|\cos\theta\right] = \sin^2\lambda\theta$$

$$\Rightarrow \frac{1}{4}[1 + 1 - 2\cos\theta] = \sin^2\lambda\theta$$

$$\Rightarrow \frac{1}{4}[2 - 2\cos\theta] = \sin^2\lambda\theta$$

$$\Rightarrow \frac{2}{4}[1 - \cos\theta] = \sin^2\lambda\theta$$

$$\Rightarrow \frac{1}{2}[1 - \cos\theta] = \sin^2\lambda\theta$$

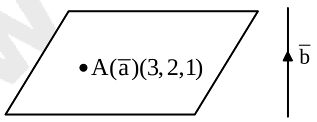
$$\Rightarrow \frac{1}{2}\left[2\sin^2\frac{\theta}{2}\right] = \sin^2\lambda\theta$$

$$\Rightarrow \sin^2\frac{\theta}{2} = \sin^2\lambda\theta$$

$$\Rightarrow \frac{\theta}{2} = \lambda\theta \Rightarrow \lambda = \frac{1}{2}$$

6. Find the equation of the plane through the point (3, –2, 1) and perpendicular to the vector (4, 7, –4).

Sol. Let $\bar{a} = 3\bar{i} - 2\bar{j} + \bar{k}$, $\bar{b} = 4\bar{i} - 7\bar{j} - 4\bar{k}$



Equation of the required plane will be in the form $\bar{r}\cdot\bar{b} = \bar{a}\cdot\bar{b}$

$\bar{r}\cdot(4\bar{i}+7\bar{j}-4\bar{k})=$

$(3\bar{i}-2\bar{j}+\bar{k})\cdot(4\bar{i}+7\bar{j}-4\bar{k})$

$\Rightarrow \bar{r}\cdot(4\bar{i}+7\bar{j}-4\bar{k})=12-14-4$

$\Rightarrow \bar{r}\cdot(4\bar{i}+7\bar{j}-4\bar{k})=-6$

7. If $|\bar{p}|=2, |\bar{q}|=3$ and $(\bar{p},\bar{q})=\frac{\pi}{6}$**, then find** $|\bar{p}\times\bar{q}|^2$**.**

Sol. Given $|\bar{p}|=2, |\bar{q}|=3$ and $(\bar{p},\bar{q})=\frac{\pi}{6}$

$|\bar{p}\times\bar{q}|^2=[|\bar{p}||\bar{q}|\sin(\bar{p},\bar{q})]^2$

$=\left[2\cdot 3\sin\frac{\pi}{6}\right]^2=\left[2\cdot 3\cdot\frac{1}{2}\right]^2$

$|\bar{p}\times\bar{q}|^2=[3]^2=9$

$\Rightarrow |\bar{p}\times\bar{q}|^2=9$

8. If $\bar{a}=2\bar{i}-3\bar{j}+\bar{k}$ and $\bar{b}=\bar{i}+4\bar{j}-2\bar{k}$**, then find** $(\bar{a}+\bar{b})\times(\bar{a}-\bar{b})$**.**

Sol. $\bar{a}+\bar{b}=3\bar{i}+\bar{j}-\bar{k},\ \bar{a}-\bar{b}=\bar{i}-7\bar{j}+3\bar{k}$

$$(\bar{a}+\bar{b})\times(\bar{a}-\bar{b})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 1 & -1\\ 1 & -7 & 3\end{vmatrix}$$

$=\bar{i}(3-7)-\bar{j}(9+1)+\bar{k}(-21-1)$

$(\bar{a}+\bar{b})\times(\bar{a}-\bar{b})=-4\bar{i}-10\bar{j}-22\bar{k}$

9. If $4\bar{i}+\frac{2p}{3}\bar{j}+p\bar{k}$ **is parallel to the vector** $\bar{i}+2\bar{j}+3\bar{k}$**, find p.**

Sol. Let $\bar{a}=4\bar{i}+\frac{2p}{3}\bar{j}+p\bar{k},\ \bar{b}=\bar{i}+2\bar{j}+3\bar{k}$

From hyp. $\bar{a}$ is parallel to $\bar{b}$ then $\bar{a}=\lambda\bar{b}$, λ is a scalar.

$\Rightarrow 4\bar{i}+\frac{2p}{3}\bar{j}+p\bar{k}=\lambda[\bar{i}+2\bar{j}+3\bar{k}]$

Comparing $\bar{i},\bar{j},\bar{k}$ on both sides

$4=\lambda \Rightarrow \lambda=4$

$\frac{2p}{3} = 2\lambda \Rightarrow p = 3\lambda \Rightarrow p = 12$

10. Compute $\bar{a}\times(\bar{b}+\bar{c})+\bar{b}\times(\bar{c}+\bar{a})+\bar{c}\times(\bar{a}+\bar{b})$.

Sol. $\bar{a}\times(\bar{b}+\bar{c})+\bar{b}\times(\bar{c}+\bar{a})+\bar{c}\times(\bar{a}+\bar{b})$

$= \bar{a}\times\bar{b}+\bar{a}\times\bar{c})+\bar{b}\times\bar{c}+\bar{b}\times\bar{a})+\bar{c}\times\bar{a}+\bar{c}\times\bar{b}$

$= \bar{a}\times\bar{b}-\bar{c}\times\bar{a}-\bar{c}\times\bar{b}-\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+\bar{c}\times\bar{b}=0$

11. Compute $2\bar{j}\times(3\bar{i}-4\bar{k})+(\bar{i}+2\bar{j})\times\bar{k}$.

Sol. $2\bar{j}\times(3\bar{i}-4\bar{k})+(\bar{i}+2\bar{j})\times\bar{k}$

$= 6(\bar{j}\times\bar{i})-8(\bar{j}\times\bar{k})+(\bar{i}\times\bar{k})+2(\bar{j}\times\bar{k})$

$= -6\bar{k}-8\bar{i}-\bar{j}+2\bar{i}$

$= -6\bar{i}-\bar{j}-6\bar{k}$

12. Find unit vector perpendicular to both $\bar{i}+\bar{j}+\bar{k}$ and $2\bar{i}+\bar{j}+3\bar{k}$.

Sol. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{b}=2\bar{i}+\bar{j}+3\bar{k}$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 1 & 1\\ 2 & 1 & 3\end{vmatrix}$$

$= \bar{i}(3-1)-\bar{j}(3-2)+\bar{k}(1-2)$

$= 2\bar{i}-\bar{j}-\bar{k}$

$|\bar{a}\times\bar{b}|=\sqrt{6}$

Unit vector perpendicular to

$\bar{a}$ and $\bar{b} = \pm\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \pm\frac{2\bar{i}-\bar{j}-\bar{k}}{\sqrt{6}}$

13. If θ is the angle between the vectors $\bar{i}+\bar{j}$ and $\bar{j}+\bar{k}$, then find sin θ.

Sol. Let $\bar{a}=\bar{i}+\bar{j}$ and $\bar{b}=\bar{j}+\bar{k}$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}$$

$$=\bar{i}(1-0)-\bar{j}(1-0)+\bar{k}(1-0)$$

$$\bar{a}\times\bar{b}=\bar{i}-\bar{j}+\bar{k}$$

$$|\bar{a}\times\bar{b}|=\sqrt{3}, |\bar{a}|=\sqrt{2}, |\bar{b}|=\sqrt{2}$$

$$\sin\theta=\frac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}=\frac{\sqrt{3}}{\sqrt{2}\sqrt{2}}$$

$$\Rightarrow \sin\theta=\frac{\sqrt{3}}{2}$$

14. Find the area of the parallelogram having $\bar{a}=2\bar{j}-\bar{k}$ and $\bar{b}=-\bar{i}+\bar{k}$ as adjacent sides.

Sol. Given $\bar{a}=2\bar{j}-\bar{k}$ and $\bar{b}=-\bar{i}+\bar{k}$

$\therefore$ Area of parallelogram = $|\bar{a}\times\bar{b}|$

$$=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 0 & 2 & -1\\ -1 & 0 & 1\end{vmatrix}=|2\bar{i}-\bar{j}+2\bar{k}|=\sqrt{9}=3$$

15. Find the area of the parallelogram whose diagonals are $3\bar{i}+\bar{j}-2\bar{k}$ and $\bar{i}-3\bar{j}+4\bar{k}$.

Sol. Given $\overline{AC}=3\bar{i}+\bar{j}-2\bar{k}, \overline{BD}=\bar{i}-3\bar{j}+4\bar{k}$

Area of parallelogram = $\frac{1}{2}|\overline{AC}\times\overline{BD}|$

$$=\frac{1}{2}\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}$$

$$=\left|\frac{1}{2}\left[\bar{i}(4-6)-\bar{j}(12+2)+\bar{k}(-9-1)\right]\right|$$

$$=\left|\frac{1}{2}[-2\bar{i}-14\bar{j}-10\bar{k}]\right|$$

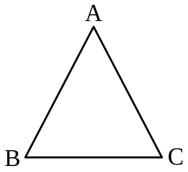
$=|-\overline{i}-7\overline{j}-5\overline{k}|$

$=\sqrt{1+49+25}=\sqrt{75}$

$\therefore$ Area of parallelogram = $5\sqrt{3}$ sq.units.

16. Find the area of the triangle having $3\overline{i}+4\overline{j}$ and $-5\overline{i}+7\overline{j}$ as two of its sides.

Sol.



Given $\overline{AB}=3\overline{i}+4\overline{j}, \overline{BC}-5\overline{i}+7\overline{j}$

We know that,

$\overline{AB}+\overline{BC}+\overline{CA}=0$

$\overline{CA}=-\overline{AB}-\overline{BC}=-3\overline{i}-4\overline{j}+5\overline{i}-7\overline{j}$

$\overline{CA}=2\overline{i}-11\overline{j}$

$\therefore$ Area of ΔABC = $\frac{1}{2}|\overline{AB}\times\overline{AC}|$

$$=\frac{1}{2}\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 3 & 4 & 0\\ 2 & -11 & 0\end{vmatrix}=\frac{1}{2}\left[\overline{k}(-33-8)\right]$$

$$=\left|\frac{-41\overline{k}}{2}\right|=\frac{41}{2}$$

17. Find unit vector perpendicular to the plane determined by the vectors $\overline{a}=4\overline{i}+3\overline{j}-\overline{k}$ and $\overline{b}=2\overline{i}-6\overline{j}-3\overline{k}$.

Sol. Given $\overline{a}=4\overline{i}+3\overline{j}-\overline{k}$, $\overline{b}=2\overline{i}-6\overline{j}-3\overline{k}$

$$\overline{a}\times\overline{b}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 4 & 3 & -1\\ 2 & -6 & -3\end{vmatrix}$$

$=\overline{i}(-9-6)-\overline{j}(-12+2)+\overline{k}(-24-6)$

$=-15\overline{i}+10\overline{j}-30\overline{k}=5(-3\overline{i}+2\overline{j}-6\overline{k})$

$|\bar{a}\times\bar{b}| = 5\sqrt{9+4+36} = 5\times 7 = 35$

$\therefore$ Unit vector perpendicular to both

$\bar{a}$ and $\bar{b} = \pm\dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \pm\dfrac{-15\bar{i}+10\bar{j}-30\bar{k}}{35}$

18. If $|\bar{a}|=13, |\bar{b}|=5$ and $\bar{a}\cdot\bar{b}=60$, then find $|\bar{a}\times\bar{b}|$.

Sol. Given $|\bar{a}|=13, |\bar{b}|=5$ and $\bar{a}\cdot\bar{b}=60$

We know that

$$|\bar{a}\times\bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - (\bar{a}\cdot\bar{b})^2$$
$$= 169\cdot 25 - 3600$$
$$= 25(169-144) = 625$$
$$|\bar{a}\times\bar{b}|^2 = 625$$
$$\therefore |\bar{a}\times\bar{b}| = 25$$

19. If $\bar{a}=\bar{i}-2\bar{j}-3\bar{k}, \bar{b}-2\bar{i}+\bar{j}-\bar{k}$, $\bar{c}=\bar{i}+3\bar{j}-2\bar{k}$ then compute $\bar{a}\cdot(\bar{b}\times\bar{c})$.

Sol. Given $\bar{a}=\bar{i}-2\bar{j}-3\bar{k}, \bar{b}-2\bar{i}+\bar{j}-\bar{k}$, $\bar{c}=\bar{i}+3\bar{j}-2\bar{k}$

$$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} = \bar{i}+3\bar{j}+5\bar{k}$$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{i}-2\bar{j}-3\bar{k})\cdot(\bar{i}+3\bar{j}+5\bar{k})$$
$$= 1-6-15 = -20$$
$$\therefore \bar{a}\cdot(\bar{b}\times\bar{c}) = -20$$

20. Simplify $(\bar{i}-2\bar{j}+3\bar{k})\times(2\bar{i}+\bar{j}-\bar{k})\cdot(\bar{j}+\bar{k})$.

Sol. $(\bar{i}-2\bar{j}+3\bar{k})\times(2\bar{i}+\bar{j}-\bar{k})\cdot(\bar{j}+\bar{k})$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$
$$= 1(1+1)+2(2-0)+3(2-0)$$
$$= 2+4+6 = 12$$

21. Find the volume of parallelepiped having co-terminous edges $\bar{i}+\bar{j}+\bar{k}$, $\bar{i}-\bar{j}$ and $\bar{i}+2\bar{j}-\bar{k}$.

Sol. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k},\ \bar{b}=\bar{i}-\bar{j}, \bar{c}=\bar{i}+2\bar{j}-\bar{k}$

Volume of parallelepiped = $\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]$

$$=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 2 & -1\end{vmatrix}$$

$$=1(1-0)-1(-1-0)+1(2+1)$$

$$=1+1+3=5\text{Cubic units}$$

22. Compute $\left[\bar{i}-\bar{j}\quad\bar{j}-\bar{k}\quad\bar{k}-\bar{i}\right]$.

Sol. $\left[\bar{i}-\bar{j}\quad\bar{j}-\bar{k}\quad\bar{k}-\bar{i}\right]=\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{vmatrix}$

$$=1(1-0)+1(0-1)+0(0+1)$$

$$=1-1=0$$

23. For non-coplanar vectors $\bar{a},\bar{b}$ and $\bar{c}$ determine the value of p in order that $\bar{a}+\bar{b}+\bar{c}$, $\bar{a}+p\bar{b}+2\bar{c}$ and $-\bar{a}+\bar{b}+\bar{c}$ are coplanar.

Sol. Let

$$\bar{A}=\bar{a}+\bar{b}+\bar{c}, \bar{B}=\bar{a}+p\bar{b}+2\bar{c}, \bar{C}=-\bar{a}+\bar{b}+\bar{c}$$

From hyp. Given vectors are coplanar.

Then $\begin{vmatrix}1 & 1 & 1\\ 1 & p & 2\\ -1 & 1 & 1\end{vmatrix}\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]=0$

$$\Rightarrow [1(p-2)-1(1+2)+1(1+p)][\bar{a}\ \bar{b}\ \bar{c}]=0$$

$$\Rightarrow [p-2-3+1+p][\bar{a}\ \bar{b}\ \bar{c}]=0$$

$$[\because [\bar{a}\ \bar{b}\ \bar{c}]\neq 0]$$

$$\Rightarrow 2p-4=0$$

$[\because \bar{a},\bar{b},\bar{c}$ are non-coplanar vectors$]$

$$\Rightarrow 2p=4$$

$$\therefore p=2$$

24. Find the volume of tetrahedron having the edges $\bar{i}+\bar{j}+\bar{k}$, $\bar{i}-\bar{j}$ and $\bar{i}+2\bar{j}+\bar{k}$.

Sol. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k},\ \bar{b}=\bar{i}-\bar{j}, \bar{c}=\bar{i}+2\bar{j}+\bar{k}$

$\therefore$ Volume of the tetrahedraon

$$=\frac{1}{6}\left[\bar{a}\quad \bar{b}\quad \bar{c}\right]$$

$$=\frac{1}{6}\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 2 & 1\end{vmatrix}$$

$$=\frac{1}{6}\left[1(-1-0)-1(1-0)+1(2+1)\right]$$

$$=\frac{1}{6}[-1-1+3]$$

$$=\frac{1}{6}[1]=\frac{1}{6}\text{ cubic units}$$

25. Let $\bar{a},\bar{b}$ and $\bar{c}$ be non-coplanar vectors and $\alpha=\bar{a}+2\bar{b}+3\bar{c}, \beta=2\bar{a}+\bar{b}-2\bar{c}$ and $\gamma=3\bar{a}-7\bar{c}$ then find $[\alpha\quad \beta\quad \gamma]$.

Sol. $$[\alpha\quad \beta\quad \gamma]=\begin{vmatrix}1 & 2 & 3\\ 2 & 1 & -2\\ 3 & 0 & -7\end{vmatrix}\left[\bar{a}\quad \bar{b}\quad \bar{c}\right]$$

$$=\left[1(-7-0)-2(-14+6)+3(0-3)\right]\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$=[-7+16-9]\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$=0\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$$

26. Prove that $\bar{a}\times\left[\bar{a}\times(\bar{a}\times\bar{b})\right]=(\bar{a}\cdot\bar{a})(\bar{b}\times\bar{a})$.

Sol. $$\bar{a}\times\left[\bar{a}\times(\bar{a}\times\bar{b})\right]=\bar{a}\times[(\bar{a}\cdot\bar{b})\bar{a}-(\bar{a}\cdot\bar{a})\bar{b}]$$

$$=(\bar{a}\cdot\bar{b})\bar{a}\times\bar{a}-(\bar{a}\cdot\bar{a})\bar{a}\times\bar{b}\ (\because \bar{b}\times\bar{a}=-\bar{a}\times\bar{b})$$

$$=(\bar{a}\cdot\bar{b})(0)+(\bar{a}\cdot\bar{a})(\bar{b}\times\bar{a})$$

$$\bar{a}\times\left[\bar{a}\times(\bar{a}\times\bar{b})\right]=(\bar{a}\cdot\bar{a})(\bar{b}\times\bar{a})$$

27. If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ **are coplanar vectors then show that** $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = 0$.

Sol. If $\bar{a}, \bar{b}, \bar{c}$ are coplanar $\Leftrightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [(\bar{a}\times\bar{b})\cdot\bar{d}]\cdot\bar{c} - [(\bar{a}\times\bar{b})\cdot\bar{c}]\bar{d}$$

$$= [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$$

$$= \bar{0}\cdot\bar{c} - \bar{0}\cdot\bar{d} \quad [\because \text{ a, b, c, d are coplanar}]$$

$$\therefore (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = \bar{0}$$

28. Show that $[(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})]\cdot\bar{d} = (\bar{a}\cdot\bar{d})[\bar{a}\ \bar{b}\ \bar{c}]$.

Sol. $[(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})]\cdot\bar{d}$

$$= [(\bar{a}\times\bar{b})\cdot\bar{c}]\bar{a} - [(\bar{a}\times\bar{b})\cdot\bar{a}\ \bar{c}]\cdot\bar{d}$$

$$= \left[[\bar{a}\ \bar{b}\ \bar{c}]\bar{a} - [\bar{a}\ \bar{b}\ \bar{a}]\bar{c}\right]\cdot\bar{d}$$

$$= \left[[\bar{a}\ \bar{b}\ \bar{c}]\bar{a} - 0\cdot\bar{c}\right]\cdot\bar{d}$$

$$= [\bar{a}\ \bar{b}\ \bar{c}]\bar{a}\cdot\bar{d} \quad (\because \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a})$$

$$\therefore [(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})]\cdot\bar{d} = (\bar{a}\cdot\bar{d})[\bar{a}\ \bar{b}\ \bar{c}]$$

29. Show that $\bar{a}\cdot\left[(\bar{b}+\bar{c})\times[\bar{a}+\bar{b}+\bar{c}]\right] = 0$.

Sol. $\text{L.H.S.} = \bar{a}\cdot\left[(\bar{b}+\bar{c})\times[\bar{a}+\bar{b}+\bar{c}]\right]$

$$= \bar{a}\cdot[\bar{b}\times\bar{a} + \bar{b}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{c}\times\bar{b} + \bar{c}\times\bar{c}]$$

$$= \bar{a}\cdot[\bar{b}\times\bar{a} + 0 - \bar{c}\times\bar{b} + \bar{c}\times\bar{a} + \bar{c}\times\bar{b} + 0]$$

$$= \bar{a}\cdot[\bar{b}\times\bar{a} + \bar{c}\times\bar{a}]$$

$$= \bar{a}\cdot(\bar{b}\times\bar{a}) + \bar{a}\cdot(\bar{c}\times\bar{a})$$

$$= [\bar{a}\ \bar{b}\ \bar{a}] + [\bar{a}\ \bar{c}\ \bar{a}]$$

$$= 0 + 0 = 0 = \text{R.H.S.}$$

30. If $\bar{a}, \bar{b}$ and $\bar{c}$ **are unit vectors then find** $\left[2\bar{a}-\bar{b}\quad 2\bar{b}-\bar{c}\quad 2\bar{c}-\bar{a}\right]$.

Sol. $\left[2\bar{a}-\bar{b}\quad 2\bar{b}-\bar{c}\quad 2\bar{c}-\bar{a}\right]$

$$= \begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix}\left[\bar{a}\quad \bar{b}\quad \bar{c}\right]$$

$$= [2(4-0) + 1(0-1) + 0(0-2)]\left[\bar{a}\quad \bar{b}\quad \bar{c}\right]$$

$$= [2\times 4 - 1](0)$$

$$= [8-1](0)$$

$$= [7](0) = 0$$

31. Show that $(\bar{a}+\bar{b})\cdot(\bar{b}+\bar{c})\times(\bar{c}+\bar{a})=2\left[\bar{a}\ \bar{b}\ \bar{c}\right]$**.**

Sol. We know that $\bar{a}\cdot(\bar{b}\times\bar{c})=(\bar{a}\ \ \bar{b}\ \ \bar{c})$

$$(\bar{a}+\bar{b})\cdot(\bar{b}+\bar{c})\times(\bar{c}+\bar{a})=\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{vmatrix}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

$$=\left[1(1-0)-1(0-1)+0(0-1)\right]\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]=(1+1)\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

$$=2\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

32. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\bar{r}\cdot(\bar{i}+\bar{j}+\bar{k})=2$**.**

Sol. Cartesian form of the given plane is

$x + y + z = 2$

Equation of the required plane will be in the form $x + y + z = k$

Since it is passing through (a, b, c)

$a + b + c = k$

Required plane is

$x + y + z = a + b + c$

Its vector form is : $\bar{r}\cdot(\bar{i}+\bar{j}+\bar{k})=a+b+c$.

33. Let $\bar{a}$ and $\bar{b}$ **be non-zero, non collinear vectors. If** $|\bar{a}+\bar{b}|=|\bar{a}-\bar{b}|$**, then find the angle between** $\bar{a}$ and $\bar{b}$**.**

Sol. $|\bar{a}+\bar{b}|=|\bar{a}-\bar{b}|$

$$\Rightarrow|\bar{a}+\bar{b}|^2=|\bar{a}-\bar{b}|^2$$

$$\Rightarrow(\bar{a}+\bar{b})(\bar{a}+\bar{b})=(\bar{a}-\bar{b})(\bar{a}-\bar{b})$$

$$\Rightarrow\bar{a}^2+2\bar{a}\bar{b}+\bar{b}^2=\bar{a}^2-2\bar{a}\bar{b}+\bar{b}^2$$

$$\Rightarrow 4\bar{a}\bar{b}=0\Rightarrow\bar{a}\cdot\bar{b}=0$$

Angle between $\bar{a}$ and $\bar{b}$ is 90°.

34. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be unit vectors such that $\bar{b}$ is not parallel to $\bar{c}$ and $\bar{a}\times(\bar{b}\times\bar{c}) = \frac{1}{2}\bar{b}$. Find the angles made y $\bar{a}$ with each of $\bar{b}$ and $\bar{c}$.

Sol. $\frac{1}{2}\bar{b} = \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

Such $\bar{b}$ and $\bar{c}$ are non-coplanar vectors, equating corresponding coefficients on both sides, $\bar{a}\cdot\bar{c} = \frac{1}{2}$ and $\bar{a}\cdot\bar{b} = 0$.

$\therefore$ $\bar{a}$ makes angle π/3 with $\bar{c}$ and is perpendicular to $\bar{b}$.

35. For any four vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$, prove that $(\bar{b}\times\bar{c})\cdot(\bar{a}\times\bar{d}) + (\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{d}) + (\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = 0$.

Sol. L.H.S. =

$$= \begin{vmatrix} \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{d} \\ \bar{c}\cdot\bar{a} & \bar{c}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{c}\cdot\bar{b} & \bar{c}\cdot\bar{d} \\ \bar{a}\cdot\bar{b} & \bar{a}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d} \\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

$$= (\bar{b}\cdot\bar{a})(\bar{c}\cdot\bar{d}) - (\bar{b}\cdot\bar{d})(\bar{c}\cdot\bar{a}) + (\bar{c}\cdot\bar{b})(\bar{a}\cdot\bar{d}) - (\bar{a}\cdot\bar{b})(\bar{c}\cdot\bar{d}) + (\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c}) = 0$$

36. Find the distance of a point (2, 5, –3) from the plane $\bar{r}\cdot(6\bar{i} - 3\bar{j} + 2\bar{k}) = 4$.

Sol. Here $\bar{a} = 2\bar{i} + 5\bar{j} - 3\bar{k}$, $N = 6\bar{i} - 3\bar{j} + 2\bar{k}$, and d = 4.

$\therefore$ The distance of the point (2, 5, –3) from the given plane is

$$\frac{|(2\bar{i} + 5\bar{j} - 3\bar{k})(6\bar{i} - 3\bar{j} + 2\bar{k}) - 4|}{|6\bar{i} - 3\bar{j} + 2\bar{k}|}$$

$$= \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$$

37. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane 10x + 2y – 11z = 3.

Sol. Let φ be the angle between the given line and the normal to the plane.

Converting the given equations into vector form, we have

$\bar{r} = (-\bar{i} + 3\bar{k}) + \lambda(2\bar{i} + 3\bar{j} + 6\bar{k})$

and $\bar{r} \cdot (10\bar{i} + 2\bar{j} - 11\bar{k}) = 3$

Here,

$b = 2\bar{i} + 3\bar{j} + 6\bar{k}$ and $n = 10\bar{i} + 2\bar{j} - 11\bar{k}$

$$\sin\phi = \frac{(2\bar{i} + 3\bar{j} + 6\bar{k}) \cdot (10\bar{i} + 2\bar{j} - 11\bar{k})}{\sqrt{2^2 + 3^2 + 6^2}\sqrt{10^2 + 2^2 + 11^2}}$$

$$= \left|\frac{-40}{7 \times 15}\right| = \frac{8}{21}$$

$$\Rightarrow \phi = \sin^{-1}\left(\frac{8}{21}\right)$$

38. If $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$ and $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}, a \neq 0$ then show that $\bar{b} = \bar{c}$.

Sol. Given that,

$\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c} \Rightarrow \bar{a}(\bar{b} - \bar{c}) = 0$ ...(1)

$\bar{a} \times \bar{b} = \bar{a} \times \bar{c} \Rightarrow \bar{a} \times (\bar{b} - \bar{c}) = 0$ ...(2)

From (1) and (2) it is evident that, the vector $(\bar{b} - \bar{c})$ cannot be both perpendicular to $\bar{a}$ and parallel to $\bar{a}$.

Unless it is zero

$\therefore \bar{b} - \bar{c} = 0 \; (\bar{a} \neq 0)$

$\therefore \bar{b} = \bar{c}$

SAQ'S

39. If $|\bar{a}| = 2, |\bar{b}| = 3$ and $|\bar{c}| = 4$ and each of $\bar{a}, \bar{b}, \bar{c}$ is perpendicular to the sum of the other two vectors, then find the magnitude of $\bar{a} + \bar{b} + \bar{c}$.

Sol. $\bar{a} \perp (\bar{b} + \bar{c})$

$\Rightarrow \bar{a} \cdot (\bar{b} + \bar{c}) = 0$

$\Rightarrow \bar{a} \cdot \bar{b} + \bar{c} \cdot \bar{a} = 0$ ...(1)

$\bar{b} \perp (\bar{c} + \bar{a})$

$\Rightarrow \bar{b} \cdot (\bar{c} + \bar{a})$

$\Rightarrow \bar{b} \cdot \bar{c} + \bar{b} \cdot \bar{a} = 0$ ...(2)

$\bar{c} \perp (\bar{a} + \bar{b})$

$\Rightarrow \bar{c} \cdot (\bar{a} + \bar{b}) = 0$

$\Rightarrow \bar{c} \cdot \bar{a} + \bar{c} \cdot \bar{b} = 0$ ...(3)

(1) + (2) + (3) $\Rightarrow$

$2[\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a}]=0$

$\Rightarrow \bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a}=0$ ...(4)

Consider

$|\bar{a}+\bar{b}+\bar{c}|^2=(\bar{a}+\bar{b}+\bar{c})^2$

$=(\bar{a})^2+(\bar{b})^2+(\bar{c})^2+2(\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a})$ $=2^2+3^2+4^2$

$=|\bar{a}|^2+|\bar{b}|^2+|\bar{c}|^2+0$ ($\because$ from(4))

$=4+9+16=29$

$|\bar{a}+\bar{b}+\bar{c}|=\sqrt{29}$

40. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{b}=2\bar{i}+3\bar{j}+\bar{k}$ find

i) The projection vector of $\bar{b}$ on $\bar{a}$ and its magnitude.

ii) The vector components of $\bar{b}$ in the direction of $\bar{a}$ and perpendicular to $\bar{a}$.

Sol. Given that $\bar{a}=\bar{i}+\bar{j}+\bar{k}$, $\bar{b}=2\bar{i}+3\bar{j}+\bar{k}$

i) Then projection of $\bar{b}$ on $\bar{a}=\dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|^2}\cdot\bar{a}$

$=\dfrac{(\bar{i}+\bar{j}+\bar{k})\cdot(2\bar{i}+3\bar{j}+3\bar{k})}{|\bar{i}+\bar{j}+\bar{k}|^2}\cdot|\bar{i}+\bar{j}+\bar{k}|$

$=\dfrac{2+3+1}{(\sqrt{3})^2}\cdot\bar{i}+\bar{j}+\bar{k}$

$=\dfrac{6(\bar{i}+\bar{j}+\bar{k})}{3}=2(\bar{i}+\bar{j}+\bar{k})$

Magnitude $=\dfrac{|\bar{a}\cdot\bar{b}|}{|\bar{a}|}=\dfrac{|(\bar{i}+\bar{j}+\bar{k})\cdot(2\bar{i}+3\bar{j}+\bar{k})|}{|\bar{i}+\bar{j}+\bar{k}|}$

$=\dfrac{|2+3+1|}{|\sqrt{3}|}=\dfrac{6}{\sqrt{3}}=2\sqrt{3}$ unit

ii) The component vector of $\bar{b}$ in the direction of $\bar{a}=\dfrac{(\bar{a}\cdot\bar{b})}{|\bar{a}|^2}\cdot\bar{a}$

$=2(\bar{i}+\bar{j}+\bar{k})$ ($\because$ from 10(i))

The vector component of $\bar{b}$ perpen-dicular to $\bar{a}$.

$= \bar{b} - \frac{(\bar{a}\cdot\bar{b})\bar{a}}{|\bar{a}|^2} = (2\bar{i} + 3\bar{j} + \bar{k}) - 2(\bar{i} + \bar{j} + \bar{k})$

$= 2\bar{i} + 3\bar{j} + \bar{k} - 2\bar{i} - 2\bar{j} - 2\bar{k} = \bar{j} - \bar{k}$

41. If $\bar{a}+\bar{b}+\bar{c}=0, |\bar{a}|=3, |\bar{b}|=5$ and $|\bar{c}|=7$ then find the angle between $\bar{a}$ and $\bar{b}$.

Sol. Given $|\bar{a}|=3, |\bar{b}|=5, |\bar{c}|=7$ and

$\bar{a}+\bar{b}+\bar{c}=0$

$\bar{a}+\bar{b}=-\bar{c}$

Squaring on both sides

$\bar{a}^2+\bar{b}^2+2\bar{a}\cdot\bar{b}=\bar{c}^2$

$\Rightarrow |\bar{a}|^2+|\bar{b}|^2| 2[|\bar{a}||\bar{b}|\cos(\bar{a},\bar{b})]=|\bar{c}|^2$

$\Rightarrow 9+25+2[3.5\cos(\bar{a},\bar{b})]=49$

$\Rightarrow 2[15\cos(\bar{a},\bar{b})]=49-34$

$\Rightarrow \cos(\bar{a},\bar{b})=\frac{15}{30}$

$\Rightarrow \cos(\bar{a},\bar{b})=\frac{1}{2}=\cos\frac{\pi}{3}$

$\Rightarrow (\bar{a},\bar{b})=\frac{\pi}{3}$

$\therefore$ Angle between $\bar{a}$ and $\bar{b}$ is 60°.

42. Find the equation of the plane passing through the point $\bar{a}=2\bar{i}+3\bar{j}-\bar{k}$ and perpendicular to the vector $3\bar{i}-2\bar{j}-2\bar{k}$ and the distance of this plane from the origin.

Sol. Let $\bar{a}=2\bar{i}+3\bar{j}-\bar{k}$ and $\bar{b}=3\bar{i}-2\bar{j}-2\bar{k}$

Equation of the required plane is

$\bar{r}\cdot\bar{b}=\bar{a}\cdot\bar{b}$

$\bar{r}.(3\bar{i}-2\bar{j}-2\bar{k})=$

$(2\bar{i}+3\bar{j}-\bar{k})\cdot(3\bar{i}-2\bar{j}-2\bar{k})$

$=6-6+2$

$\bar{r}.(3\bar{i}-2\bar{j}-2\bar{k})=2$

Its Cartesian form is

$(x\bar{i}+y\bar{j}+z\bar{k})\cdot(3\bar{i}-2\bar{j}-2\bar{k})=2$

$\Rightarrow 3x-2y-2z=2$

Perpendicular distance from the origin to the above plane is

$$\frac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|}=\frac{2}{\sqrt{9+4+4}}=\frac{2}{\sqrt{17}}$$

43. If $\bar{a}=2\bar{i}+\bar{j}-\bar{k},\ \bar{b}=-\bar{i}+2\bar{j}-4\bar{k}$ and $\bar{c}=\bar{i}+\bar{j}+\bar{k}$ then find $(\bar{a}\times\bar{b})\cdot(\bar{b}\times\bar{c})$.

Sol. $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 1 & -1\\ -1 & 2 & -4\end{vmatrix}$

$=\bar{i}(-4+2)-\bar{j}(-8-1)+\bar{k}(4+1)$

$=-2\bar{i}+9\bar{j}+5\bar{k}$

$\bar{b}\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -1 & 2 & -4\\ 1 & 1 & 1\end{vmatrix}$

$=\bar{i}(2+4)-\bar{j}(-1+4)+\bar{k}(-1-2)$

$=6\bar{i}-3\bar{j}-3\bar{k}$

$(\bar{a}\times\bar{b})\cdot(\bar{b}\times\bar{c})$

$=(-2\bar{i}+9\bar{j}+5\bar{k})\cdot(6\bar{i}-3\bar{j}-3\bar{k})$

$=-12-27-15=-54$

44. If $\bar{a},\bar{b},\bar{c}$ are unit vectors such that $\bar{a}$ is perpendicular to the plane of $\bar{b},\bar{c}$ and the angle between $\bar{b}$ and $\bar{c}$ is $\pi/3$, then find $|\bar{a}+\bar{b}+\bar{c}|$.

Sol. $\bar{a}$ perpendicular to plane contain $\bar{b}$ and $\bar{c}$.

$\Rightarrow \bar{a}\cdot\bar{b}=0,\ \bar{a}\cdot\bar{c}=0$

Consider

$|\bar{a}+\bar{b}+\bar{c}|^2=(\bar{a}+\bar{b}+\bar{c})^2$

$=\bar{a}^2+\bar{b}^2+\bar{c}^2+2\bar{a}\bar{b}+2\bar{b}\bar{c}+2\bar{c}\,\bar{a}$

$=|\bar{a}|^2+|\bar{b}|^2+|\bar{c}|^2+0$

$+2|\bar{b}||\bar{c}|\cos(\bar{b},\bar{c})+0$

$$= 1+1+2+2(1)(1)\cos\frac{\pi}{3}$$

$$= 3+2\times\frac{1}{2} = 3+1 = 4$$

$$\therefore |\bar{a}+\bar{b}+\bar{c}| = 2$$

45. If $\bar{a} = 2\bar{i}+3\bar{j}+4\bar{k}, \bar{b} = \bar{i}+\bar{j}-\bar{k}$ and $\bar{c} = \bar{i}-\bar{j}+\bar{k}$ then compute $\bar{a}\times(\bar{b}\times\bar{c})$ and verify that it is perpendicular to a.

Sol. Given $\bar{a} = 2\bar{i}+3\bar{j}+4\bar{k}, \bar{b} = \bar{i}+\bar{j}-\bar{k}$, $\bar{c} = \bar{i}-\bar{j}+\bar{k}$

$$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \bar{i}(1-1)-\bar{j}(1+1)+\bar{k}(-1-1) = -2\bar{j}-2\bar{k}$$

$$\bar{a}\times(\bar{b}\times\bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & 4 \\ 0 & -2 & -2 \end{vmatrix} = \bar{i}(-6+8)-\bar{j}(-4-0)+\bar{k}(-4-0) = 2\bar{i}+4\bar{j}-4\bar{k}$$

$$(\bar{a}\times(\bar{b}\times\bar{c}))\cdot\bar{a} = (2\bar{i}+4\bar{j}-4\bar{k})\cdot(2\bar{i}+3\bar{j}+4\bar{k})$$

$$= 4+12-16 = 16-16 = 0$$

$\therefore \bar{a}\times(\bar{b}\times\bar{c})$ is perpendicular to $\bar{a}$.

46. Let $\bar{a},\bar{b}$ and $\bar{c}$ are non-coplanar vectors prove that if $\left[\bar{a}+2\bar{b} \quad 2\bar{b}+\bar{c} \quad 5\bar{c}+\bar{a}\right] = \lambda\left[\bar{a}\ \bar{b}\ \bar{c}\right]$, then find λ.

Sol. Given

$$\left[\bar{a}+2\bar{b} \quad 2\bar{b}+\bar{c} \quad 5\bar{c}+\bar{a}\right] = \lambda\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 5 \end{vmatrix}\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \lambda\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$\Rightarrow [1(10-0)-2(0-1)+0(0-2)]$$

$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \lambda\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$\Rightarrow (10+2)\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \lambda\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$\therefore \lambda = 12$$

47. If $\bar{a}=\bar{i}-2\bar{j}-3\bar{k}, \bar{b}=2\bar{i}+\bar{j}-\bar{k}$ and $\bar{c}=\bar{i}+3\bar{j}-2\bar{k}$ verify that $\bar{a}\times(\bar{b}\times\bar{c})\neq(\bar{a}\times\bar{b})\times\bar{c}$.

Sol. $\bar{b}\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 1 & -1\\ 1 & 3 & -2\end{vmatrix}$

$=\bar{i}(-2+3)-\bar{j}(-4+1)+\bar{k}(6-1)$

$\bar{b}\times\bar{c}=\bar{i}+3\bar{j}+5\bar{k}$

$\bar{a}\times(\bar{b}\times\bar{c})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & -3\\ 1 & 3 & 5\end{vmatrix}=\bar{i}(-10+9)-\bar{j}(5+3)+\bar{k}(3+2)$

$\bar{a}\times(\bar{b}\times\bar{c})=-\bar{i}-8\bar{j}+5\bar{k}$

$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & -3\\ 2 & 1 & -1\end{vmatrix} \quad =\bar{i}(2+3)-\bar{j}(-1+6)+\bar{k}(1+4)$

$\bar{a}\times\bar{b}=5\bar{i}-5\bar{j}+5\bar{k}$

$(\bar{a}\times\bar{b})\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 5 & -5 & 5\\ 1 & 3 & -2\end{vmatrix}=\bar{i}(10-15)-\bar{j}(-10-5)+\bar{k}(15+5)$

$(\bar{a}\times\bar{b})\times\bar{c}=-5\bar{i}+15\bar{j}+20\bar{k}$

$\therefore \bar{a}\times(\bar{b}\times\bar{c})\neq(\bar{a}\times\bar{b})\times\bar{c}$

48. Let $\bar{b}=2\bar{i}+\bar{j}-\bar{k}, \bar{c}=\bar{i}+3\bar{k}$. If $\bar{a}$ is a unit vector then find the maximum value of $[\bar{a}\ \bar{b}\ \bar{c}]$.

Sol. Consider $\bar{b}\times\bar{c}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 1 & -1\\ 1 & 0 & 3\end{vmatrix}$

$=\bar{i}(3)-\bar{j}(6+1)+\bar{k}(0-1)$

$=3\bar{i}-7\bar{j}-\bar{k}$

$|\bar{b}\times\bar{c}| = \sqrt{9+49+1} = \sqrt{59}$

Let $(\bar{a}, \bar{b}\times\bar{c}) = \theta$

Consider $[\bar{a}\ \bar{b}\ \bar{c}] = \bar{a}\cdot\bar{b}\times\bar{c}$

$= |\bar{a}||\bar{b}\times\bar{c}|\cos[\bar{a}, \bar{b}\times\bar{c}]$

$= (1)(\sqrt{59})\cos\theta$

$= \sqrt{59}\cos\theta$

We know that $-1 \le \cos\theta \le 1$

$\therefore$ Maximum value of $[\bar{a}\ \bar{b}\ \bar{c}] = \sqrt{59}$.

49. Let $\bar{a}, \bar{b}, \bar{c}$ be mutually orthogonal vectors of equal magnitudes. Prove that the vector $\bar{a}+\bar{b}+\bar{c}$ is equally inclined to each of $\bar{a}, \bar{b}, \bar{c}$, the angle of inclination being $\cos^{-1}\frac{1}{\sqrt{3}}$.

Sol. Let $|\bar{a}| = |\bar{b}| = |\bar{c}| = \lambda$

Now, $|\bar{a}+\bar{b}+\bar{c}|^2 = \bar{a}^2 + \bar{b}^2 + \bar{c}^2 + 2\Sigma\bar{a}\cdot\bar{b}$

$= 3\lambda^2 (\because \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{c} = \bar{c}\cdot\bar{a} = 0)$

Let θ be the angle between $\bar{a}$ and $\bar{a}+\bar{b}+\bar{c}$

Then $\cos\theta = \frac{\bar{a}\cdot(\bar{a}+\bar{b}+\bar{c})}{|\bar{a}||\bar{a}+\bar{b}+\bar{c}|} = \frac{\bar{a}\cdot\bar{a}}{\lambda(\lambda\sqrt{3})} = \frac{1}{\sqrt{3}}$

Similarly, it can be proved that $\bar{a}+\bar{b}+\bar{c}$ inclines at an angle of $\cos^{-1}\frac{1}{\sqrt{3}}$ with b and c.

50. In ΔABC, if $\overline{BC} = \bar{a}, \overline{CA} = \bar{b}$ and $\overline{AB} = \bar{c}$, then show that $\bar{a}\times\bar{b} = \bar{b}\times\bar{c} = \bar{c}\times\bar{a}$.

Sol. $\bar{a}+\bar{b}+\bar{c} = \overline{BC}+\overline{CA}+\overline{AB} = \overline{BB} = \bar{0}$

$\therefore \bar{a}+\bar{b} = -\bar{c}$

$\therefore \bar{a}\times(\bar{a}+\bar{b}) = \bar{a}\times(-\bar{c})$

$\therefore \bar{a}\times\bar{b} = -(\bar{a}\times\bar{c}) = \bar{c}\times\bar{a}$

Also $(\bar{a}+\bar{b})\times\bar{b} = (-\bar{c})\times\bar{b}$

$\therefore \bar{a}\times\bar{b} = -(\bar{c}\times\bar{b}) = \bar{b}\times\bar{c}$

$\therefore \bar{b}\times\bar{c} = \bar{a}\times\bar{b} = \bar{c}\times\bar{a}$

51. Let $\bar{a} = 2\bar{i} + \bar{j} - 2\bar{k}, \bar{b} = \bar{i} + \bar{j}$. If $\bar{c}$ is a vector such that $\bar{a}\cdot\bar{c} = |\bar{c}|, |\bar{c} - \bar{a}| = 2\sqrt{2}$ and the angle between $\bar{a}\times\bar{b}$ and $\bar{c}$ is 30°, then find the value of $|(\bar{a}\times\bar{b})\times\bar{c}|$.

Sol. $|\bar{a}| = 3, |\bar{b}| = \sqrt{2}$ and $\bar{a}\cdot\bar{c} = |\bar{c}|$

$2\sqrt{2} = |\bar{c} - \bar{a}|$

$\Rightarrow 8 = |\bar{c} - \bar{a}|^2 = |\bar{c}|^2 + |\bar{a}|^2 - 2(\bar{a}\cdot\bar{c})$

$\therefore 8 = |\bar{c}|^2 + 9 - 2|\bar{c}|$

$\therefore (|\bar{c}| - 1)^2 = 0$

$\therefore |\bar{c}| = 1$

Now, $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\bar{i} - 2\bar{j} + \bar{k}$

$\therefore |(\bar{a}\times\bar{b})\times\bar{c}| = |\bar{a}\times\bar{b}||\bar{c}|\sin 30°$

$= 3(1)\left(\frac{1}{2}\right) = \frac{3}{2}$

52. If $\bar{a}$ is a non-zero vector and $\bar{b}, \bar{c}$ are two vectors such that $\bar{a}\times\bar{b} = \bar{a}\times\bar{c}$ and $\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c}$ then prove that $\bar{b} = \bar{c}$.

Sol. $\bar{a}\times\bar{b} = \bar{a}\times\bar{c} \Rightarrow \bar{a}\times(\bar{b} - \bar{c}) = 0$

$\Rightarrow$ either $\bar{b} = \bar{c}$ or $\bar{b} - \bar{c}$ is collinear with $\bar{a}$

Again $\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c} \Rightarrow \bar{a}\cdot(\bar{b} - \bar{c}) = 0$

$\Rightarrow$ $\bar{b} = \bar{c}$ or $\bar{b} - \bar{c}$ is perpendicular to $\bar{a}$

$\therefore$ If $\bar{b} \neq \bar{c}$, then $\bar{b} - \bar{c}$ is parallel to $\bar{a}$ and is perpendicular to $\bar{a}$ which is impossible.

$\therefore$ $\bar{b} = \bar{c}$.

53. Prove that for any three vectors $\bar{a}, \bar{b}, \bar{c}$, $\left[\bar{b}+\bar{c} \quad \bar{c}+\bar{a} \quad \bar{a}+\bar{b}\right] = 2\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$.

Sol. $\left[\bar{b}+\bar{c} \quad \bar{c}+\bar{a} \quad \bar{a}+\bar{b}\right]$

$= (\bar{b}+\bar{c})\cdot\{(\bar{c}+\bar{a})\times(\bar{a}+\bar{b})\}$

$= (\bar{b}+\bar{c})\cdot\{\bar{c}\times\bar{a} + \bar{c}\times\bar{b} + \bar{a}\times\bar{b}\}$

$= \bar{b}(\bar{c}\times\bar{a}) + \bar{b}(\bar{c}\times\bar{b}) + \bar{b}(\bar{a}\times\bar{b})$

$\quad + \bar{c}(\bar{c}\times\bar{a}) + \bar{c}(\bar{c}\times\bar{b}) + \bar{c}(\bar{a}\times\bar{b})$

$= [\bar{b}\ \bar{c}\ \bar{a}] + 0 + 0 + 0 + 0 + [\bar{c}\ \bar{a}\ \bar{b}]$

$= 2[\bar{a}\ \bar{b}\ \bar{c}]$

54. For any three vectors $\bar{a}, \bar{b}, \bar{c}$ prove that $\left[\bar{b}\times\bar{c} \quad \bar{c}\times\bar{a} \quad \bar{a}\times\bar{b}\right] = [\bar{a}\ \bar{b}\ \bar{c}]^2$.

Sol. $\left[\bar{b}\times\bar{c} \quad \bar{c}\times\bar{a} \quad \bar{a}\times\bar{b}\right]$

$= (\bar{b}\times\bar{c})\cdot\{(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b})\}$

$= (\bar{b}\times\bar{c})\cdot\left\{[\bar{c}\ \bar{a}\ \bar{b}]\bar{a} - [\bar{a}\ \bar{a}\ \bar{b}]\bar{c}\right\}$

$= (\bar{b}\times\bar{c})\cdot\bar{a}[\bar{c}\ \bar{a}\ \bar{b}] = [\bar{a}\ \bar{b}\ \bar{c}]^2$

55. For any four vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$, $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$ and $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$.

Sol. Let m = c × d

$\therefore (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = (\bar{a}\times\bar{b})\times m$

$= (\bar{a}\cdot m)\bar{b} - (\bar{b}\cdot m)\bar{a}$

$= (\bar{a}\cdot(\bar{c}\times\bar{d}))\bar{b} - (\bar{b}\cdot(\bar{c}\times\bar{d}))\bar{a}$

$= [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$

Again, Let $\bar{a}\times\bar{b} = n$, then

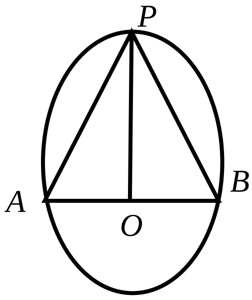
$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = n\times(\bar{c}\times\bar{d})$

$= (n\cdot\bar{d})\bar{c} - (n\cdot\bar{c})\bar{d}$

$= ((\bar{a}\times\bar{b})\cdot\bar{d})\bar{c} - ((\bar{a}\times\bar{b})\bar{c})\bar{d}$

$= [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

56. The angle in semi circle is a right angle



Proof: Let APB be a semi circle with centre at O.

OA = OB = OP also $\overrightarrow{OB} = -\overrightarrow{OA}$

$\overrightarrow{AP}.\overrightarrow{BP} = (\overrightarrow{OP} - \overrightarrow{OA}).(\overrightarrow{OP} - \overrightarrow{OA})$

$= (\overrightarrow{OP} - \overrightarrow{OA}).(\overrightarrow{OP} + \overrightarrow{OA}) \qquad \because \overrightarrow{OB} = -\overrightarrow{OA}$

$= (\overrightarrow{OP})^2 - (\overrightarrow{OA})^2 \qquad \left\{\because (\vec{a} + \vec{b}0.(\vec{a} - \vec{b}) = (\vec{a})^2 - (\vec{b})^2\right\}$

$= \left|\overrightarrow{OP}\right|^2 - \left|\overrightarrow{OA}\right|^2 = OP^2 - OP^2 = 0 \qquad \{\because OA = OP\}$

$\overrightarrow{AP}.\overrightarrow{BP} = 0 \qquad \therefore \overrightarrow{AP} \perp r\ \overrightarrow{PB}$ Hence $\angle APB = 90^0$

Hence angle in semi –cricle is 90^0

57. For any two vectors $\vec{a}$ and $\vec{b}$ prove that $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2=|\vec{a}|^2\,|\vec{b}|^2$

Sol: $(\vec{a}\times\vec{b})^2+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2\,|\vec{b}|^2\sin^2(\vec{a},\vec{b})+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2\,|\vec{b}|^2\{1-\cos^2(\vec{a},\vec{b})\}+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2\,|\vec{b}|^2-|\vec{a}|^2\,|\vec{b}|^2\cos^2(\vec{a},\vec{b})+(\vec{a}.\vec{b})^2$

$|\vec{a}|^2\,|\vec{b}|^2-\cancel{(\vec{a}.\vec{b})^2}+\cancel{(\vec{a}.\vec{b})^2}$

$=|\vec{a}|^2\,|\vec{b}|^2=R.H.S$

58. **If** $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ $\quad\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ **, then** $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$

Proof : $\bar{a}\times\bar{b}=(a_1\bar{i}+a_2\bar{j}+a_3\bar{k})\times(b_1\bar{i}+b_2\bar{j}+b_3\bar{k})$

$=a_1b_1(\bar{i}\times\bar{i})+a_1b_2(\bar{i}\times\bar{j})+a_1b_3(\bar{i}\times\bar{k})+a_2b_1(\bar{j}\times\bar{i})+a_2b_2(\bar{j}\times\bar{j})+a_2b_3(\bar{j}\times\bar{k})$

$+a_3b_1(\bar{k}\times\bar{i})+a_3b_2(\bar{k}\times\bar{j})+a_3b_3(\bar{k}\times\bar{k})$

$=a_1b_1(\bar{0})+a_1b_2(\bar{k})+a_1b_3(-\bar{j})\ +a_2b_1(-\bar{k})+a_2b_2(\bar{0})+a_2b_3(\bar{i})\ +a_3b_1(\bar{j})+a_3b_2(-\bar{i})+a_3b_3(\bar{0})$

$=\bar{i}(a_2b_3-a_3b_2)-\bar{j}(a_1b_3-a_3b_1)\ +\bar{k}(a_1b_2-a_2b_1)$

$$=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$$

59. If $\vec{a},\vec{b},\vec{c},\vec{d}$ are four vectors then $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c}&\vec{a}.\vec{d}\\\vec{b}.\vec{c}&\vec{b}.\vec{d}\end{vmatrix}$

Proof : $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\{(\vec{a}\times\vec{b})\times\vec{c}\}.\vec{d}$ $\{\because$ dot and cross are inter changeable $\}$

$\{(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}\}.\vec{d}=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{b}.\vec{c})(\vec{a}.\vec{d})$

$$=\begin{vmatrix}\vec{a}.\vec{c}&\vec{a}.\vec{d}\\\vec{b}.\vec{c}&\vec{b}.\vec{d}\end{vmatrix}$$

60. If $\vec{a},\vec{b},\vec{c},\vec{d}$ **are four vectors then** $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}\,]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}\,]\vec{d}$

$=[\vec{a}\,\vec{c}\,\vec{d}\,]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}\,]\vec{a}$

Proof :- $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{a}\times\vec{b}.\vec{d})\vec{c}-(\vec{a}\times\vec{b}.\vec{c})\vec{d}=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

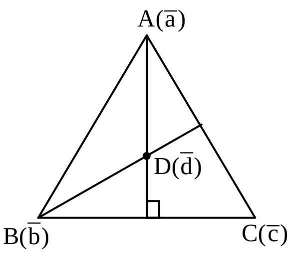
$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{c}\times\vec{d}\,.\vec{a}]\vec{b}-(\vec{c}\times\vec{d}\,.\vec{b}]\vec{a}$

$=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}-(\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

LAQ'S

61. a, b, c and d are the position vectors of four coplanar points such that (a – d).(b – c) = (b – d).(c – a) = 0 show that the point 'd' represents the orthocenter of the triangle with a, b and c as its vertices.

Sol.



Let O be the origin and

$\overline{OA}=\bar{a},\overline{OB}=\bar{b},\overline{OC}=\bar{c},\overline{OD}=\bar{d}$

Given that $(\bar{a}-\bar{d})\cdot(\bar{b}-\bar{c})=0$

$\Rightarrow(\overline{OA}-\overline{OD})\cdot(\overline{OB}-\overline{OC})=0$

$\Rightarrow\overline{DA}\cdot\overline{CB}=0$

$\Rightarrow\overline{DA}$ perpendicular to $\overline{CB}$

D is an altitudes of ΔABC

Consider $(\bar{b}-\bar{d})\cdot(\bar{c}-\bar{a})=0$

$(\overline{OB}-\overline{OD})\cdot(\overline{OC}-\overline{OA})=0$

$\overline{DB}\cdot\overline{AC}=0$

$\Rightarrow\overline{DB}$ perpendicular to $\overline{AC}$

$\Rightarrow$ $\overline{DB}$ is also an altitude of ΔABC

The altitudes $\overline{DA},\overline{DB}$ intersect at D.

$\Rightarrow$ D is the orthocenter of ΔABC.

62. Let $\bar{a} = 4\bar{i} + 5\bar{j} - \bar{k}$, $\bar{b} = \bar{i} - 4\bar{j} + 5\bar{k}$ and $\bar{c} = 3\bar{i} + \bar{j} - \bar{k}$. Find the vector which is perpendicular to both $\bar{a}$ and $\bar{b}$ whose magnitude is twenty one times the magnitude of $\bar{c}$.

Sol. Given that $\bar{a} = 4\bar{i} + 5\bar{j} - \bar{k}$, $\bar{b} = \bar{i} - 4\bar{j} + 5\bar{k}$ and $\bar{c} = 3\bar{i} + \bar{j} - \bar{k}$

$$|\bar{c}| = \sqrt{9+1+1} = \sqrt{11}$$

The unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ is = $\dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$

Now $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$

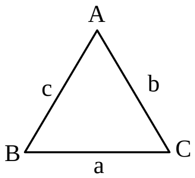
$$= \bar{i}(25-4) - \bar{j}(20+1) + \bar{k}(-16-5)$$
$$= 21\bar{i} - 21\bar{j} - 21\bar{k}$$
$$= 21(\bar{i} - \bar{j} - \bar{k})$$
$$|\bar{a}\times\bar{b}| = 21\sqrt{1+1+1} = 21\sqrt{3}$$

The vector perpendicular both $\bar{a}$ and $\bar{b}$ and having the magnitude 21 times magnitude of $\bar{c}$ is

$$= \pm\frac{21|\bar{c}|(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|}$$
$$= \pm\frac{21\sqrt{11}\times 21(\bar{i} - \bar{j} - \bar{k})}{21\sqrt{3}}$$
$$= \frac{\pm 21\sqrt{11}(\bar{i} - \bar{j} - \bar{k})}{\sqrt{3}}$$
$$= \frac{\pm 7\cdot 3\sqrt{11}(\bar{i} - \bar{j} - \bar{k})}{\sqrt{3}}$$
$$= \pm 7\sqrt{3}\sqrt{11}(\bar{i} - \bar{j} - \bar{k})$$
$$= \pm 7\sqrt{33}(\bar{i} - \bar{j} - \bar{k})$$

63. G is the centroid ΔABC and a, b, c are the lengths of the sides BC, CA and AB respectively. Prove that $a^2 + b^2 + c^2 = 3(OA^2 + OB^2 + OC^2) - 9(OG)^2$ where O is any point.

Sol.



Given that BC = a, CA = b, AB = c

Let O be the origin

$\overline{OA} + \overline{OB} + \overline{OC} = 3\overline{OG}$

$a^2 = \overline{BC}^2 = (\overline{OC} - \overline{OB})^2$

$= \overline{OC}^2 + \overline{OB}^2 - 2\overline{OC} \cdot \overline{OB}$

$b^2 = \overline{CA}^2 = (\overline{OA} - \overline{OC})^2$

$= \overline{OA}^2 + \overline{OC}^2 - 2\overline{OA} \cdot \overline{OC}$

$c^2 = \overline{AB}^2 = (\overline{OB} - \overline{OA})^2$

$= \overline{OB}^2 + \overline{OA}^2 - 2\overline{OB} \cdot \overline{OA}$

Consider

$a^2 + b^2 + c^2 = 2\left[\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2\right] - 2\left[\overline{OA} \cdot \overline{OB} + \overline{OB} \cdot \overline{OC} + \overline{OC} \cdot \overline{OA}\right] \ldots(1)$

We have $\overline{OA} + \overline{OB} + \overline{OC} = 3\overline{OG}$

Squaring on both sides

$\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2 + 2\left[\overline{OA} \cdot \overline{OB} + \overline{OB} \cdot \overline{OC} + \overline{OC} \cdot \overline{OA}\right] = 9\overline{OG}^2$

$\Rightarrow -2(\overline{OA} \cdot \overline{OB} + \overline{OB} \cdot \overline{OC} + \overline{OC} \cdot \overline{OA})$

$= \overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2 - 9\overline{OG}^2 \quad \ldots(2)$

Substituting in eq.(1), we get

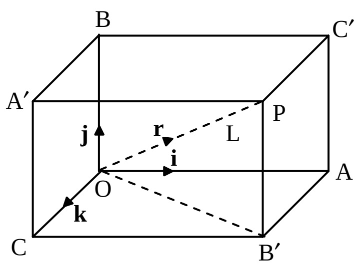
$a^2 + b^2 + c^2 = 2\left[\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2\right] + [\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2] - 9\overline{OG}^2$

$a^2 + b^2 + c^2 = 3\left[\overline{OA}^2 + \overline{OB}^2 + \overline{OC}^2\right] - 9\overline{OG}^2$

64. A line makes angles θ_1, θ_2, θ_3 and θ_4 with the diagonals of a cube. Show that

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 = \frac{4}{3}.$$

Sol.



Let OAB′C, BC′PA′ be a unit cube.

Let $\overline{OA} = \overline{i}, \overline{OB} = \overline{j}$ and $\overline{OC} = \overline{k}$

$\overline{OP}, AA', BB', CC'$ be its diagonals.

Let $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ be a unit vector along a line L.

Which makes angles θ_1, θ_2, θ_3 and θ_4 with $\overline{AA'}, \overline{BB'}, \overline{CC'}$ and $\overline{OP}$.

$$\Rightarrow |\overline{r}| = \sqrt{x^2 + y^2 + z^2} = 1$$

We have $\overline{OB'} = \overline{OA} - \overline{OC} = \overline{i} + \overline{k}$

$$\overline{OP} = \overline{OB'} - \overline{B'P} = \overline{i} + \overline{k} + \overline{j} \, [\because \overline{B'O} = \overline{OB} = \overline{j}]$$

$$= \overline{i} + \overline{j} + \overline{k}$$

$$\overline{AA'} = \overline{OA'} - \overline{OA} = \overline{j} + \overline{k} - \overline{i} = -\overline{i} + \overline{j} + \overline{k}$$

$$\overline{BB'} = \overline{OB'} - \overline{OB} = \overline{i} + \overline{k} - \overline{j} = \overline{i} - \overline{j} + \overline{k}$$

$$\overline{CC'} = \overline{OC'} - \overline{OC} = \overline{i} + \overline{j} - \overline{k}$$

Let $(\overline{r}, \overline{OP}) = \theta_1$

$$\cos\theta_1 = \frac{\overline{r} \cdot \overline{OP}}{|\overline{r}||\overline{OP}|} = \frac{(x\overline{i} + y\overline{j} + z\overline{k}) \cdot (\overline{i} + \overline{j} + \overline{k})}{1 \cdot \sqrt{1+1+1}}$$

$$= \frac{x + y + z}{\sqrt{3}} \qquad \text{...(1)}$$

Similarly $(\overline{r}, AA') = \theta_2$

$$\Rightarrow \cos\theta_2 = \frac{\overline{r} \cdot \overline{AA'}}{|\overline{r}||\overline{AA'}|} = \frac{(x\overline{i} + y\overline{j} + z\overline{k}) \cdot (-\overline{i} + \overline{j} + \overline{k})}{1 \cdot \sqrt{1+1+1}}$$

$$= \frac{-x + y + z}{\sqrt{3}} \qquad \text{...(2)}$$

$(\bar{r}, \overline{BB'}) = \theta_3$

$$\Rightarrow \cos\theta_3 = \frac{\bar{r}\cdot\overline{BB'}}{|\bar{r}||\overline{BB'}|}$$

$$= \frac{(x\bar{i}+y\bar{j}+z\bar{k})\cdot(\bar{i}-\bar{j}+\bar{k})}{1\cdot\sqrt{1+1+1}}$$

$$= \frac{x-y+z}{\sqrt{3}} \qquad ...(3)$$

$(\bar{r}, \overline{CC'}) = \theta_4$

$$\Rightarrow \cos\theta_3 = \frac{\bar{r}\cdot\overline{CC'}}{|\bar{r}||\overline{CC'}|}$$

$$= \frac{(x\bar{i}+y\bar{j}+z\bar{k})\cdot(\bar{i}+\bar{j}-\bar{k})}{1\cdot\sqrt{1+1+1}}$$

$$= \frac{x+y-z}{\sqrt{3}} \qquad ...(4)$$

$\therefore \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4$

$$= \left(\frac{x+y+z}{\sqrt{3}}\right)^2 + \left(\frac{-x+y+z}{\sqrt{3}}\right)^2 + \left(\frac{x-y+z}{\sqrt{3}}\right)^2 + \left(\frac{x+y-z}{\sqrt{3}}\right)^2$$

$$(x+y+z)^2 + (-x+y+z)^2 = \frac{(x+y+z)^2 + (x-y+z)^2 + (x+y-z)^2}{3}$$

$$= \frac{2(x+y)^2 + 2z^2 + 2(x-y)^2 + 2z^2}{3} = \frac{2\left[(x+y)^2 + (x-y)^2\right] + 4z^2}{3}$$

$$= \frac{2\left[2x^2 + 2y^2\right] + 4z^2}{3}$$

$$= \frac{4x^2 + 4y^2 + 4z^2}{3} = \frac{4}{3}\left[x^2 + y^2 + z^2\right] = \frac{4}{3}(1) = \frac{4}{3}$$

65. If $\bar{a}+\bar{b}+\bar{c}=0$ then prove that $\bar{a}\times\bar{b} = \bar{b}\times\bar{c} = \bar{c}\times\bar{a}$.

Sol. Given $\bar{a}+\bar{b}+\bar{c}=0$

$\bar{a}+\bar{b} = -\bar{c}$

$(\bar{a}+\bar{b})\times\bar{b} = -\bar{c}\times\bar{b}$

$\bar{a}\times\bar{b} + \bar{b}\times\bar{b} = \bar{b}\times\bar{c}$

$\bar{a}\times\bar{b} + 0 = \bar{b}\times\bar{c}$

$\bar{a}\times\bar{b} = \bar{b}\times\bar{c} \qquad ...(1)$

Given $\bar{a}+\bar{b}+\bar{c}=0$

$$\bar{a}+\bar{b}=-\bar{c}$$
$$(\bar{a}+\bar{b})\times\bar{a}=-\bar{c}\times\bar{a}$$
$$\bar{a}\times\bar{a}+\bar{b}\times\bar{a}=-\bar{c}\times\bar{a}$$
$$0-\bar{a}\times\bar{b}=-\bar{c}\times\bar{a}$$
$$\bar{a}\times\bar{b}=\bar{c}\times\bar{a} \qquad ...(2)$$

From (1) and (2)

$$\bar{a}\times\bar{b}=\bar{b}\times\bar{c}=\bar{c}\times\bar{a}$$

66. Let $\bar{a}$ and $\bar{b}$ be vectors, satisfying $|\bar{a}|=|\bar{b}|=5$ and $(\bar{a},\bar{b})=45°$. Find the area of the triangle have $\bar{a}-2\bar{b}$ and $3\bar{a}+2\bar{b}$ as two of its sides.

Sol. Given $\bar{a}$ and $\bar{b}$ are two vectors.

$|\bar{a}|=|\bar{b}|=5$ and $(\bar{a},\bar{b})=45°$

$\bar{c}=\bar{a}-2\bar{b}$ and $\bar{d}=3\bar{a}+2\bar{b}$

The area of Δle having $\bar{c}$ and $\bar{d}$ as adjacent sides is $\frac{|\bar{c}\times\bar{d}|}{2}$

$$|\bar{c}\times\bar{d}|=|(\bar{a}-2\bar{b})\times(3\bar{a}+2\bar{b})|$$
$$=|3(\bar{a}\times\bar{a})+2(\bar{a}\times\bar{b})-6(\bar{b}\times\bar{a})-4(\bar{b}\times\bar{b})|$$
$$=|3(0)+2(\bar{a}\times\bar{b})+6(\bar{a}\times\bar{b})-4(0)|$$
$$=|8(\bar{a}\times\bar{b})|$$
$$=8|\bar{a}\times\bar{b}|$$
$$=8|\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})$$
$$=8\cdot 5\cdot 5\sin 45°$$
$$=200\cdot\frac{1}{\sqrt{2}}=100\sqrt{2}$$
$$\therefore \text{Area}=\frac{|\bar{c}\times\bar{d}|}{2}=\frac{100\sqrt{2}}{2}=50\sqrt{2}\,\text{sq.units.}$$

67. Find a unit vector perpendicular to the plane determined by the points P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1).

Sol. Let O be the origin and

$$\overline{OP} = \bar{i} - \bar{j} + 2\bar{k}, \overline{OQ} = 2\bar{i} - \bar{k}, \overline{OR} = 2\bar{j} + \bar{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = \bar{i} - 2\bar{k}$$

$$\overline{PR} = \overline{OR} - \overline{OP} = -\bar{i} + 3\bar{j} - \bar{k}$$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -2 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= \bar{i}(0+6) - \bar{j}(-1-2) + \bar{k}(3-0)$$

$$\overline{PQ} \times \overline{PR} = 6\bar{i} + 3\bar{j} + 3\bar{k}$$

$$|\overline{PQ} \times \overline{PR}| = 3\sqrt{4+1+1} = 3\sqrt{6}$$

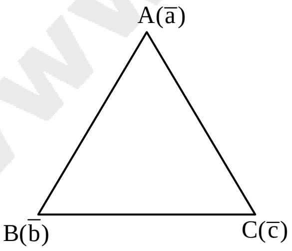
∴ The unit vector perpendicular to the plane passing through

P, Q and R is $= \pm \dfrac{\overline{PQ} \times \overline{PR}}{|\overline{PQ} \times \overline{PR}|}$

$$= \pm \frac{3(2\bar{i} + \bar{j} + \bar{k})}{3\sqrt{6}} = \pm \frac{2\bar{i} + \bar{j} + \bar{k}}{\sqrt{6}}$$

68. If $\bar{a}, \bar{b}$ and $\bar{c}$ represent the vertices A, B and C respectively of ΔABC, then prove that $|(\bar{a}\times\bar{b})+(\bar{b}+\bar{c})+(\bar{c}\times\bar{a})|$ is twice the area of ΔABC.

Sol.



Let O be the origin,

$$\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$$

Area of ΔABC is $\Delta = \frac{1}{2}\left|(\overline{AB} \times \overline{AC})\right|$

$$=\frac{1}{2}\left|(\overline{OB}-\overline{OA})\times(\overline{OC}-\overline{OA})\right|$$

$$=\frac{1}{2}\left|(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})\right|$$

$$=\frac{1}{2}\left|\overline{b}\times\overline{c}-\overline{b}\times\overline{a}-\overline{a}\times\overline{c}+\overline{a}\times\overline{a}\right|$$

$$=\frac{1}{2}\left|\overline{b}\times\overline{c}+\overline{a}\times\overline{b}+\overline{c}\times\overline{a}+\overline{0}\right|$$

$$=\frac{1}{2}\left|\overline{b}\times\overline{c}+\overline{a}\times\overline{b}+\overline{c}\times\overline{a}\right|$$

$2\Delta=\left|\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}\right|$. Hence proved.

69. If $\overline{a}=4\overline{i}-2\overline{j}+3\overline{k},\ \overline{b}=2\overline{i}+8\overline{k}$ **and** $\overline{c}=\overline{i}+\overline{j}+\overline{k}$ **then** $\overline{a}\times\overline{b},\overline{a}\times\overline{c}$ **and** $\overline{a}\times(\overline{b}+\overline{c})$**.**

Verify whether the cross product is distributive over vector addition.

Sol. Given

$\overline{a}=4\overline{i}-2\overline{j}+3\overline{k},\ \overline{b}=2\overline{i}+8\overline{k},\ \overline{c}=\overline{i}+\overline{j}+\overline{k}$

$$\overline{a}\times\overline{b}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 7 & -2 & 3\\ 2 & 0 & 8\end{vmatrix}$$

$$=\overline{i}(-16-0)-\overline{j}(56-6)+\overline{k}(0+4)$$

$$\overline{a}\times\overline{b}=-16\overline{i}-50\overline{j}+4\overline{k}$$

$$\overline{a}\times\overline{c}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 7 & -2 & 3\\ 1 & 1 & 1\end{vmatrix}$$

$$=\overline{i}(-2-3)-\overline{j}(7-3)+\overline{k}(7+2)$$

$$\overline{a}\times\overline{c}=-4\overline{i}-4\overline{j}+9\overline{k}$$

$$\overline{a}\times(\overline{b}+\overline{c})=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 7 & -2 & 3\\ 3 & 1 & 9\end{vmatrix}$$

$$=\overline{i}(-18-3)-\overline{j}(63-9)+\overline{k}(7+6)$$

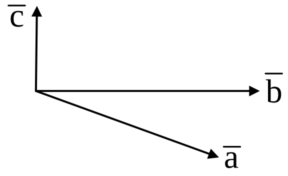
$$\therefore\ \overline{a}\times(\overline{b}+\overline{c})=-21\overline{i}-54\overline{j}+13\overline{k}$$

$$\overline{a}\times\overline{b}+\overline{a}\times\overline{c}=-21\overline{i}-54\overline{j}+13\overline{k}$$

$$\therefore\ \overline{a}\times(\overline{b}+\overline{c})=\overline{a}\times\overline{b}+\overline{a}\times\overline{c}$$

70. If $\bar{a} = \bar{i} + \bar{j} + \bar{k}$, $\bar{c} = \bar{j} - \bar{k}$, then find vector b such that $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

Sol.



Let $\bar{b} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \bar{i}(z-y) - \bar{j}(z-x) + \bar{k}(y-x) = \bar{c}(\text{given})$$

$\bar{a} \times \bar{b} = \bar{c}$

$\Rightarrow \bar{i}(z-y) - \bar{j}(z-x) + \bar{k}(y-x) = \bar{j} - \bar{k}$

$z - y = 0 \quad \ldots(1)$

$x - z = 1 \quad \ldots(2)$

$y - x = -1 \Rightarrow x - y = 1 \ldots(3)$

$\bar{a} \cdot \bar{b} = 3$

$(\bar{i} + \bar{j} + \bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = 3$

$x + y + z = 3 \quad \ldots(4)$

Put $y = z$ in (4)

$x + z + z = 3$

$x + 2z = 3 \quad \ldots(5)$

From (2) and (5)

x + 2y = 3

x – z = 1

3z = 2 $\Rightarrow z = \frac{2}{3} \Rightarrow y = \frac{2}{3}$

Now, we have

x + y + z = 3

$x+\frac{2}{3}+\frac{2}{3}=3$

$x+\frac{4}{3}=3$

$x=3-\frac{4}{3}=\frac{5}{3}$

$\therefore \bar{b}=\frac{5}{3}\bar{i}+\frac{2}{3}\bar{j}+\frac{2}{3}\bar{k}=\frac{1}{3}\left[5\bar{i}+2\bar{j}+2\bar{k}\right]$

71. $\bar{a},\bar{b},\bar{c}$ **are three vectors of equal magnitudes and each of them is inclined at an angle of 60° to the others. If** $|\bar{a}+\bar{b}+\bar{c}|=\sqrt{6}$**, then find** $|\bar{a}|$**.**

Sol. $|\bar{a}+\bar{b}+\bar{c}|=\sqrt{6}$

$\Rightarrow |\bar{a}+\bar{b}+\bar{c}|^2=6$

$\Rightarrow \bar{a}^2+\bar{b}^2+\bar{c}^2+2\bar{a}\bar{b}+2\bar{b}\bar{c}+2\bar{c}\,\bar{a}=6$

Let $|\bar{a}|=|\bar{b}|=|\bar{c}|=a$

$\Rightarrow a^2+a^2+a^2+2a^2\cos(\bar{a},\bar{b})+2a^2\cos(\bar{b},\bar{c})+2a^2\cos(\bar{c},\bar{a})=6$

$\Rightarrow 3a^2+2a^2\cos 60°+2a^2\cos 60°+2a^2\cos 60°=6$

$\Rightarrow 3a^2+6a^2\cos 60°=6$

$\Rightarrow 3a^2+6a^2\times\frac{1}{2}=6$

$\Rightarrow 3a^2+3a^2=6$

$\Rightarrow 6a^2=6$

$\Rightarrow a^2=1\Rightarrow a=1\Rightarrow|\bar{a}|=1$

72. $\bar{a}=3\bar{i}-\bar{j}+2\bar{k},\bar{b}=-\bar{i}+3\bar{j}+2\bar{k}$**,** $\bar{c}=4\bar{i}+5\bar{j}-2\bar{k}$ and $\bar{d}=\bar{i}+3\bar{j}+5\bar{k}$**,then compute the following.**

i) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})$

ii) $(\bar{a}\times\bar{b})\cdot\bar{c}-(\bar{a}\times\bar{d})\cdot\bar{b}$

Sol. i) $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & -1 & 2\\ -1 & 3 & 2\end{vmatrix}$

$=\bar{i}(-2-6)-\bar{j}(6+2)+\bar{k}(9-1)$

$\bar{a}\times\bar{b}=-8\bar{i}-8\bar{j}+8\bar{k}$

$$\bar{c}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 4 & 5 & -2\\ 1 & 3 & 5\end{vmatrix}$$

$$=\bar{i}(25+6)-\bar{j}(20+2)+\bar{k}(12-5)$$

$$=31\bar{i}-22\bar{j}+7\bar{k}$$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -8 & -8 & 8\\ 31 & -22 & 7\end{vmatrix}$$

$$=\bar{i}(-56+176)-\bar{j}(-56-248)+\bar{k}(176+248)$$

$$\therefore(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=120\bar{i}+304\bar{j}+424\bar{k}$$

ii) $(\bar{a}\times\bar{b})\cdot\bar{c}=(-8\bar{i}-8\bar{j}+8\bar{k})\cdot(3\bar{i}-\bar{j}+2\bar{k})$

$$=-24+8+16$$

$$(\bar{a}\times\bar{b})\cdot\bar{c}=0$$

$$\bar{a}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & -1 & 2\\ 1 & 3 & 5\end{vmatrix}$$

$$=\bar{i}(-5-6)-\bar{j}(15-2)+\bar{k}(9+1)$$

$$=-11\bar{i}-13\bar{j}+10\bar{k}$$

$$(\bar{a}\times\bar{d})\cdot\bar{b}(-11\bar{i}-13\bar{j}+10\bar{k})\cdot(-\bar{i}+3\bar{j}+2\bar{k})$$

$$=11-39+20=-8$$

$$\therefore(\bar{a}\times\bar{b})\cdot\bar{c}-(\bar{a}\times\bar{d})\cdot\bar{b}=0-(-8)$$

$$=0+8=8$$

73. If $\bar{a},\bar{b}$ and $\bar{c}$ are mutually perpendicular unit vectors then find the value of $\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2$.

Sol. **Case (i) :** Let $\bar{a},\bar{b},\bar{c}$ form a right hand system

$$\Rightarrow\bar{b}\times\bar{c}=\bar{a}$$

$$\Rightarrow\left[\bar{a}\ \bar{b}\ \bar{c}\right]=\bar{a}\cdot(\bar{b}\times\bar{c})=\bar{a}\cdot\bar{a}=|\bar{a}|^2=1$$

$$\therefore\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2=1$$

Case (ii) : Let $\bar{a}, \bar{b}, \bar{c}$ form a left hand system

$\Rightarrow (\bar{b} \times \bar{c}) = -\bar{a}$

$\Rightarrow \left[\bar{a}\ \bar{b}\ \bar{c}\right] = \bar{a} \cdot (\bar{b} \times \bar{c})$

$= -(\bar{a} \cdot \bar{a}) = -|\bar{a}|^2 = -1$

$\therefore \left[\bar{a}\ \bar{b}\ \bar{c}\right]^2 = 1$

$\therefore$ In both cases we have $\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2 = 1$.

74. If $\bar{a}, \bar{b}$ and $\bar{c}$ are non-zero vectors and a is perpendicular to both b and c. If $|\bar{a}| = 2, |\bar{b}| = 3, |\bar{c}| = 4$ and $(\bar{b}, \bar{c}) = \frac{2\pi}{3}$, then find $\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$.

Sol. If $\bar{a}$ is perpendicular to $\bar{b}$ and $\bar{c}$.

$\Rightarrow \bar{a}$ is parallel to $\bar{b} \times \bar{c}$

$\Rightarrow \left[\bar{a}, \bar{b} \times \bar{c}\right] = 0$

$\Rightarrow \bar{b} \times \bar{c} = |\bar{b}||\bar{c}| \sin(\bar{b}, \bar{c})\hat{a}$

$\Rightarrow |\bar{b} \times \bar{c}| = 3 \times 4 \sin\frac{2\pi}{3}\hat{a}$

$\Rightarrow |\bar{b} \times \bar{c}| = 12 \sin 120.1 = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$

$\therefore \left[|\bar{a}\ \bar{b}\ \bar{c}|\right] = |\bar{a} \cdot (\bar{b} \times \bar{c})| = |\bar{a}||\bar{b} \times \bar{c}| \cos(\bar{a}\ \bar{b}\ \bar{c})$

$= (2 \cdot 6\sqrt{3}) \cos 0 = 12\sqrt{3}$

$\therefore |\bar{a} \cdot \bar{b} \times \bar{c}| = (2 \cdot 6\sqrt{3}) = 12\sqrt{3}$

75. If $\left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right] + \left[\bar{c}\ \ \bar{a}\ \ \bar{d}\right] + \left[\bar{a}\ \ \bar{b}\ \ \bar{d}\right] = \left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$ then show that $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar.

Sol. Let O be the origin, then

$\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}, \overline{OD} = \bar{d}$ are position vectors.

Then $\overline{AB} = \bar{b} - \bar{a}, \overline{AC} = \bar{c} - \bar{a}$ and $\overline{AD} = \bar{d} - \bar{a}$

The vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\therefore \left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right] = 0$

$\Rightarrow \left[\bar{b}-\bar{a}\quad \bar{c}-\bar{a}\quad \bar{d}-\bar{a}\right] = 0$

$\Rightarrow (\bar{b}-\bar{a})\times(\bar{c}-\bar{a})\cdot(\bar{d}-\bar{a}) = 0$

$\Rightarrow (\bar{b}\times\bar{c}-\bar{b}\times\bar{a}-\bar{a}\times\bar{c}+\bar{a}\times\bar{a})\cdot(\bar{d}-\bar{a}) = 0$

$\Rightarrow (\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a})\cdot(\bar{d}-\bar{a}) = 0$

$(\because \bar{a}\times\bar{a} = 0)$

$\Rightarrow (\bar{b}\times\bar{c})\cdot\bar{d}+(\bar{a}\times\bar{b})\cdot\bar{d}+(\bar{c}\times\bar{a})\cdot\bar{d}-(\bar{b}\times\bar{c})\cdot\bar{a}-(\bar{a}\times\bar{b})\cdot\bar{a}-(\bar{c}\times\bar{a})\cdot\bar{a} = 0$

$\Rightarrow (\bar{b}\times\bar{c})\cdot\bar{d}+(\bar{a}\times\bar{b})\cdot\bar{d}+(\bar{c}\times\bar{a})\cdot\bar{d}-(\bar{b}\times\bar{c})\cdot\bar{a} = 0$

$\Rightarrow \left[\bar{b}\quad \bar{c}\quad \bar{d}\right]+\left[\bar{a}\quad \bar{b}\quad \bar{d}\right]+\left[\bar{c}\quad \bar{a}\quad \bar{d}\right] = \left[\bar{a}\quad \bar{b}\quad \bar{c}\right]$

76. If $\bar{a},\bar{b},\bar{c}$ non-coplanar vectors then prove that the four points with position vectors $2\bar{a}+3\bar{b}-\bar{c}, \bar{a}-2\bar{b}+3\bar{c}, 3\bar{a}+4\bar{b}-2\bar{c}$ and $\bar{a}-6\bar{b}+6\bar{c}$ are coplanar.

Sol. Let A, B, C, D be the position vectors of given vectors.

Then $\overline{OA} = 2\bar{a}+3\bar{b}-\bar{c}, \overline{OB} = \bar{a}-2\bar{b}+3\bar{c}$

$\overline{OC} = 3\bar{a}+4\bar{b}-2\bar{c}, \overline{OD} = \bar{a}-6\bar{b}+6\bar{c}$

$\overline{AB} = \overline{OB}-\overline{OA} = -\bar{a}-5\bar{b}+4\bar{c}$

$\overline{AC} = \overline{OC}-\overline{OA} = \bar{a}+\bar{b}-\bar{c}$

$\overline{AD} = \overline{OD}-\overline{OA} = -\bar{a}-9\bar{b}+7\bar{c}$

Let $\overline{AB} = x\overline{AC}+y\overline{AD}$ where x, y are scalars.

$-\bar{a}-5\bar{b}+4\bar{c} = x(\bar{a}+\bar{b}-\bar{c})+y(-\bar{a}-9\bar{b}+7\bar{c})$

$-\bar{a}-5\bar{b}+4\bar{c} = (x-y)\bar{a}+(x-9y)\bar{b}+(-x+7y)\bar{c}$

Comparing $\bar{a},\bar{b},\bar{c}$ coefficients on both sides

$x - y = -1$ …(1)

$x - 9y = -5$ …(2)

$-x + 7y = 4$ …(3)

(1) – (2) $\Rightarrow 8y = 4 \Rightarrow y = \frac{1}{2}$

From (1) : $x = -\frac{1}{2}$

$\Rightarrow \frac{1}{2}+\frac{7}{2}=4 \Rightarrow \frac{8}{2}=4 \Rightarrow 4=4$

$\therefore$ Given vectors are coplanar.

77. Show that the equation of the plane passing through the points with position vectors $3\bar{i}-5\bar{j}-\bar{k}, -\bar{i}+5\bar{j}+7\bar{k}$ and parallel to the vector $3\bar{i}-\bar{j}+7\bar{k}$ is $3x + 2y - z = 0$.

Sol. Let $\overline{OA}=3\bar{i}-5\bar{j}-\bar{k}, \overline{OB}=-\bar{i}+5\bar{j}+7\bar{k}$

$\overline{OC}=3\bar{i}-\bar{j}+7\bar{k}$

Let $P(x\bar{i}+y\bar{j}+z\bar{k})$ be any point on the plane with position vector.

Such that $\overline{OP}=x\bar{i}+y\bar{j}+z\bar{k}$

$\overline{AP}=\overline{OP}-\overline{OA}=x\bar{i}+y\bar{j}+z\bar{k}-3\bar{i}+5\bar{j}+\bar{k}$

$=(x-3)\bar{i}+(y+5)\bar{j}+(z+1)\bar{k}$

$\overline{AB}=\overline{OB}-\overline{OA}=-\bar{i}+5\bar{j}+7\bar{k}-3\bar{i}+5\bar{j}+\bar{k}$ $\therefore$ The vector equation of the plane passing

$=-4\bar{i}+10\bar{j}+8\bar{k}$

$\bar{C}=3\bar{i}-\bar{j}+7\bar{k}$

through A, B and parallel to $\bar{C}$ is :

$[\overline{AP}\ \ \overline{AB}\ \ \bar{C}]=0$

$$\Rightarrow \begin{vmatrix} x-3 & y+5 & z+1 \\ -4 & 10 & 8 \\ 3 & -1 & 7 \end{vmatrix}=0$$

$\Rightarrow (x-3)[70+8]-(y+5)[-28-24] \quad +(z+1)[4-30]=0$

$\Rightarrow (x-3)78+(y+5)52+(z+1)(-26)=0$

$\Rightarrow 26\left[(x+1)3+(y+5)2+(z+1)(-1)\right]=0$

$\Rightarrow 3x-9+2y+10-z-1=0$

$\Rightarrow 3x+2y-z=0$

78. Find the vector equation of the plane passing through the intersection of planes $\bar{r}\cdot(2\bar{i}+2\bar{j}-3\bar{k})=7, \bar{r}\cdot(2\bar{i}+5\bar{j}+3\bar{k})=9$ and through the point (2, 1, 3).

Sol. Cartesian form the given planes is

$2x + 2y - 3z = 7$ …(1) and

$2x + 5y + 3z = 9$ …(2)

Equation of the required plane will be in the form

$(2x + 2y - 3z - 7) + \lambda(2x + 5y + 3z - 9) = 0$

Since it is passing through the point (2,1,3)

$[2(2) + 2(1) - 3(3) - 7] + \lambda[2(2) + 5(1) + 3(3) - 9] = 0$

$(4 + 2 - 9 - 7) + \lambda(4 + 5 + 9 - 9) = 0$

$-10 + 9\lambda = 0$

$9\lambda = 10 \Rightarrow \lambda = \frac{10}{9}$

Required plane is :

$(2x+2y-3z-7)+\frac{10}{9}(2x+5y+3z-9)=0$

$18x+18y-27z-63+20x+50y+30z-90=0$

$38x+68y+3z-153=0$

Its vector form is

$\bar{r}\cdot(38\bar{i}+68\bar{j}+3\bar{k})=153$.

79. Find the shortest distance between the lines $\bar{r}=6\bar{i}+2\bar{j}+2\bar{k}+\lambda(\bar{i}-2\bar{j}+2\bar{k})$ and $\bar{r}=-4\bar{i}-\bar{k}+\mu(3\bar{i}-2\bar{j}-2\bar{k})$.

Sol. Given lines are

$\bar{r}=6\bar{i}+2\bar{j}+2\bar{k}+\lambda(\bar{i}-2\bar{j}+2\bar{k})$

$\bar{r}=-4\bar{i}-\bar{k}+\mu(3\bar{i}-2\bar{j}-2\bar{k})$

Let $\bar{a}=6\bar{i}+2\bar{j}+2\bar{k}, \bar{b}=\bar{i}-2\bar{j}+2\bar{k}$

$\bar{c}=-4\bar{i}-\bar{k}, \bar{d}=3\bar{i}-2\bar{j}-2\bar{k}$

Shortest distance between the given lines is

$$\frac{\left|[\bar{a}-\bar{c}\ \ \bar{b}\ \ \bar{d}]\right|}{|\bar{b}\times\bar{d}|}$$

$\bar{a}-\bar{c}=10\bar{i}+2\bar{j}+3\bar{k}$

$$\left|[\bar{a}-\bar{c}\ \ \bar{b}\ \ \bar{d}]\right|=\begin{vmatrix}10 & 2 & 3\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$$

$=10(4+4)-2(-2-6)+3(-2+6)$

$=80+16+12=108$

$$[\bar{b}\times\bar{d}]=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$$

$=\bar{i}(4+4)-\bar{j}(-2-6)+\bar{k}(-2+6)$

$=8\bar{i}+8\bar{j}+4\bar{k}$

$|\bar{b}\times\bar{d}|=\sqrt{64+64+16}=\sqrt{144}=12$

$\therefore$ Distance = $\frac{108}{12}=9$ units.

80. If $\bar{a},\bar{b},\bar{c}$ are the position vectors of the points A, B and C respectively. Then prove that the vector $\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}$ is perpendicular to the plane of ΔABC.

Sol. We have

$\overline{AB}=\bar{b}-\bar{a},\overline{BC}=\bar{c}-\bar{b}$ and $\overline{CA}=\bar{a}-\bar{c}$

Let $\bar{r}=\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}$

then $\bar{r}\cdot\overline{AB}=\bar{r}\cdot(\bar{b}-\bar{a})$

$=(\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a})\cdot(\bar{b}-\bar{a})$

$=\bar{a}\times\bar{b}\cdot\bar{b}-\bar{a}\times\bar{b}\cdot\bar{a}+\bar{b}\times\bar{c}\cdot\bar{b}\ -\bar{b}\times\bar{c}\cdot\bar{a}+\bar{c}\times\bar{a}\cdot\bar{b}-\bar{c}\times\bar{a}\cdot\bar{a}$

$=\left[\bar{a}\ \ \bar{b}\ \ \bar{b}\right]-\left[\bar{a}\ \ \bar{b}\ \ \bar{a}\right]+\left[\bar{b}\ \ \bar{c}\ \ \bar{b}\right]-\left[\bar{b}\ \ \bar{c}\ \ \bar{a}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{b}\right]-\left[\bar{c}\ \ \bar{a}\ \ \bar{a}\right]$

$=-\left[\bar{b}\ \ \bar{c}\ \ \bar{a}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{b}\right]$ $(\because\left[\bar{a}\bar{b}\bar{b}\right]=0)$

$=0$ $(\because\left[\bar{c}\ \ \bar{a}\ \ \bar{b}\right]=\left[\bar{b}\ \ \bar{c}\ \ \bar{a}\right])$

Thus $\bar{r}$ is perpendicular to $\overline{AB}$

($\because$ neither of them is zero vector)

Similarly we can show that $\bar{r}\cdot\overline{BC}=0$ and hence $\bar{r}$ is also perpendicular to $\overline{BC}$. Since $\bar{r}$ is perpendicular to two lines in the plane ΔABC, it is perpendicular to the plane ΔABC.

81. Show that $(\bar{a}\times(\bar{b}\times\bar{c}))\times\bar{c}=(\bar{a}\cdot\bar{c})(\bar{b}\times\bar{c})$ **&** $(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{c})+(\bar{a}\cdot\bar{b})(\bar{a}\cdot\bar{c})=(\bar{a}\cdot\bar{a})(\bar{b}\cdot\bar{c})$**.**

Sol. $[\bar{a}\times(\bar{b}\times\bar{c})]\times\bar{c}=[(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}]\times\bar{c}$

$=(\bar{a}\cdot\bar{c})(\bar{b}\times\bar{c})=(\bar{a}\cdot\bar{b})(\bar{c}\times\bar{c})$

$=(\bar{a}\cdot\bar{c})(\bar{b}\times\bar{c})-(\bar{a}\cdot\bar{b})(0)$

$[\bar{a}\times(\bar{b}\times\bar{c})]\times\bar{c}=(\bar{a}\cdot\bar{c})(\bar{b}\times\bar{c})$

$(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{c})+(\bar{a}\cdot\bar{b})(\bar{a}\cdot\bar{c})=(\bar{a}\cdot\bar{b})(\bar{b}\cdot\bar{c})$

$(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{c})=\begin{vmatrix}\bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{c}\\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{c}\end{vmatrix}=(\bar{a}\cdot\bar{a})(\bar{b}\cdot\bar{c})-(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{a})$

$\text{L.H.S.}=(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{c})+(\bar{a}\cdot\bar{b})(\bar{a}\cdot\bar{c})$

$=(\bar{a}\cdot\bar{a})(\bar{b}\cdot\bar{c})-(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{a})+(\bar{a}\cdot\bar{b})(\bar{a}\cdot\bar{c})$

$=(\bar{a}\cdot\bar{a})(\bar{b}\cdot\bar{c})=\text{R.H.S.}$

82. If A = (1, –2, –1), B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5) find the shortest distance between AB and CD.

Sol. Let O be the origin

Let $\overline{OA}=\bar{i}-2\bar{j}-\bar{k},\overline{OB}=4\bar{i}-3\bar{k}$

$\overline{OC}=\bar{i}+2\bar{j}-\bar{k},\overline{OD}=2\bar{i}-4\bar{j}-5\bar{k}$

The vector equation of a line passing through A, B is

$\bar{r}=(1-t)\bar{a}+t\bar{b},t\in R$

$=\bar{a}+t(\bar{b}-\bar{a})$

$=\bar{i}-2\bar{j}-\bar{k}+t(4\bar{i}-3\bar{k}-\bar{i}+2\bar{j}+\bar{k})$

$=\bar{i}-2\bar{j}-\bar{k}+t(3\bar{i}+2\bar{j}-2\bar{k})$

$=\bar{a}+t\bar{b}$

where $\bar{a}=\bar{i}-2\bar{j}-\bar{k},\bar{b}=3\bar{i}+2\bar{j}-2\bar{k}$

The vector equation of a line passing through C, D is

$\bar{r} = (1-s)\bar{c} + s\bar{d}, s \in R$

$\bar{r} = \bar{c} + s(\bar{d} - \bar{c})$

$= \bar{i} + 2\bar{i} - \bar{k} + s[2\bar{i} - 4\bar{j} - 5\bar{k} - \bar{i} - 2\bar{j} + \bar{k}]$

$= \bar{i} + 2\bar{j} - \bar{k} + s[\bar{i} - 6\bar{j} - 4\bar{k}]$

$= \bar{c} + s\bar{d}$

where $\bar{c} = \bar{i} + 2\bar{j} - \bar{k}, \bar{d} = \bar{i} - 6\bar{j} - 4\bar{k}$

$$\bar{b}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix}$$

$= \bar{i}[-8-12] - \bar{j}[-12+2] + \bar{k}[-18-2]$

$= -20\bar{i} + 10\bar{j} - 20\bar{k} = 10[-2\bar{i} + \bar{j} - 2\bar{k}]$

$|\bar{b}\times\bar{d}| = 10\sqrt{4+1+4} = 10\cdot 3 = 30$

$\bar{a} - \bar{c} = \bar{i} - 2\bar{j} - \bar{k} - \bar{i} - 2\bar{j} + \bar{k} = -4\bar{j}$

$$\frac{[\bar{a}-\bar{c}\ \bar{b}\cdot\bar{d}]}{|\bar{b}\times\bar{d}|} = \frac{(\bar{a}-\bar{c})\cdot(\bar{b}\times\bar{d})}{|\bar{b}\times\bar{d}|}$$

$$= \frac{-4\bar{j}\cdot 10[-2\bar{i} + \bar{j} - 2\bar{k}]}{30} = \frac{10[4]}{30} = \frac{40}{30} = \frac{4}{3}$$

∴ The shortest distance between the lines = 4/3.

83. If $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k}, \bar{b} = \bar{i} - 2\bar{j} + \bar{k}$**,** $\bar{c} = -\bar{i} + \bar{j} - 4\bar{k}$ and $\bar{d} = \bar{i} + \bar{j} + \bar{k}$ **then compute** $|(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})|$**.**

Sol. $$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$= \bar{i}(1-6) - \bar{j}(2+3) + \bar{k}(-4-1)$

$\bar{a}\times\bar{b} = -5\bar{i} - 5\bar{j} - 5\bar{k}$

$$\bar{c}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 1 & -4 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}(1+4) - \bar{j}(-1+4) + \bar{k}(-1-1)$$

$\bar{c}\times\bar{d}=5\bar{i}-3\bar{j}-2\bar{k}$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -5 & -5 & -5\\ 5 & -3 & -2\end{vmatrix}=\bar{i}(10-15)-\bar{j}(10+25)+\bar{k}(15+25)$$

$$=-5\bar{i}-35\bar{j}+40\bar{k}$$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=+5\left[-\bar{i}-7\bar{j}+8\bar{k}\right]$$

$$|(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})|=+5\sqrt{1+49+64}$$

$$\therefore|(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})|=+5\sqrt{114}$$

84. If $\bar{A}=(1\ \bar{a}\ \bar{a}^2),\ \bar{B}=(1\ \bar{b}\ \bar{b}^2)$ and $\bar{c}=(1\ \bar{c}\ \bar{c}^2)$ are non-coplanar vectors and $\begin{vmatrix}\bar{a} & \bar{a}^2 & 1+\bar{a}^3\\ \bar{b} & \bar{b}^2 & 1+\bar{b}^3\\ \bar{c} & \bar{c}^2 & 1+\bar{c}^3\end{vmatrix}=0$ then show that $(\bar{a}\ \bar{b}\ \bar{c}+1)=0$.

Sol. Given $\begin{vmatrix}\bar{a} & \bar{a}^2 & 1+\bar{a}^3\\ \bar{b} & \bar{b}^2 & 1+\bar{b}^3\\ \bar{c} & \bar{c}^2 & 1+\bar{c}^3\end{vmatrix}=0$

$$\begin{vmatrix}\bar{a} & \bar{a}^2 & 1\\ \bar{b} & \bar{b}^2 & 1\\ \bar{c} & \bar{c}^2 & 1\end{vmatrix}+\begin{vmatrix}\bar{a} & \bar{a}^2 & \bar{a}^3\\ \bar{b} & \bar{b}^2 & \bar{b}^3\\ \bar{c} & \bar{c}^2 & \bar{c}^3\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}\bar{a} & \bar{a}^2 & 1\\ \bar{b} & \bar{b}^2 & 1\\ \bar{c} & \bar{c}^2 & 1\end{vmatrix}+\bar{a}\,\bar{b}\,\bar{c}\begin{vmatrix}1 & \bar{a} & \bar{a}^2\\ 1 & \bar{b} & \bar{b}^2\\ 1 & \bar{c} & \bar{c}^2\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}\bar{a} & \bar{a}^2 & 1\\ \bar{b} & \bar{b}^2 & 1\\ \bar{c} & \bar{c}^2 & 1\end{vmatrix}+\underset{(-)}{\bar{a}\,\bar{b}\,\bar{c}}\begin{vmatrix}\bar{a} & 1 & \bar{a}^2\\ \bar{b} & 1 & \bar{b}^2\\ \bar{c} & 1 & \bar{c}^2\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}\bar{a} & \bar{a}^2 & 1\\ \bar{b} & \bar{b}^2 & 1\\ \bar{c} & \bar{c}^2 & 1\end{vmatrix}+\underset{(-)(-)}{\bar{a}\,\bar{b}\,\bar{c}}\begin{vmatrix}\bar{a} & \bar{a}^2 & 1\\ \bar{b} & \bar{b}^2 & 1\\ \bar{c} & \bar{c}^2 & 1\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}\bar{a} & \bar{a}^2 & 1\\ \bar{b} & \bar{b}^2 & 1\\ \bar{c} & \bar{c}^2 & 1\end{vmatrix}(1+\bar{a}\,\bar{b}\,\bar{c})=0$$

$$\bar{a}\,\bar{b}\,\bar{c}+1=0$$

$(\because \bar{a}\,\bar{b}\,\bar{c}$ are non-coplanar vectors)

$\Rightarrow \bar{a}, \bar{b}, \bar{c} = -1$

85. If $\bar{a}, \bar{b}, \bar{c}$ are non-zero vectors $|(\bar{a}\times\bar{b})\bar{c}| = |\bar{a}||\bar{b}||\bar{c}| \Leftrightarrow \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{c} = \bar{c}\cdot\bar{a} = 0$.

Sol. Given $\bar{a} \neq 0, \bar{b} \neq 0$ and $\bar{c} \neq 0$

$|\bar{a}\times\bar{b}\cdot\bar{c}| = |\bar{a}||\bar{b}||\bar{c}|$

$\Rightarrow |\bar{a}\times\bar{b}||\bar{c}|\cos\left((\bar{a}\times\bar{b})\cdot\bar{c}\right) = |\bar{a}||\bar{b}||\bar{c}|$

$\Rightarrow |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})\cdot\cos(\bar{a}\times\bar{b}\cdot\bar{c}) = |\bar{a}||\bar{b}|$

$\Rightarrow \sin(\bar{a},\bar{b})\cdot\cos(\bar{a}\times\bar{b}\cdot\bar{c}) = 1$

$\Rightarrow \sin(\bar{a},\bar{b}) = 1$ and $\cos(\bar{a}\times\bar{b}\cdot\bar{c}) = 1$

$\Rightarrow \bar{a}\cdot\bar{b} = 90°$ and $\bar{a}\times\bar{b}\cdot\bar{c} = 0$

$\Rightarrow \bar{a}\cdot\bar{b} = 90°$ and $\bar{a}\times\bar{b}$ parallel to $\bar{c}$

$\bar{a}\cdot\bar{b} = 90°$ and $\bar{a}, \bar{b}$ are perpendicular to $\bar{c}$

$\Rightarrow \bar{a}\cdot\bar{b} = 0°$ and $\bar{a}\cdot\bar{c} = \bar{b}\cdot\bar{c} = 0$

$\Rightarrow \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{c} = \bar{c}\cdot\bar{a} = 0$

86. If $|\bar{a}| = 1, |\bar{b}| = 1, |\bar{c}| = 2$ and $\bar{a}\times(\bar{a}\times\bar{c}) + \bar{b} = 0$, then find the angle between $\bar{a}$ and $\bar{c}$.

Sol. Given that $|\bar{a}| = 1, |\bar{b}| = 1, |\bar{c}| = 2$

Let $(\bar{a}, \bar{c}) = \theta$

Consider $\bar{a}\cdot\bar{c} = |\bar{a}||\bar{c}|\cos\theta$

$= (1)(2)\cos\theta$

$= 2\cos\theta$ ...(1)

Consider $\bar{a}\times(\bar{a}\times\bar{c}) + \bar{b} = 0$

$(\bar{a}\cdot\bar{c})\bar{a} - (\bar{a}\cdot\bar{a})\bar{c} + \bar{b} = \bar{0}$

$(2\cos\theta)\bar{a} - (1)\bar{c} + \bar{b} = 0$ ...(2)

$(2\cos\theta)\bar{a} - \bar{c} = -\bar{b}$

Squaring on both sides

$[(2\cos\theta)\bar{a} - \bar{c}]^2 = (-\bar{b})^2$

$\Rightarrow (4\cos^2\theta)(\bar{a})^2 + (\bar{c})^2 - 4\cos\theta(\bar{a}\cdot\bar{c}) = \bar{b}^2$

$\Rightarrow 4\cos^2\theta(1) + (2)^2 - 4\cos\theta(2\cos\theta) = 1$

$\Rightarrow 4\cos^2\theta + 4 - 8\cos^2\theta = 1$

$\Rightarrow 4 - 4\cos^2\theta = 1$

$\Rightarrow 4\cos^2\theta = 3$

$$\Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$$

Case I :

$$\text{If } \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\Rightarrow (\bar{a}, \bar{c}) = \frac{\pi}{6} = 30°$$

Case II :

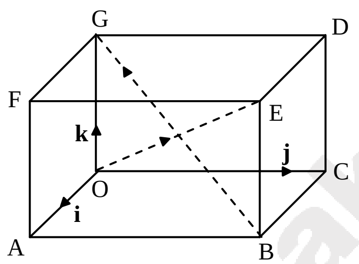
$$\text{If } \cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} = 150°$$

$$\Rightarrow (\bar{a}, \bar{c}) = \frac{5\pi}{6} = 150°$$

87. Prove that the smaller angle θ between any tow diagonals of a cube is given by cos θ = 1/3.

Sol. Without loss of generality we may assume that the cube is a unit cube.

∴ Let $\overline{OA} = \bar{i}, \overline{OC} = \bar{j}$ and $\overline{OG} = \bar{k}$ be coterminus edges of the cube.



∴ Diagonal $\overline{OE} = \bar{i} + \bar{j} + \bar{k}$ and diagonal $\overline{BG} = -\bar{i} - \bar{j} + \bar{k}$.

Let θ be the smaller angle between the diagonals OE and BG.

Then $\cos\theta = \frac{|\overline{OE} \cdot \overline{BG}|}{|\overline{OE}||\overline{BG}|} = \frac{|-1-1+1|}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$

88. The altitudes of a triangle are concurrent

Proof : Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$ be the position vectors of the vertices of of a triangle ABC

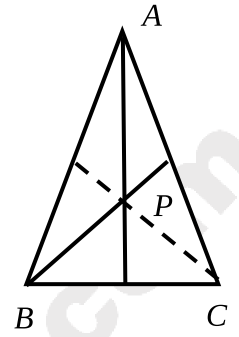
Let the altitudes through A and B meet at p. let $\overrightarrow{OP} = \vec{r}$ now

$\overrightarrow{AP} \perp^r \overrightarrow{BC} \Rightarrow \overrightarrow{AP}.\overrightarrow{BC} = 0$

$(\vec{r}-\vec{a}).(\vec{c}-\vec{b})=0 \Rightarrow \vec{r}.(\vec{c}-\vec{b})=\vec{a}.(\vec{c}-\vec{b}) \rightarrow (1)$

Also $\overrightarrow{BP} \perp^r \overrightarrow{BC} \Rightarrow \overrightarrow{BP}.\overrightarrow{CA}=0$

$(\vec{r}-\vec{a}).(\vec{a}-\vec{c})=0 \Rightarrow \vec{r}.(\vec{a}-\vec{c})=\vec{b}.(\vec{a}-\vec{c}) \rightarrow (2)$



(1)+(2) $\Rightarrow \vec{r}.(\vec{c}-\vec{b})+\vec{r}(\vec{a}-\vec{c})=\vec{a}.(\vec{c}-\vec{b})+\vec{b}.(\vec{a}-\vec{c})$

$\vec{r}.(\vec{a}-\vec{b})=\vec{c}.(\vec{b}-\vec{a})$

$\vec{r}.(\vec{b}-\vec{a})-\vec{c}.(\vec{b}-\vec{a})=0$

$(\vec{r}-\vec{c}).(\vec{b}-\vec{a})=0$

$\overrightarrow{CP}.\overrightarrow{AB}=0 \qquad \therefore \overrightarrow{CP} \perp^r \overrightarrow{AB}$

$\therefore$ Altitude through C also passes through

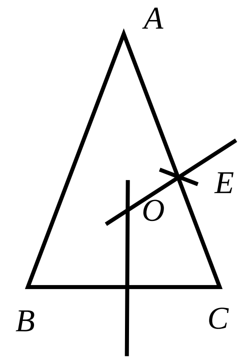
$\therefore$ Altitudes are concurrent

89. The perpendicular bisectors of sides of a triangle are concurrent.

Proof: Let A ,B, C be the vertices of a triangle with position vectors $\vec{a},\vec{b},\vec{c}$.

Let D, E, F be the mid points of BC, CA, AB respectively Let 'O' be point of intersection of perpendicular bisectors of BC and AC

$\overrightarrow{OD}=\frac{\vec{b}+\vec{c}}{2} \qquad \overrightarrow{OE}=\frac{\vec{a}+\vec{c}}{2}$



$\overrightarrow{OD} \perp^r \overrightarrow{BC} \Rightarrow \overrightarrow{OD}.\overrightarrow{BC}=0$

$\left(\frac{\vec{b}+\vec{c}}{2}\right).(\vec{c}-\vec{b})=0$

$(\vec{c})^2-(\vec{b})^2=0 \rightarrow (1)$

$\overrightarrow{OE}=\frac{\vec{a}+\vec{c}}{2} \qquad \overrightarrow{AC}=\vec{c}-\vec{a}$

$\overrightarrow{OE} \perp^r \overrightarrow{CA} \Rightarrow \overrightarrow{OE}.\overrightarrow{CA}=0$

$\left(\frac{\vec{a}+\vec{c}}{2}\right).(\vec{a}-\vec{c})=0$

$\Rightarrow (\vec{a})^2-(\vec{c})^2=0 \rightarrow (2)$

(1)+(2) we have $(\vec{a})^2-(\vec{b})^2=0 \Rightarrow (\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$

$\left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{a}-\vec{b})=0 \Rightarrow \overrightarrow{OF}.\overrightarrow{BA}=0$

$\overrightarrow{OF} \perp^r \overrightarrow{BA}$

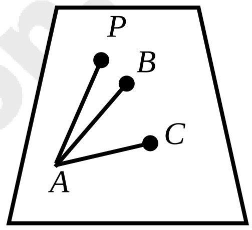
$\therefore \perp^r$ bisector of AB also passes through O

Hence perpendicular bisectors are concurrent.

90. The vector equation of plane passing through the points A, B, C having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $[\vec{r}-\vec{a} \;\; \vec{b}-\vec{a} \;\; \vec{c}-\vec{a}] = 0$ **(or)** $\vec{r}.\{(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})\} = [\vec{a}\;\vec{b}\;\vec{c}]$

Sol: Let $\overrightarrow{OP} = \vec{r}$ be any point on the plane $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ are the given points

$\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar



$[\overrightarrow{AP}\;\overrightarrow{AB}\;\overrightarrow{AC}] = 0$

$[\vec{r}-\vec{a} \;\; \vec{b}-\vec{a} \;\; \vec{c}-\vec{a}] = 0$

$(\vec{r}-\vec{a}).\,(\vec{b}-\vec{a})\times(\vec{c}-\vec{a}) = 0$

$(\vec{r}-\vec{a}).\,\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\} = 0$

$\vec{r}.\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\} - \vec{a}.\vec{b}\times\vec{c} - \vec{a}.\vec{c}\times\vec{a} - \vec{a}.\vec{a}\times\vec{b} = 0$

$\vec{r}.\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\} = [\vec{a}\,\vec{b}\,\vec{c}]\left\{\therefore \vec{a}.\vec{c}\times\vec{a} = 0 \;\; \vec{a}.\vec{a}\times\vec{b} = 0\right\}$

91. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then

i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ **ii)** $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{b}$

Proof : i) Let $\vec{a} = a_1\vec{l} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{l} + b_2\vec{j} + b_3\vec{k}$, $\vec{c} = c_1\vec{l} + c_2\vec{j} + c_3\vec{k}$ be three vectors $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{l}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \vec{l}\{c_3(a_3b_1 - a_1b_3) - c_2(a_1b_2 - a_2b_1)\} - \vec{j}\{c_3(a_2b_3 - a_3b_2 - c_1(a_1b_2 - a_2b_1)\} + \vec{k}\{c_2(a_2b_3 - a_3b_2) - c_1(a_3b_1 - a_1b_3)\}$$

$$(\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a} = (a_1c_1 + a_2c_2 + a_3c_3)\{b_1\vec{l} + b_2\vec{j} + b_3\vec{k})$$

$$(\cancel{a_1b_1c_1} + a_2b_1c_2 + a_3b_1c_3 - \cancel{a_1b_1c_2} - a_1b_3c_3)\vec{l} + (a_1b_2c_1 + a_2b_3c_2 + a_3b_3c_3 - a_2b_1c_1 - \cancel{a_2b_2c_2} - a_2b_3c_3)\vec{j}$$
$$+(a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_3b_1c_1 - a_3b_2c_2 - a_3b_3c_3)\vec{k}$$

$$\Rightarrow \{c_3(a_3b_1-a_1b_3)-c_2(a_1b_2-a_2b_1)\}\vec{l}+\vec{j}\{c_3(a_2b_3-a_3b_2)-c_1(a_1b_2-a_2b_1)+\vec{k}\{c_2(a_2b_3-a_3b_2)-c_1(a_3b_1-a_1b_3)\}$$

Hence proved

Proof ii ; $\vec{b}\times\vec{c}=\begin{vmatrix} i & j & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=\vec{l}(b_2c_3-b_3c_2)-\vec{j}(b_1c_3-b_3c_1)+\vec{k}(b_1c_2-b_2c_1)$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix} \vec{i} & \vec{j} & k \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1 \end{vmatrix}$$

$$=\vec{l}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}-\vec{j}\{a_1(b_1c_2-b_2c_1)-a_3(b_1c_2-b_2c_1)\}+\vec{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$

R.H.S. $(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$$(a_1c_1+a_2c_2+a_3c_3)\{b_1\vec{l}+b_2\vec{j}+b_3\vec{k}\}-(a_1b_1+a_2b_2+a_3b_3)\{c_1\vec{l}+c_2\vec{j}+c_3\vec{k}\}$$

$$\Rightarrow \vec{l}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}-\vec{j}\{a_1cb_1c_2-b_2c_1)-a_3(b_1c_2-b_2c_1)\}+\vec{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$

92. If $\vec{a},\vec{b},\vec{c},\vec{d}$ are four vectors then $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

Proof : $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\{(\vec{a}\times\vec{b})\times\vec{c}\}.\vec{d}$ $\{\because$ dot and cross are inter changeable $\}$

$$\{(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}\}.\vec{d}=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{b}.\vec{c})(\vec{a}.\vec{d})=\begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$$

93. If $\vec{a},\vec{b},\vec{c},\vec{d}$ are four vectors then

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}\,]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}\,]\vec{d}=[\vec{a}\,\vec{c}\,\vec{d}\,]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}\,]\vec{a}$$

Proof :- $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{a}\times\vec{b}.\vec{d})\vec{c}-(\vec{a}\times\vec{b}.\vec{c})\vec{d}$

$$=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=(\vec{c}\times\vec{d}.\vec{a}]\vec{b}-(\vec{c}\times\vec{d}.\vec{b}]\vec{a}$$

$$=[\vec{a}\ \vec{c}\ \vec{d}]\vec{b}-(\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

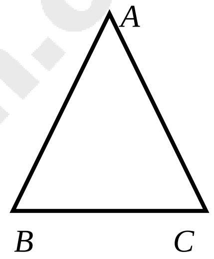
94. **The vector area of a triangle ABC is** $\frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}=\frac{1}{2}\overrightarrow{BC}\times\overrightarrow{BA},\ =\frac{1}{2}\overrightarrow{CA}\times\overrightarrow{CB}$

Sol: In a triangle $\overrightarrow{ABC}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ are the vectors represented by the sides AB, BC, CA

$A=(\overrightarrow{AB},\overrightarrow{AC})\quad B=(\overrightarrow{BA},\overrightarrow{BC})\quad C=(\overrightarrow{CB},\overrightarrow{CA})$

Let $\vec{n}$ be the unit vector $\perp^r$ $\overrightarrow{AB},\overrightarrow{AC}$ and $\overrightarrow{AB},\overrightarrow{AC},\vec{n}$ form right handed system area of triangle ABC

$$\Delta=\frac{1}{2}AB.AC\sin A$$



$$\Delta=\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin A$$

$$\Delta\vec{n}=\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\vec{n}\sin A$$

$$\Delta\vec{n}=\frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}$$

$$\Delta\vec{n}=\frac{1}{2}\overrightarrow{BC}\times\overrightarrow{BA}=\frac{1}{2}\overrightarrow{CA}\times\overrightarrow{CB}$$

95. If $\vec{a},\vec{b},\vec{c}$ **are the prove that of the vertices of the triangle ABC then vector area** $=\frac{1}{2}\{\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\}$

Sol: $\overrightarrow{OA}=\vec{a}$ $\quad\overrightarrow{OB}=\vec{b}$ $\quad\overrightarrow{OC}=\vec{c}$ be the given vertices

Vector area $=\frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}$

$$=\frac{1}{2}\left\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right\}$$

$$=\frac{1}{2}\left\{\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}\right\}$$

$$=\frac{1}{2}\left\{\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right\}$$

96. **In ΔABC the length of the median through the vertex *A* is $\frac{1}{2}\left(2b^2+2c^2-a^2\right)^{1/2}$**

Proof: Let *D* be the mid point of the side *BC*. Take '*A*' as the origin. Let $\overline{AB}=\overline{\alpha}$ and $\overline{AC}=\beta$ so that $(\overline{\alpha},\overline{\beta})=\angle A$

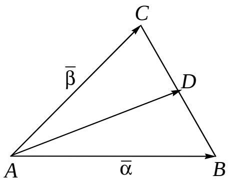


Fig. 2.16

Since $\overline{AD}=\frac{\overline{\alpha}+\overline{\beta}}{2}$, we have $4\overline{AD}^2=\overline{\alpha}^2+\overline{\beta}^2+2\overline{\alpha}.\overline{\beta}=\overline{AB}^2+\overline{AC}^2+2\left|\overline{AB}\right|\left|\overline{AC}\right|\cos\left(\overline{AB},\overline{AC}\right)$

$= c^2 + b^2 + 2bc\cos A = c^2 + b^2 + (b^2+c^2-a^2) = 2b^2 + 2c^2 - a^2$

$\therefore AD=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

97. Theorem : If $\vec{a},\vec{b},\vec{c}$ are three vectors then

i) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$ **ii)** $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{c}.\vec{a})\vec{b}-(\vec{c}.\vec{b})\vec{b}$

Proof : i) Let $\vec{a}=a_1\vec{l}+a_2\vec{j}+a_3\vec{k}$, $\vec{b}=b_1\vec{l}+b_2\vec{j}+b_3\vec{k}$, $\vec{c}=c_1\vec{l}+c_2\vec{j}+c_3\vec{k}$ be three vectors $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\vec{l}(a_2b_3-a_3b_2)-\vec{j}(a_1b_3-a_3b_1)+\vec{k}(a_1b_2-a_2b_1)$

$$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_2b_3-a_3b_2 & a_3b_1-a_1b_3 & a_1b_2-a_2b_1\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\vec{l}\{c_3(a_3b_1-a_1b_3)-c_2(a_1b_2-a_2b_1)\}-\vec{j}\{c_3(a_2b_3-a_3b_2-c_1(a_1b_2-a_2b_1)\}+\vec{k}\{c_2(a_2b_3-a_3b_2)-c_1(a_3b_1-a_1b_3)\}$$

$$(\vec{c}.\vec{a})\vec{b}-(\vec{c}.\vec{b})\vec{a}=(a_1c_1+a_2c_2+a_3c_3)\{b_1\vec{l}+b_2\vec{j}+b_3\vec{k})$$

$$(\cancel{a_1b_1c_1}+a_2b_1c_2+a_3b_1c_3-\cancel{a_1b_1c_2}-a_1b_3c_3)\vec{l}+(a_1b_2c_1+a_2b_3c_2+a_3b_3c_3-a_2b_1c_1-\cancel{a_2b_2c_2}-a_2b_3c_3)\vec{j}$$
$$+(a_1b_3c_1+a_2b_3c_2+a_3b_3c_3-a_3b_1c_1-a_3b_2c_2-a_3b_3c_3)\vec{k}$$

$$\Rightarrow\{c_3(a_3b_1-a_1b_3)-c_2(a_1b_2-a_2b_1)\}\vec{l}+\vec{j}\{c_3(a_2b_3-a_3b_2)-c_1(a_1b_2-a_2b_1)+\vec{k}\{c_2(a_2b_3-a_3b_2)-c_1(a_3b_1-a_1b_3)\}$$

Hence proved

Proof ii ; $\vec{b}\times\vec{c}=\begin{vmatrix} i & j & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=\vec{l}(b_2c_3-b_3c_2)-\vec{j}(b_1c_3-b_3c_1)+\vec{k}(b_1c_2-b_2c_1)$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix} \vec{i} & \vec{j} & k \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1 \end{vmatrix}$$

$$=\vec{l}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}-\vec{j}\{a_1(b_1c_2-b_2c_1)-a_3(b_1c_2-b_2c_1)\}+\vec{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$

R.H.S. $(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$$(a_1c_1+a_2c_2+a_3c_3)\{b_1\vec{l}+b_2\vec{j}+b_3\vec{k}\}-(a_1b_1+a_2b_2+a_3b_3)\{c_1\vec{l}+c_2\vec{j}+c_3\vec{k}\}$$

$$\Rightarrow\vec{l}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}-\vec{j}\{a_1cb_1c_2-b_2c_1)-a_3(b_1c_2-b_2c_1)\}+\vec{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$