VECTORS

1. ADDITION OF VECTORS
2. SCALAR PRODUCT OF VECTORS
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ADDITIONS OF VECTORS

Definitions and key points:

Scalar :- A quantity which has only magnitude but no directions is called scalar quantity.

Ex :- Length, mass, time………..

Vector :- A quantity which has both magnitude and direction is called a vector quantity

Ex:- Displacement, Velocity, Force ………………….

A vector can also be denoted by a single letter $\vec{a},\vec{b},\vec{c}$…… or bold letter a, b, c

Length of $\vec{a}$ is dinded by $|\vec{a}|$.Length of $\vec{a}$ is called magnitude of $\vec{a}$.

Zero vector (Null Vector) :- The vector of length O and having any direction is called null vector. It is denoted by $\vec{O}$

Note : 1) If A is any point in the space then $\overrightarrow{AA}=\vec{O}$

2) A non zero vector is called a proper vector.

Free vector :-A vector which is independent of its position is called free vector

Localisedvector :-If $\vec{a}$ is a vector P is a point then the ordere of pair, (P,a) is called localized vector at P

Multiplication of a vector by a scalar:-

i) Let m be any scalar and $\vec{a}$ be any vector then the vector m $\vec{a}$ is defined as

ii) Length of m $\vec{a}$ is $|m|$ times of length of $\vec{a}$ i.e. $|m\vec{a}| = |m||a|$

iii) The line of support of $m\vec{a}$ is same or parallel to that of $\vec{a}$

The sense the direction of $m\vec{a}$ is same as that of $\vec{a}$ if m is positive, the direction of $m\vec{a}$ is opposite to that $\vec{a}$ if m is negative

Note : 1) $o\vec{a} = \vec{o}$ 2) $m\vec{o} = \vec{o}$ 3) $(mn)\vec{a} = m(n\vec{a}) = n(m\vec{a})$ 4) $(-1)\vec{a}$

Negative of a vector :- $\vec{a}, \vec{b}$ are two vectors having same length but their directions are opposite to each other then each vector is called the negative of the other vector.

Here $\vec{a} = -\vec{b}$ and $\vec{b} = -\vec{a}$

Unit vector :- A vector whose magnitude is unity is called unit vector.

Note : If $\vec{a}$ is any vector then unit vector is $\frac{\vec{a}}{|\vec{a}|}$ this is denoted by $\hat{a}$ {read as 'a' cap}

Collinear or parallel vectors:- Two or more vectors are said to the collinear vectors if the have same line of support. The vectors are said to be parallel if they have parallel lines of support.

Like parallel vectors: - Vectors having same direction are called like parallel vectors.

Unlike parallel vectors: - Vectors having different direction are called unlike parallel vectors.

Note :1) If $\vec{a}, \vec{b}$ are two non-zero collinear or parallel vectors then there exists a non zero scalar m such that $\vec{a} = m\vec{b}$

2) Conversly if there exists a relation of the type $\vec{a} = m\vec{b}$ between two non zero two non zero vectors $\vec{a}, \vec{b}$ then $\vec{a}, \vec{b}$ must be parallel or collinear.

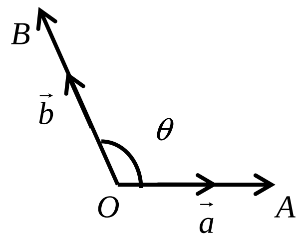
Cointialvectors :- The vectors having the same initial point are called co-initial vectors.

Co planar vectors :- Three or more vectors are said to be coplanar if they lie on a plane parallel to same plane. Other wise the vectors are non coplanar vectors.

Angle between two non-zero vectors :-

Let $\overline{OA} = \vec{a}$ and $\overline{OB} = \vec{b}$ be two non-zero vectors. Then the angle between $\vec{a}$ and $\vec{b}$ is defined as that angle AOB where $O \le \angle AOB \le 180^0$

The angle between $\vec{a}$ and $\vec{b}$ is denoted by $(\vec{a}, \vec{b})$



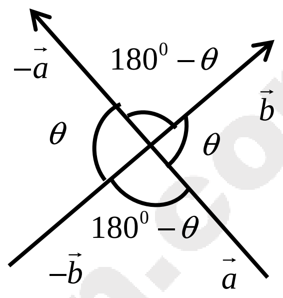
Note :- If $\vec{a}$ and $\vec{b}$ are any two vectors such that $(\vec{a},\vec{b})=\theta$ then

$o \le \theta \le 180^0$ {this is the range of θ i.e. angle beween vectors}

$(\vec{a},\vec{b})=(\vec{b},\vec{a})$

$(\vec{a},-\vec{b})=(-\vec{a},\vec{b})=180^0-(\vec{a},\vec{b})$

$(-\vec{a},-\vec{b})=(\vec{a},\vec{b})$



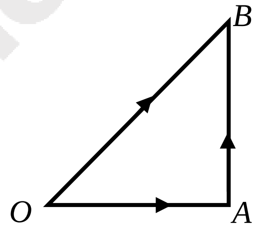
If k > 0; l > 0 then $(k\vec{a},l\vec{b})=(\vec{a},\vec{b})$

If $(\vec{a},\vec{b})=0$ then $\vec{a}$, $\vec{b}$ are like parallel vectors

If $(\vec{a},\vec{b})=180^0$ then $\vec{a}$, $\vec{b}$ are unlike parallel vectors

If $(\vec{a},\vec{b})=90^0$ then $\vec{a}$, $\vec{b}$ are orthogonal or perpendicular vectors.

Addition of vectors:- Let $\vec{a}$ and $\vec{b}$ be any two given vectors. If three points O, A, B are taken such that $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{AB}=\vec{b}$ then vectors $\overrightarrow{OB}$ is called the vector sum or resultant of the given vectors $\vec{a}$ and $\vec{b}$ we write $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$



Triangular law of vectors :-Trianglular law states that if two vectors are represented in magnitude and direction by two sides of a triangle taken in order, then their sum or resultant is represented in magnitude and direction by the third side of the triangle taken in opposite direction.

Position vector :- Let O be a fixed point in the space called origin. If P is any point in the space then $\overrightarrow{OP}$ is called position vector of P relative to O.

Note : If $\vec{a}$ and $\vec{b}$ be two non collinear vectors, then there exists a unique plane through $\vec{a},\vec{b}$ this plane is called plane generated by a, b. If $\overrightarrow{OA}=\vec{a}$; $\overrightarrow{OB}=\vec{b}$ then the plane generated by $\vec{a},\vec{b}$ is denoted by $\overleftrightarrow{AOB}$.

* Two non zero vectors $\vec{a},\vec{b}$ are collinear if $m\vec{a}+n\vec{b}=0$ for some scalars m,n not both zero

* Let $\vec{a},\vec{b},\vec{c}$ be the position vectors of the points A, B, C respectively. Then A, B, C are collinear iff $m\vec{a}+n\vec{b}+p,\vec{c}=0$ for some scalars m, n, p not all zero such that m + n + p = 0

* Let $\vec{a},\vec{b}$ be two non collinear vectors, If $\vec{r}$ is any vector in the plane generated by $\vec{a},\vec{b}$ then there exist a unique pair of real numbers x, y such that $\vec{r} = x\vec{a} + y\vec{b}$.

* Let $\vec{a},\vec{b}$ be two non collinear vectors. If $\vec{r}$ is any vector such that $\vec{r} = x\vec{a} + y\vec{b}$ for some real numbers then $\vec{r}$ lies in the plane generated by $\vec{a},\vec{b}$.

* Three vectors $\vec{a},\vec{b},\vec{c}$ coplanar iff $x\vec{a} + y\vec{b} + z\vec{c} = 0$ for some scalars x, y, z not all zero.

* Let $\vec{a},\vec{b},\vec{c},\vec{d}$ be the position vectors of the points A, B, C, D in which no three of them are collinear. Then A, B, C, D are coplanar iff $m\vec{a} + n\vec{b} + p\vec{c} + q\vec{d} = 0$ for some scalars m,n,p,q not all zero such that m + n + p + q = 0

* If $\vec{a},\vec{b},\vec{c}$ are three non coplanar vectors and $\vec{r}$ is any vector then there exist a unique traid of real numbers. x, y, z such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

Right handed system of orthonormal vectors :-

A triced of three non-coplanar vectors $\vec{i},\vec{j},\vec{k}$ is said to be a right hand system of orthonormal triad of vectors if

i) $\vec{i},\vec{j},\vec{k}$ from a right handed system

ii) $\vec{i},\vec{j},\vec{k}$ are unit vectors

iii) $(i, j) = 90^0 = (\vec{j},\vec{k}) = (\vec{k}, i)$

Right handed system :- Let $\overline{OA} = \vec{a}$, $\overline{OB} = \vec{b}$ and $\overline{OC} = \vec{c}$ be three non coplanar vectors. If we observe from the point C that a rotation from $\overline{OA}$ to $\overline{OB}$ through an angle not greater than 180^0 is in the anti clock wise direction then the vectors $\vec{a},\vec{b},\vec{c}$ are said to form 'Right handed system'.

Left handed system :- If we observe from C that a rotation from OA to OB through an angle not greater than 180^0 is in the clock –wise direction then the vectors $\vec{a},\vec{b},\vec{c}$ are said to form a "Left handed system".

* If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $|\vec{r}| = \sqrt{x_2 + y_2 + z_2}$.

Direction Cosines :- If a given directed line makes angles α, β, γ with positive direction of axes of x,y, and z respectively then $\cos\alpha, \cos\beta, \cos\gamma$ are called direction cosines of the line and these are denoted by l,m,n.

Direction ratios :-Thre real numbers a,b,c are said to be direction ratios of a line if a:b:c = l:m:n where l,m,n are the direction cosines of the line.

Linear combination :- Let $\overline{a_1},\overline{a_2},\overline{a_3}......\overline{a_n}$ be n vectors and $l_1, l_2, l_3....l_n$ be n scalars then $l_1\overline{a_1} + l_2\overline{a_2} + l_3\overline{a_3} + ... + l_n\overline{a_n}$ is called a. linear combination of $\overline{a_1},\overline{a_2}.......\overline{a_n}$

Linear dependent vectors :-

The vectors $\overrightarrow{a_1}, \overrightarrow{a_2} ....\overrightarrow{a_n}$ are said to be linearly dependent if there exist scalars $l_1, l_2, l_3 ....l_n$ not all zero such that $l_1\overrightarrow{a_1} + l_2\overrightarrow{a_2} + ... + l_n\overrightarrow{a_n} = 0$

Linear independent The vectors $a_1, a_2, a_3 .....a_n$ are said to be linearly independt if $l_1, l_2, l_3 .....l_n$ are scalars, $l_1\overrightarrow{a_1} + l_2\overrightarrow{a_2} + ...l_n\overrightarrow{a_n} = 0$

$\Rightarrow l_1 = 0\ l_2 = 0 \ldots.\ l_n = 0$

* Let $\vec{a}, \vec{b}$ be the position vectors of A, B respectively the position vector of the point P which divides $\overline{AB}$ in the ratio m:n is $\frac{m\vec{b} + n\vec{a}}{m+n}$. Conversely the point P with position vector $\frac{m\vec{b} + n\vec{a}}{m+n}$ lies on the lines $\overleftrightarrow{AB}$ and divides $\overleftrightarrow{AB}$ in the ratio m:n.

* The medians of a triangle are concurrent. The point of concurrence divides each median in the ratio 2:1

* Let ABC be a triangle and D be a point which is not in the plane of $\overleftrightarrow{ABC}$ the lines joining O, A, B,C with the centroids of triangle ABC, triangle BCD, triangle CDA and triangle DAB respectively are concurrent and the point of concurrence divides each line segment in the ratio 3:1

* The equation of the line passing through the point $A = (x_1, y_1, z_1)$ and parallel to the vector $\vec{b} = (l, m, n)$ is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = t$

* The equation of the line having through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$

* The unit vector bisecting the angle between the vectors $\vec{a}, \vec{b}$ is $\frac{\hat{a} + \hat{b}}{\left|\hat{a} + \hat{b}\right|}$

* The internal bisector the angle between $\vec{a}, \vec{b}$ is $\frac{\overline{OP}}{OP} = \frac{\hat{a} + \hat{b}}{\left|\hat{a} + \hat{b}\right|}$ are concurrent.

IMP THEOREMS

Theorem 1:

The vector equation of a line parallel to the vector $\vec{b}$ and passing through the point A with position vector $\vec{a}$ is $\vec{r} = \vec{a} + t\vec{b}$ where t is a scalar.

Proof : Let $\overrightarrow{OA} = \vec{a}$ be the given point and $\overrightarrow{OP} = \vec{r}$ be any point on the line

$\overrightarrow{AP} = t\vec{b}$ where 't' is a scalar

$\vec{r} - \vec{a} = t\vec{b} \Rightarrow \vec{r} = \vec{a} + t\vec{b}$

Theorem 2:

The vector equation of the line passing through the points A, B whose position vectors $\vec{a}, \vec{b}$ respectively is $\vec{r} = \vec{a}(1-t) + t\vec{b}$ where t is a scalar.

Proof: let P a a point on the line joining of A, B

$\overrightarrow{OA} = \vec{a}$ $\overrightarrow{OB} = \vec{b}$ $\overrightarrow{OP} = \vec{r}$

$\overrightarrow{AP}, \overrightarrow{AB}$ are collinear

$\therefore \overrightarrow{AP} = t\overrightarrow{AB}$

$\Rightarrow \vec{r} - \vec{a} = t(\vec{b} - \vec{a})$

$\therefore \vec{r} = (1-t)\vec{a} + t\vec{b}$

Theorem 3:

The vector equation of the plane passing through the point A with position vector $\vec{a}$ and parallel to the vectors $\vec{b}, \vec{c}$ is $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$ where s,t are scalars

Proof : Given that $\overrightarrow{OA} = \vec{a}$

Let $\overrightarrow{OP} = \vec{r}$ be the position vector of P

$\overrightarrow{AP}, \vec{b}, \vec{c}$ are coplanar

$\therefore \overrightarrow{AP} = s\vec{b} + t\vec{c} \Rightarrow \vec{r} - \vec{a} = s\vec{b} + t\vec{c}$

$\therefore \vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

Theorem 4:

The vector equation of the plane passing through the points A, B with position vectors $\vec{a},\vec{b}$ and parallel to the vector $\vec{c}$ is $\vec{r}=(1-s)\vec{a}+s\vec{b}+t\vec{c}$

Proof : Let P be a point on the plane and $\overrightarrow{OP}=\vec{r}$

$\overrightarrow{OA}=\vec{a}\ \overrightarrow{OB}=\vec{b}$ be the given points $\overrightarrow{AP},\overrightarrow{AB},\vec{C}$ are coplanar

$$\overrightarrow{AP}=s\overrightarrow{AB}+t\vec{C}$$

$$\vec{r}-\vec{a}=s(\vec{b}-\vec{a})+t\vec{c}$$

$$\vec{r}=(1-s)\vec{a}+s\vec{b}+t\vec{c}$$

Theorem 5:

The vector equation of the plane passing through the points A, B, C having position vectors $\vec{a},\vec{b},\vec{c}$ is $\vec{r}=(1-s-t)\vec{a}+s\vec{b}+t\vec{c}$ where s,t are scalars

Proof:

Let P be a point on the plane and $\overrightarrow{OP}=\vec{r}$

$\overrightarrow{OA}=\vec{a}\ \overrightarrow{OB}=\vec{b},\ \overline{OC}=\bar{c}$

Now the vectors $\overline{AB},\overline{AC},\overline{AP}$ are coplanar.

Therefore, $\overline{AP}=s\overline{AB}+t\overline{AC}$, t,s are scalars.

$$\bar{r}-\bar{a}=s\left(\bar{b}-\bar{a}\right)+t\left(\bar{c}-\bar{a}\right)$$
$$\Rightarrow \bar{r}=\left(1-s-t\right)\bar{a}+s\ \bar{b}+t\ \bar{c}$$

PROBLEMS

VSAQ'S

1. Let $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{b} = 3\bar{i} + \bar{j}$ find the unit vector in the direction of $\bar{a} + \bar{b}$.

Sol. Given $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{b} = 3\bar{i} + \bar{j}$

$$\bar{a} + \bar{b} = \bar{i} + 2\bar{j} + 3\bar{k} + 3\bar{i} + \bar{j}$$
$$= 4\bar{i} + 3\bar{j} + 3\bar{k}$$

$\therefore$ The unit vector in the direction of $\bar{a} + \bar{b} = \pm\dfrac{\bar{a} + \bar{b}}{|\bar{a} + \bar{b}|}$

$$= \pm\frac{4\bar{i} + 3\bar{j} + 3\bar{k}}{|4\bar{i} + 3\bar{j} + 3\bar{k}|}$$
$$= \pm\frac{4\bar{i} + 3\bar{j} + 3\bar{k}}{\sqrt{16 + 9 + 9}}$$
$$= \pm\frac{4\bar{i} + 3\bar{j} + 3\bar{k}}{\sqrt{34}}$$

2. If the vectors $-3\bar{i} + 4\bar{j} + \lambda\bar{k}$ and $\mu\bar{i} + 8\bar{j} + 6\bar{k}$ are collinear vectors then find λ and μ.

Sol. Let $\bar{a} = -3\bar{i} + 4\bar{j} + \lambda\bar{k}$, $\bar{b} = \mu\bar{i} + 8\bar{j} + 6\bar{k}$

From hyp. $\bar{a}, \bar{b}$ are collinear then $\bar{a} = t\bar{b}$

$$\Rightarrow -3\bar{i} + 4\bar{j} + \lambda\bar{k} = t(\mu\bar{i} + 8\bar{j} + 6\bar{k})$$
$$-3\bar{i} + 4\bar{j} + \lambda\bar{k} = \mu t\bar{i} + 8t\bar{j} + 6t\bar{k}$$

Comparing i, j, k coefficients on both sides

$$\mu t = -3 \Rightarrow \mu = -\frac{3}{t} = -\frac{3}{1/2} = -6 \Rightarrow \mu = -6$$
$$8t - 4 \Rightarrow t = \frac{4}{8} \Rightarrow t = \frac{1}{2}$$
$$6t = \lambda \Rightarrow \lambda = 6 \cdot \frac{1}{2} \Rightarrow \lambda = 3$$

3. ABCDE is a pentagon. If the sum of the vectors AB, AE, BC, DE, ED and AC is λAC, then find the value of λ.

Sol. Given that,

$$\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} + \overline{AC} = \lambda\overline{AC}$$
$$\Rightarrow (\overline{AB} + \overline{BC}) + (\overline{AE} + \overline{ED}) + (\overline{DC} + \overline{AC}) = \lambda\overline{AC}$$

$\Rightarrow \overline{AC}+\overline{AD}+\overline{DC}+\overline{AC}=\lambda\overline{AC}$

$\Rightarrow \overline{AC}+\overline{AC}+\overline{AC}=\lambda\overline{AC}$

$\Rightarrow 3\overline{AC}=\lambda\overline{AC}$

$\therefore \lambda=3$

4. If the position vectors of the points A, B, C are $-2\overline{i}+\overline{j}-\overline{k}, -4\overline{i}+2\overline{j}+2\overline{k}$ **and** $6\overline{i}-3\overline{j}-13\overline{k}$ **respectively and** $\overline{AB}=\lambda\overline{AC}$ **then find the value of** λ.

Sol. Let O be the origin and $\overline{OA}=-2\overline{i}+\overline{j}-\overline{k}$, $\overline{OB}=-4\overline{i}+2\overline{j}+2\overline{k}$, $\overline{OC}=6\overline{i}-3\overline{j}-13\overline{k}$

Given $\overline{AB}=\lambda\overline{AC}$

$\overline{OB}-\overline{OA}=\lambda\left[\overline{OC}-\overline{OA}\right]$

$-4\overline{i}+2\overline{j}+2\overline{k}+2\overline{i}-\overline{j}+\overline{k}=$

$\lambda\left[6\overline{i}-3\overline{j}-13\overline{k}+2\overline{i}-\overline{j}+\overline{k}\right]$

$-2\overline{i}+\overline{j}+3\overline{k}=\lambda\left[8\overline{i}-\overline{j}-12\overline{k}\right]$

Comparing $\overline{i}$ coefficient on both sides

$-2=\lambda 8 \Rightarrow \lambda=-\frac{2}{8}\Rightarrow \lambda=\frac{-1}{4}$

5. If $\overline{OA}=\overline{i}+\overline{j}+\overline{k}, \overline{AB}=3\overline{i}-2\overline{j}+\overline{k}, \overline{BC}=\overline{i}+2\overline{j}-2\overline{k}$ **and** $\overline{CD}=2\overline{i}+\overline{j}+3\overline{k}$ **then find the vector of** $\overline{OD}$**.**

Sol. $\overline{OD}=\overline{OA}+\overline{AB}+\overline{BC}+\overline{CD}$

$=\overline{i}+\overline{j}+\overline{k}+3\overline{i}-2\overline{j}+\overline{k}+\overline{i}+2\overline{j}-2\overline{k}+2\overline{i}+\overline{j}+3\overline{k}$

$\overline{OD}=7\overline{i}+2\overline{j}+3\overline{k}$

6. Let $\overline{a}=2\overline{i}+4\overline{j}-5\overline{k}$**,** $\overline{b}=\overline{i}+\overline{j}+\overline{k}$ **and** $\overline{c}=\overline{j}+2\overline{k}$**, find the unit vector in the opposite direction of** $\overline{a}+\overline{b}+\overline{c}$**.**

Sol. $\overline{a}+\overline{b}+\overline{c}=2\overline{i}+4\overline{j}-5\overline{k}+\overline{i}+\overline{j}+\overline{k}+\overline{j}+2\overline{k}$

$=3\overline{i}+6\overline{j}-2\overline{k}$

$\therefore$Unit vector in the direction of

$\overline{a}+\overline{b}+\overline{c}=\pm\frac{\overline{a}+\overline{b}+\overline{c}}{|\overline{a}+\overline{b}+\overline{c}|}$

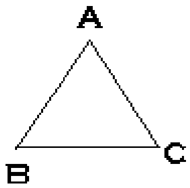
$=\pm\frac{3\overline{i}+6\overline{j}-2\overline{k}}{\sqrt{49}}=\pm\frac{3\overline{i}+6\overline{j}-2\overline{k}}{7}$

7. Is the triangle formed by the vectors $3\bar{i}+5\bar{j}+2\bar{k},\ 2\bar{i}-3\bar{j}-5\bar{k}$ **and** $-5\bar{i}-2\bar{j}+3\bar{k}$ **is equilateral.**

Sol. Given vectors are

$\overline{AB}=3\bar{i}+5\bar{j}+2\bar{k},\ \overline{BC}=2\bar{i}-3\bar{j}-5\bar{k}$

$\overline{CA}=-5\bar{i}-2\bar{j}+3\bar{k}$



From given vectors $\overline{AB}+\overline{BC}+\overline{CA}=0$

Therefore, given vectors are the sides of the triangle.

$|\overline{AB}|=\sqrt{3^2+5^2+2^2}=\sqrt{9+25+4}=\sqrt{38}$

$|\overline{BC}|=\sqrt{2^2+3^2+5^2}=\sqrt{38}$

$|\overline{CA}|=\sqrt{5^2+2^2+3^2}=\sqrt{38}$

$\therefore |\overline{AB}|=|\overline{BC}|=|\overline{CA}|$

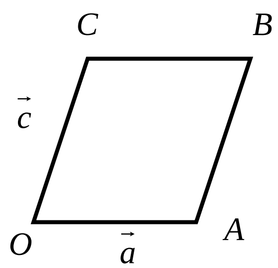
Therefore, given vectors forms an equilateral triangle.

8. OABC is a parallelogram if $\overline{OA}=\vec{a}$ **and** $\overline{OC}=\vec{c}$ **find the vector equation of side** $\overline{BC}$**.**

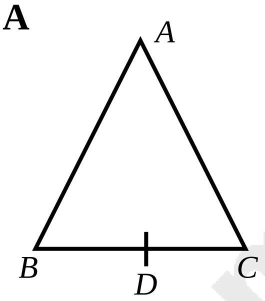
Sol: let o be the origin.. $\overline{OA}=\vec{a}$ and $\overline{OC}=\vec{c}$

The side BC is parallel to $\overline{OA}$ i.e. $\vec{a}$ and passing through C is $\vec{c}$

$\therefore \vec{r}=\vec{c}+t\vec{a}$ where $t\in R$



9. . If $\vec{a},\vec{b},\vec{c}$ are the position vectors of the vertices A, B and C respectively of ΔABC then find the vector equation of median through the vertex A



Sol: $\overline{OA}=\vec{a}$, $\overline{OB}=\vec{b}$, $\overline{OC}=\vec{c}$ be the given vertices

Let D be the mid point of $BC=\dfrac{\vec{b}+\vec{c}}{2}$

The vector equation of the line passing through the

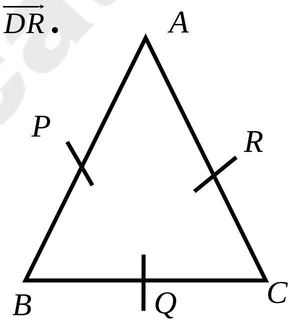
points $\vec{a}$, $\vec{b}$ is $\vec{r}=(1-t)\vec{a}+t\vec{b}$

$\therefore$vector equation is $\vec{r}=(1=t)\vec{a}+t\left(\dfrac{\vec{b}+\vec{c}}{2}\right)$ where $t\in R$

10. In ΔABC P, Q and R are the mid points of the sides AB, BC and CA respectively if D is any point

i) Then express $\overline{DA}+\overline{DB}+\overline{DC}$ in terms of $\overline{DP},\overline{DQ}$ and $\overline{DR}$.

ii) If $\overline{PA}+\overline{QB}+\overline{RC}=a$ then find a



Sol : D is any proof let D the origin of vectors

$\overline{DA}=\vec{a}$ $\qquad \overline{DB}=\vec{b}$ $\qquad \overline{DC}=\vec{c}$

$\overline{DP}=\dfrac{\vec{a}+\vec{b}}{2}$; $\overline{DQ}=\dfrac{\vec{b}+\vec{c}}{2}$; $\overline{DR}=\dfrac{\vec{a}+\vec{c}}{2}$

$$\overline{DP}+\overline{DQ}+\overline{DR}=\frac{\vec{a}+\vec{b}+\vec{b}+\vec{c}+\vec{a}+\vec{c}}{2}=\vec{a}+\vec{b}+\vec{c}=\overline{DA}+\overline{DB}+\overline{DC}$$

ii) $\overline{PA}+\overline{QB}+\overline{RC}=\vec{a}-\left(\dfrac{\vec{a}+\vec{b}}{2}\right)+\vec{b}-\left(\dfrac{\vec{b}+\vec{c}}{2}\right)+\vec{c}-\dfrac{\vec{c}+\vec{a}}{2}$

$$\frac{\vec{a}-\vec{b}+\vec{b}-\vec{c}+\vec{c}-\vec{a}}{2}=\vec{O}$$

$$\vec{a}=\bar{0}$$

11. Using the vector equation of the straight line passing through two points, prove that the points whose position vectors are $\bar{a},\bar{b}$ and $3\bar{a}-2\bar{b}$ are collinear.

Sol. The equation of the line passing through two points $\bar{a}$ and $\bar{b}$ is $\bar{r}=(1-t)\bar{a}+t\bar{b}$. The line also passes through the point $3\bar{a}-2\bar{b}$, if $3\bar{a}-2\bar{b}=(1-t)\bar{a}+t\bar{b}$ for some scalar t.

Equating corresponding coefficients,

1 – t = 3 and t = –2

$\therefore$ The three given points are collinear.

12. Write direction ratios of the vector $a = \bar{i} + \bar{j} - 2\bar{k}$ and hence calculate its direction cosines.

Sol. Note that direction ratios a, b, c of a vector $r = x\bar{i} + y\bar{j} + z\bar{k}$ are just the respective components c, y and z of the vector. So, for the given vector, we have a = 1, b=1, c= –2. Further, if l, m and n the direction cosines of the given vector, then

$$l = \frac{a}{|r|} = \frac{1}{\sqrt{6}}, m = \frac{b}{|r|} = \frac{1}{\sqrt{6}}, n = \frac{c}{|r|} = -\frac{2}{\sqrt{6}} \text{ as} |r| = \sqrt{6}$$

Thus, the direction cosines are

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right).$$

13. Find the vector equation of the plane passing through the points

(0, 0, 0), (0, 5, 0) and (2, 0, 1).

Sol. Let $\bar{a}$ = (0, 0, 0), $\bar{b}$ = (0, 5, 0), $\bar{c}$ =(2, 0, 1)

$$\bar{a} = 0,\ \bar{b} = 5\bar{j},\ \bar{c} = 2\bar{i} + \bar{k}$$

The vector equation of the plane passing through the points

$\bar{a}, \bar{b}, \bar{c}$ is $\bar{r} = \bar{a} + s(\bar{b} - \bar{a}) + t(\bar{c} - \bar{a}), s, t \in R$

$\bar{r} = s(5\bar{j}) + t(2\bar{i} + \bar{k}),\ s, t \in R$

14. Find unit vector in the direction of vector $a = 2\bar{i} + 3\bar{j} + \bar{k}$.

Sol. The unit vector in the direction of a vector $\hat{a}$ is given by $\hat{a} = \frac{1}{|a|}a$

Now $|a| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

Therefore $\hat{a} = \frac{1}{\sqrt{14}}(2\bar{i} + 3\bar{j} + \bar{k})$

$$= \frac{2}{\sqrt{14}}\bar{i} + \frac{3}{\sqrt{14}}\bar{j} + \frac{1}{\sqrt{14}}\bar{k}$$

15. Find a vector in the direction of vector $a = \bar{i} - 2\bar{j}$ that has magnitude 7 units.

Sol. The unit vector in the direction of the given vector ‘a’ is

$$\hat{a} = \frac{1}{|a|}a = \frac{1}{\sqrt{5}}(\bar{i} - 2\bar{j}) = \frac{1}{\sqrt{5}}\bar{i} - \frac{2}{\sqrt{5}}\bar{j}$$

Therefore, the vector having magnitude equal to 7 and in the direction of a is

$$7a = 7\left(\frac{1}{\sqrt{5}}\bar{i} - \frac{2}{\sqrt{5}}\bar{j}\right) = \frac{7}{\sqrt{5}}\bar{i} - \frac{14}{\sqrt{5}}\bar{j}$$

16. Find the angles made by the straight line passing through $(1,3,2),(3,5,1)$ with co-ordinate axes.

Sol. $A = (1,3,2), B = (3,5,1)$

D.RS OF $\overline{AB} = (1-3, 3-5, 2-1) = (-2,-2,1) = (2,2,-1) = (a,b,c)$

D.Rs. of $\overline{AB} = \left(\frac{a}{r}, \frac{b}{r}, \frac{c}{r}\right), r = \sqrt{a^2+b^2+c^2}$

$r = \sqrt{4+4+1} = 3$

$d.cs$ of $\overline{AB} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

Angles are $\cos^{-1}\left(\frac{2}{3}\right), \cos^{-1}\left(\frac{2}{3}\right), \cos^{-1}\left(-\frac{1}{3}\right)$.

17. In the two dimensional plane, prove by using vector methods, the equation of the line whose intercepts on the axes are $\bar{a}$ and $\bar{b}$ is $\frac{x}{a} + \frac{y}{b} = 1$.

Sol. Let A = $(\bar{a}, 0)$ and $B = (0, \bar{b})$

$\therefore \bar{A} = \bar{a}i, \bar{B} = \bar{b}j$

The equation of the line is $\bar{r} = (1-t)\bar{a}i + t(\bar{b}j)$

If $\bar{r} = x\bar{i} + y\bar{j}$, then $x = (1-t)\bar{a}$ and $y = t\bar{b}$

$\therefore \frac{x}{a} + \frac{y}{b} = 1 - t + t = 1$

SAQ'S

18. If α, β and γ are the angles made by the vector $3\bar{i}-6\bar{j}+2\bar{k}$ with the positive directions of the coordinate axes then find cosα, cosβ and cosγ.

Sol. Let $\bar{a}=3\bar{i}-6\bar{j}+2\bar{k}$

Let $\alpha=(\bar{a},\bar{i})$

$$\cos\alpha=\frac{\bar{a}\cdot\bar{i}}{|\bar{a}||\bar{i}|}=\frac{|3\bar{i}-6\bar{j}+2\bar{k}|i}{|4\bar{i}-6\bar{j}+2\bar{k}||i|}$$

$$=\frac{3}{\sqrt{9+36+4}\times 1}=\frac{3}{\sqrt{49}}=\frac{3}{7}$$

$$\cos\alpha=\frac{3}{7};\beta=(\bar{a},\bar{j}),\ \gamma=(\bar{a},\bar{k})$$

$$\Rightarrow\cos\beta=\frac{-6}{7},\ \cos\gamma=\frac{2}{7}$$

19. Show that the points $A(2\bar{i}-\bar{j}+\bar{k})$, $B(\bar{i}-3\bar{j}-5\bar{k})$, $C(3\bar{i}-4\bar{j}-4\bar{k})$ are the vertices of a right angle triangle.

Sol. We have

$$AB=(1-2)\bar{i}+(-3+1)\bar{j}+(-5-1)\bar{k}$$
$$=-\bar{i}-2\bar{j}-6\bar{k}$$

$$|AB|=\sqrt{1+4+36}=\sqrt{41}$$

$$BC=(3-1)\bar{i}+(-4+3)\bar{j}+(-4+5)\bar{k}$$
$$=2\bar{i}-\bar{j}+\bar{k}\text{ and}$$

$$|BC|=\sqrt{4+1+1}=\sqrt{6}$$

$$CA=(2-3)\bar{i}+(-1+4)\bar{j}+(1+4)\bar{k}$$
$$=-\bar{i}+3\bar{j}+5\bar{k}$$

$$|CA|=\sqrt{1+9+25}=\sqrt{35}$$

We have $|AB|^2 = |BC|^2 + |CA|^2$

Therefore, the triangle is a rt. Triangle.

20. ABCD is a parallelogram if L and M are the middle point of BC and CD respectively then find (i) AL and AM interns of AB and AD (ii) if $\overrightarrow{AL}+\overrightarrow{AM}=\overrightarrow{AC}$

Sol: Let A be the origin of vectors $\overrightarrow{AB}=\vec{a}\ \overrightarrow{AC}=\vec{b}\ \overrightarrow{AD}=\vec{c}$

$$\left.\begin{array}{l}\overrightarrow{AB}=\overrightarrow{DC}\\ \overrightarrow{BC}=\overrightarrow{AD}\end{array}\right\}\because \text{Opposite sides of parallelogram}$$

$$\vec{a}=\vec{b}-\vec{c}\Rightarrow\vec{a}+\vec{c}=\vec{b}$$

$$\overrightarrow{AL}=\frac{\vec{a}+\vec{b}}{2}\qquad\overrightarrow{AM}=\frac{\vec{b}+\vec{c}}{2}$$

$$\overrightarrow{AL}=\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}=\frac{\overrightarrow{AB}+\overrightarrow{AB}+\overrightarrow{BC}}{2}=\frac{2\overrightarrow{AB}+\overrightarrow{AD}}{2}$$

$$\boxed{\therefore\overrightarrow{AL}=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AB}}$$

$$\overrightarrow{AM}=\frac{\overrightarrow{AC}+\overrightarrow{AD}}{2}=\frac{\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{AD}}{2}=\frac{\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{AD}}{2}[\because\overrightarrow{BC}=\overrightarrow{AD}]$$

$$\overrightarrow{AM}=\frac{\overrightarrow{AB}}{2}+\overrightarrow{AD}$$

(ii) $\overrightarrow{AM}=\overrightarrow{AM}=\lambda\overrightarrow{AC}\Rightarrow\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AD}+\frac{1}{2}\overrightarrow{AB}+\overrightarrow{AD}=\lambda\overrightarrow{AC}\frac{3}{2}\{\overrightarrow{AB}+\overrightarrow{AD}\}=\lambda\overrightarrow{AC}\Rightarrow\frac{3}{2}\{\overrightarrow{AB}+\overrightarrow{BC}\}=\lambda\overrightarrow{AC}$

$$\frac{3}{2}\overrightarrow{AC}=\lambda\overrightarrow{AC}\ \therefore=3/2$$

21. If $\bar{a}+\bar{b}+\bar{c}=\alpha\bar{d}$, $\bar{b}+\bar{c}+\bar{d}=\beta\bar{a}$ and $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors then show that $\bar{a}+\bar{b}+\bar{c}+\bar{d}=0$.

Sol. $\bar{a}+\bar{b}+\bar{c}-\alpha\bar{d}=\bar{0}$ …(1)

$\beta\bar{a}-\bar{b}-\bar{c}-\alpha\bar{d}=\bar{0}$ …(2)

$(2)\times(-1)\Rightarrow$

$-\beta\bar{a}+\bar{b}+\bar{c}+\alpha\bar{d}=\bar{0}$ …(3)

(1) = (3)

$$\bar{a}+\bar{b}+\bar{c}-\alpha\bar{d}=-\beta\bar{a}+\bar{b}+\bar{c}+\bar{d}$$
$$-\beta=1\Rightarrow\beta=-1$$
$$-\alpha=1\Rightarrow\alpha=-1$$

By substituting $\alpha=-1$ in (1) we get

$\bar{a}+\bar{b}+\bar{c}+\bar{d}=0$.

22. If a, b, c are non-coplanar vectors, prove that $-\bar{a}+4\bar{b}-3\bar{c},\ 3\bar{a}+2\bar{b}-5\bar{c},\ -3\bar{a}+8\bar{b}-5\bar{c}$**,** $-3\bar{a}+2\bar{b}+\bar{c}$ **are coplanar.**

Sol. Let $\overline{OA}=-\bar{a}+4\bar{b}-3\bar{c}, \overline{OB}=3\bar{a}+2\bar{b}-5\bar{c}$

$\overline{OC}=-3\bar{a}+8\bar{b}-5\bar{c}, \overline{OD}=-3\bar{a}+2\bar{b}+\bar{c}$

$\overline{AB}=\overline{OB}-\overline{OA}$

$=3\bar{a}+2\bar{b}-5\bar{c}+\bar{a}-4\bar{b}+3\bar{c}$

$=4\bar{a}-2\bar{b}-2\bar{c}$

$\overline{AC}=\overline{OC}-\overline{OA}$

$=-3\bar{a}+8\bar{b}-5\bar{c}+\bar{a}-4\bar{b}+3\bar{c}$

$=-2\bar{a}+4\bar{b}-2\bar{c}$

$\overline{AD}=\overline{OD}-\overline{OA}$

$=-3\bar{a}+2\bar{b}+\bar{c}+\bar{a}-4\bar{b}+3\bar{c}$

$=-2\bar{a}-2\bar{b}+4\bar{c}$

$\overline{AB}=x\overline{AC}+y\overline{AD}$ where x, y are scalars.

$4\bar{a}-2\bar{b}-2\bar{c}=$

$x\left[-2\bar{a}+4\bar{b}-2\bar{c}\right]+y\left[-2\bar{a}-2\bar{b}+4\bar{c}\right]$

$4=-2x-2y \quad \ldots(1)$

$-2=4x-2y \quad \ldots(2)$

$-2=-2x+4y \quad \ldots(3)$

$(1)-(3) \Rightarrow -2x-2y=4$

$\underline{-2x+4y=-2}$

$-6y=6 \Rightarrow y=-1$

From (3) $-2x=-2-4y$

$=-2+4$

$-2x=2 \Rightarrow x=-1$

Substitute x, y in (2)

$-2=-4+2$

$-2=-2$

Equation (2) is satisfied by x = –1, y = –1

Hence given vectors are coplanar.

23. If $\overline{i}, \overline{j}, \overline{k}$ are unit vectors along the positive directions of the coordinate axes, then show that the four points $4\overline{i}+5\overline{j}+\overline{k}, -\overline{j}-\overline{k}$, $3\overline{i}+9\overline{j}+4\overline{k}$ and $-4\overline{i}+4\overline{j}+4\overline{k}$ are coplanar.

Sol. Let O be a origin, then

$\overline{OA} = 4\overline{i}+5\overline{j}+\overline{k},\ \overline{OB} - \overline{j} - \overline{k}$

$\overline{OC} = 3\overline{i}+9\overline{j}+4\overline{k}$, and $\overline{OD} = -4\overline{i}+4\overline{j}+4\overline{k}$

$\overline{AB} = \overline{OB} - \overline{OA} = -4\overline{i} - 6\overline{j} - 2\overline{k}$

$\overline{AC} = \overline{OC} - \overline{OA} = -\overline{i} + 4\overline{j} + 3\overline{k}$

$\overline{AD} = \overline{OD} - \overline{OA} = -8\overline{i} - \overline{j} + 3\overline{k}$

$$\left[\overline{AB}\quad \overline{AC}\quad \overline{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4[12+3] + 6[-3+24] - 2[1+32]$$
$$= -4\times 15 + 6\times 21 - 2\times 33$$
$$= -60 + 126 - 66$$
$$= -126 + 126 = 0$$

Hence given vectors are coplanar.

24. If a, b, c are non-coplanar vectors, then test for the collinearity of the following points whose position vectors are given.

i) Show that $\overline{a} - 2\overline{b} + 3\overline{c}, 2\overline{a} + 3\overline{b} - 4\overline{c}$, $-7\overline{b} + 10\overline{c}$ are collinear.

Sol. Let $\overline{OA} = \overline{a} - 2\overline{b} + 3\overline{c},\ \overline{OB} = 2\overline{a} + 3\overline{b} - 4\overline{c}$

$\overline{OC} = -7\overline{b} + 10\overline{c}$

$\overline{AB} = \overline{OB} - \overline{OA} = \overline{a} + 5\overline{b} - 7\overline{c}$

$\overline{AC} = \overline{OC} - \overline{OA} = -\overline{a} - 5\overline{b} + 7\overline{c}$

$\overline{AC} = -\overline{a} - 5\overline{b} + 7\overline{c} = -[\overline{a} + 5\overline{b} - 7\overline{c}]$

$\overline{AC} = -\overline{AB}$

$\overline{AC} = \lambda\overline{AB}$ where $\lambda = -1$

$\therefore$ Given vectors are collinear.

ii) $3\bar{a}-4\bar{b}+3\bar{c},\ -4\bar{a}+5\bar{b}-6\bar{c},\ 4\bar{a}-7\bar{b}+6\bar{c}$

Sol. Let $\overline{OA}=3\bar{a}-4\bar{b}+3\bar{c},\ \overline{OB}=-4\bar{a}+5\bar{b}-6\bar{c}$

$$\overline{OC}=4\bar{a}-7\bar{b}+6\bar{c}$$
$$\overline{AB}=\overline{OB}-\overline{OA}=-7\bar{a}+9\bar{b}-9\bar{c}$$
$$\overline{AC}=\overline{OC}-\overline{OA}=\bar{a}-3\bar{b}+3\bar{c}$$
$$\overline{AB}\neq\lambda\overline{AC}$$

Therefore, the points are non collinear.

25. Find the vector equation of the line passing through the point $2\bar{i}+3\bar{j}+\bar{k}$ and parallel to the vector $4\bar{i}-2\bar{j}+3\bar{k}$.

Sol. Let $\bar{a}=2\bar{i}+3\bar{j}+\bar{k}$ and $\bar{b}=4\bar{i}-2\bar{j}+3\bar{k}$

The vector equation of the line passing through $\bar{a}$ and parallel to $\bar{b}$ is

$$\bar{r}=\bar{a}+t\bar{b},t\in R$$
$$\bar{r}=2\bar{i}+3\bar{j}+\bar{k}+t(4\bar{i}-2\bar{j}+3\bar{k}),t\in R$$
$$\bar{r}=(2+4t)\bar{i}+(3-2t)\bar{j}+(1+3t)\bar{k},t\in R$$

26. Find the vector equation of the line joining the points $2\bar{i}+\bar{j}+3\bar{k}$ and $-4\bar{i}+3\bar{j}-\bar{k}$.

Sol. Let $\bar{a}=2\bar{i}+\bar{j}+3\bar{k}$ and $\bar{b}=-4\bar{i}+3\bar{j}-\bar{k}$

The vector equation of the line passing through the points $\bar{a},\ \bar{b}$ is

$$\bar{r}=(1-t)\bar{a}+t\bar{b},t\in R$$
$$=\bar{a}+t(\bar{b}-\bar{a})$$
$$=2\bar{i}+\bar{j}+3\bar{k}+t[-4\bar{i}+3\bar{j}-\bar{k}-2\bar{i}-\bar{j}-3\bar{k}]$$
$$\bar{r}=2\bar{i}+\bar{j}+3\bar{k}+t[-6\bar{i}+2\bar{j}-4\bar{k}]$$

27. Find the vector equation of the plane passing through the points $\bar{i}-2\bar{j}+5\bar{k}$, $-5\bar{j}-\bar{k}$ and $-3\bar{i}+5\bar{j}$.

Sol. Let

$$\bar{a}=\bar{i}-2\bar{j}+5\bar{k},\bar{b}=-5\bar{j}-\bar{k},\bar{c}=-3\bar{i}+5\bar{j}$$

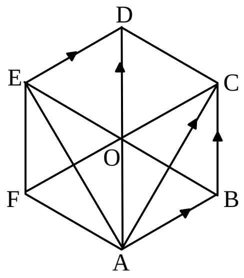
$\therefore$ The vector equation of the plane passing through the points $\bar{a},\bar{b},\bar{c}$ is

$$\bar{r}=(1-s-t)\bar{a}+s\bar{b}+t\bar{c},\text{where } s,t\in R$$
$$=\bar{a}+s(\bar{b}-\bar{a})+t(\bar{c}-\bar{a})$$
$$=\bar{i}-2\bar{j}+5\bar{k}+s(-5\bar{j}-\bar{k}-\bar{i}+2\bar{j}-5\bar{k})+t(-3\bar{i}+5\bar{j}-\bar{i}+2\bar{j}-5\bar{k})$$
$$\bar{r}=\bar{i}-2\bar{j}+5\bar{k}+s(-\bar{i}-3\bar{j}-6\bar{k})+t(-4i+7\bar{j}-5\bar{k})$$

28. Let ABCDEF be a regular hexagon with center O. Show that

$\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=3\overline{AD}=6\overline{AO}$.

Sol.



From figure,

$\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=\left(\overline{AB}+\overline{AE}\right)+\overline{AD}+\left(\overline{AC}+\overline{AF}\right)$

$=\left(\overline{AE}+\overline{ED}\right)+\overline{AD}+\left(\overline{AC}+\overline{CD}\right)\left(\because \overline{AB}=\overline{ED},\overline{AF}=\overline{CD}\right)$

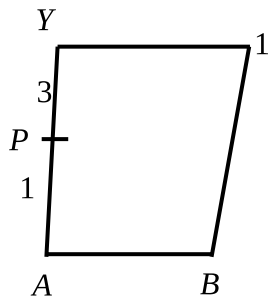
$=\overline{AD}+\overline{AD}+\overline{AD}=3\overline{AD}$

$=6\overline{AO}(\because \text{O is the center and } \overline{OD}=\overline{AO})$

29. The points O,A,B,X and Y are such that $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, $\overrightarrow{OX}=3\vec{a}$ and $\overrightarrow{OY}=3\vec{b}$.find $\overrightarrow{BX}$ and $\overrightarrow{AY}$ in terms of $\vec{a}$ and $\vec{b}$ further if P divides AY in the ratio 1:3 then express $\overrightarrow{BP}$ in terms of $\vec{a}$ and $\vec{b}$.

Sol: $\overrightarrow{BX}=\overrightarrow{OX}=\overrightarrow{OB}=3\vec{a}-\vec{b}$

$\overrightarrow{AY}=\overrightarrow{OY}=\overrightarrow{Oa}=3\vec{b}-\vec{a}$



$$\overrightarrow{OP}=\frac{1\times\overrightarrow{OY}+3\overrightarrow{OA}}{4}$$

$$\overrightarrow{OP}=\frac{3\vec{b}+3\vec{a}}{4}$$

$$\overrightarrow{BP}=\overrightarrow{OP}-\overrightarrow{OB}=\frac{3\vec{b}+3\vec{a}}{4}-\vec{b}=\frac{3\vec{b}+3\vec{a}}{4}-4\vec{b}$$

$$=\frac{1}{4}(3\vec{a}-\vec{b})$$

LAQ'S

30. If $\bar{a},\bar{b},\bar{c}$ are non-coplanar, find the point of intersection of the line passing through the points $2\bar{a}+3\bar{b}-\bar{c}, 3\bar{a}+4\bar{b}-2\bar{c}$ with the line joining the points $\bar{a}-2\bar{b}+3\bar{c}$, $\bar{a}-6\bar{b}+6\bar{c}$.

Sol. Let $\overline{OA}=2\bar{a}+3\bar{b}-\bar{c}$, $\overline{OB}=3\bar{a}+4\bar{b}-2\bar{c}$

$\overline{OC}=\bar{a}-2\bar{b}+3\bar{c}$, $\overline{OD}=\bar{a}-6\bar{b}+6\bar{c}$

The vector equation of the line joining the points $\overline{OA},\overline{OB}$ is

$$\bar{r}=\overline{OA}+t\left(\overline{OB}-\overline{OA}\right), t\in R$$

$$=2\bar{a}+3\bar{b}-\bar{c}+t(3\bar{a}+4\bar{b}-2\bar{c}-2\bar{a}-3\bar{b}+\bar{c}]$$

$$\bar{r}=2\bar{a}+3\bar{b}-\bar{c}+t(\bar{a}+\bar{b}-\bar{c}) \quad ...(1)$$

The vector equation of the line joining the points $\overline{OC},\overline{OD}$ is

$$\bar{r}=\overline{OC}+s\left(\overline{OD}-\overline{OC}\right), s\in R$$

$$=\bar{a}-2\bar{b}+3\bar{c}+s(\bar{a}-6\bar{b}+6\bar{c}-\bar{a}+2\bar{b}-3\bar{c}]$$

$$\bar{r}=\bar{a}-2\bar{b}+3\bar{c}+s(-4\bar{b}+3\bar{c}) \quad ...(2)$$

$$=\bar{a}+(-2-4s)\bar{b}+(3+3s)\bar{c}$$

$$(2+t)\bar{a}+(3+t)\bar{b}+(-1-t)\bar{c}$$

$$=\bar{a}+(-2-48)\bar{b}+(3+38)\bar{c}$$

Comparing $\bar{a},\bar{b},\bar{c}$ coefficients on both sides

$2 + t = 1 \Rightarrow t = -1$

$3 = t = -2 - 4s \Rightarrow 2 = -2 + 4s \Rightarrow s = -1$

$-1 - t = 3 + 3s \Rightarrow 3x + t = -4$

Substitute t in (1)

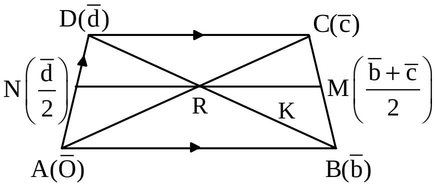
$$\bar{r}=2\bar{a}+3\bar{b}-\bar{c}+(-1)(\bar{a}+\bar{b}-\bar{c})$$

$$=2\bar{a}+3\bar{b}-\bar{c}-\bar{a}-\bar{b}+\bar{c}$$

$$\bar{r}=\bar{a}+2\bar{b}$$

31. In a quadrilateral ABCD. If the mid points of one pair of opposite sides and the point of intersection of the diagonals are collinear, using vector methods, prove that the quadrilateral ABCD is a trapezium.

Sol.



Let ABCD be the quadrilateral M, N are the mid points of the sides BC, AD respectively. R be the point of intersection of the diagonals. Given that M, R, N are collinear.

Let A be the origin

$\overline{AB}=\overline{b}, \overline{AC}=\overline{c}$ and $\overline{AD}=\overline{d}$

$\overline{AM}=\dfrac{\overline{b}+\overline{c}}{2}, \overline{AN}=\dfrac{\overline{d}}{2}$

Also $\overline{AR}=s,\overline{AC}=s\overline{c}$ where s is scalar

Since M, R, N are collinear

$\overline{MN}=t(\overline{RN})$ where t is a scalar.

$\overline{AN}-\overline{AM}=t(\overline{AN}-\overline{AR})$

$\dfrac{\overline{d}}{2}-\dfrac{\overline{b}+\overline{c}}{2}=t\left(\dfrac{\overline{d}}{2}-s\overline{c}\right)$

$\dfrac{\overline{d}-\overline{b}-\overline{c}}{2}=\dfrac{t}{2}\left[\overline{d}-2s\overline{c}\right]$

$\overline{d}-\overline{b}-\overline{c}=t(\overline{d}-2s\overline{c})$

$\overline{d}-\overline{b}-\overline{c}=t\overline{d}-2st\overline{c}$

$\overline{d}-\overline{b}-\overline{c}+2st\overline{c}=\overline{b}$

$(1-t)\overline{d}+(-1+2st)\overline{c}=\overline{b}$ …(1)

Let R be divides $\overline{BD}$ in the ratio k : 1

$\overline{AR}=\dfrac{k\overline{d}+\overline{b}}{k+1}$ but $\overline{AR}=s\overline{c}$

$\Rightarrow \dfrac{k\overline{d}+\overline{b}}{k+1}=s\overline{c}$

$\Rightarrow k\overline{d}+\overline{b}=s(k+1)\overline{c}$

$\Rightarrow \bar{b} = s(k+1)\bar{c} - k\bar{d}$ ...(2)

(1) = (2)

$(1-t)\bar{d} + (2st-1)\bar{c} = s(k+1)\bar{c} - k\bar{d}$

2st – 1 = s(k + 1) ...(i)

1 – t = –k ⇒k + 1 = t

Put k + 1 = t in (i)

⇒ 2st – 1 = st⇒st = 1

Substituting st = 1 in equation (1)

$(1-t)\bar{d} + \bar{c} = \bar{b}$

$(1-t)\bar{d} = \bar{b} - \bar{c}$

$-(t-1)\bar{d} = -(\bar{c} - \bar{b})$

$\Rightarrow (t-1)\overline{AD} = \overline{AC} - \overline{AB} = \overline{BC}$

$\Rightarrow \overline{BC} = (t-1)\overline{AD}$

$\Rightarrow \overline{AD} // \overline{BC}$

⇒ ABCD is a trapezium.

32. Find the vector equation of the plane which passes through the points $2\bar{i} + 4\bar{j} + 2\bar{k}$, $2\bar{i} + 3\bar{j} + 5\bar{k}$ and parallel to the vector $3\bar{i} - 2\bar{j} + \bar{k}$. Also find the point where this plane meets the line joining the points $2\bar{i} + \bar{j} + 3\bar{k}$ and $4\bar{i} - 2\bar{j} + 3\bar{k}$.

Sol. $\bar{r} = (1-s)\bar{a} + s\bar{b} + t\bar{c}$

$\bar{a} = 2\bar{i} + 4\bar{j} + 2\bar{k}, \bar{b} = 2\bar{i} + 3\bar{j} + 5\bar{k}$ and

$\bar{c} = 3\bar{i} - 2\bar{j} + \bar{k}$

Equation of the plane is

$\bar{r} = (1-s)(2\bar{i} + 4\bar{j} + 2\bar{k}) + s(2\bar{i} + 3\bar{j} + 5\bar{k}) + t(3t - 2\bar{j} + \bar{k})$

$= [2 - 2s + 2s + 3t]\bar{i} + [4 - 4s + 3s - 2t]\bar{j} + [2 - 2s + 5s + t]t$

$\bar{r} = [3t + 2]\bar{i} + [4 - s - 2t]\bar{j} + [2 + 3\bar{j} + t]\bar{k}$ ...(1)

Equation of the line is

$\bar{r} = (1-p)\bar{a} + p\bar{b}$ where p is a scalar.

$\bar{r} = (1-p)(2\bar{i} + \bar{j} + 3\bar{k}) + p(4\bar{i} - 2\bar{j} + 3\bar{k})$

$= (2 - 2p + 4p)\bar{i} + (1 - p - 2p)\bar{j} + (3 - 3p + 3p)\bar{k}$

$\bar{r} = [2 + 2p]\bar{i} + [1 - 3p]\bar{j} + [3]\bar{k}$ ...(2)

At the point of intersection (1) = (2)

$(3t+2)\bar{i}+(4-s-2t)\bar{j}+(2+3s+t)\bar{k}=(2+2p)\bar{i}+(1-3p)\bar{j}+3\bar{k}$

By comparing the like term, we get

$3t+2=2p+2$

$3t-2p=0$ ...(i)

$4-s-2t=1-3p$

$2t+s-3p=3$ ...(ii)

$2+3s+t=3$

$3s+t=1$ ...(iii)

Solving (ii) and (iii)

(ii) $\times$ 3 $\Rightarrow$ 6t + 3s – 9p = 9

(iii) $\Rightarrow$ t + 3s = 1

5t – 9p = 8 …(iv)

Solving (i) – (iv) :

15t – 10 p = 0

15t – 27p = 24

17p = –24 $\Rightarrow p=-\frac{24}{17}$

To find point of intersection,

Put $p=-\frac{24}{17}$ in (2)

$$\bar{r}=\left(2-\frac{48}{17}\right)\bar{i}+\left(1+\frac{72}{17}\right)\bar{j}+3\bar{k}$$

$$=\left(-\frac{14}{17}\right)\bar{i}+\left(\frac{89}{17}\right)\bar{j}+3\bar{k}$$

$\therefore$ Point of intersection is $\left(-\frac{14}{17},\frac{89}{17},3\right)$.

33. Find the vector equation of the plane passing through the points $4\bar{i}-3\bar{j}-\bar{k}$, $3\bar{i}+7\bar{j}-10\bar{k}$ and $2\bar{i}+5\bar{j}-7\bar{k}$ and show that the point $\bar{i}+2\bar{j}-3\bar{k}$ lies in the plane.

Sol. Let $\bar{a}=4\bar{i}-3\bar{j}-\bar{k},\ \bar{b}=3\bar{i}+7\bar{j}-10\bar{k}$

$\bar{c}=2\bar{i}+5\bar{j}-7\bar{k}$

The vector equation of the plane passing through $\bar{a},\bar{b},\bar{c}$ is

$$\overline{r} = (1-s-t)\overline{a} + s\overline{b} + t\overline{c},\ s, t \in R$$
$$= \overline{a} + s(\overline{b} - \overline{a}) + t(\overline{c} - \overline{a})$$
$$= 4\overline{i} - 3\overline{j} - \overline{k} + s(3\overline{i} + 7\overline{j} - 10\overline{k} - 4\overline{i} + 3\overline{j} + 1\overline{c}) + t(2\overline{i} + 5\overline{j} - 7\overline{k} - 4\overline{i} + 3\overline{j} + \overline{k})$$
$$\overline{r} = 4\overline{i} - 3\overline{j} - \overline{k} + s(-\overline{i} + 10\overline{j} - 9\overline{k}) + t(-2\overline{i} + 8\overline{j} - 6\overline{k})$$

Suppose $\overline{r} = \overline{i} + 2\overline{j} - 3\overline{k}$ lies in the plane then

$$\overline{i} + 2\overline{j} - 3\overline{k} = 4\overline{i} - 3\overline{j} - \overline{k} + s(-\overline{i} + 10\overline{j} - 9\overline{k}) + t(-2\overline{i} + 8\overline{j} - 6\overline{k})$$

Comparing $\overline{i}, \overline{j}, \overline{k}$ on both sides

$$1 = 4 - s - 2t \Rightarrow 2t + s = 3 \quad ...(1)$$
$$2 = -3 + 10s + 8t \Rightarrow 8t + 10s = 5 \quad ...(2)$$
$$-3 = -1 - 9s - 6t \Rightarrow 6t + 9s = 2 \quad ...(3)$$

$(1) \times 3 - (3) \Rightarrow 6t + 3s = 9$

$6t + 9s = 2$

$-6s = 7 \Rightarrow s = -\frac{7}{6}$

From (1) $2t = 3 + \frac{7}{6} = \frac{25}{6} \Rightarrow t = \frac{25}{12}$

Substitute s, t in (2)

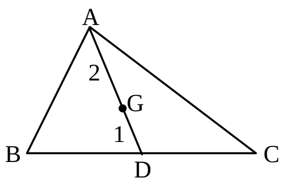
$$8 \cdot \frac{25}{12} + 10\left(-\frac{7}{6}\right) = 5$$
$$\Rightarrow \frac{50}{3} - \frac{35}{3} \Rightarrow \frac{15}{3} = 5 \Rightarrow 5 = 5$$

$\therefore$ Given vectors are collinear.

$\therefore$ The point $\overline{i} + 2\overline{j} - 3\overline{k}$ lies in the same plane.

34. In ΔABC, if $\overline{a}, \overline{b}, \overline{c}$ are position vectors of the vertices A, B and C respectively, then prove that the position vector of the centroid G is $\frac{1}{3}(\overline{a} + \overline{b} + \overline{c})$.

Sol.



Let G be the centroid of ΔABC and AD be the median through the vertex A. (see figure).

Then $\overline{AG}:\overline{GD}=2:1$

Since the position vector of $\overline{D}$ is $\frac{1}{2}(\overline{b}+\overline{c})$ by the Theorem 3.5.5, the position vector of

G is $\dfrac{\frac{2(\overline{b}+\overline{c})}{2}+1\overline{a}}{2+1}=\dfrac{\overline{a}+\overline{b}+\overline{c}}{3}$.

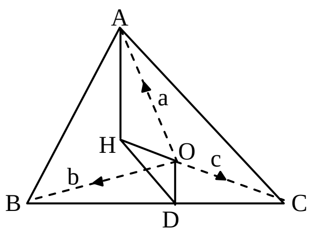
35. In ΔABC, if O is the circumcenter and H is the orthocenter, then show that

(i) $\overline{OA}+\overline{OB}+\overline{OC}=\overline{OH}$

(ii) $\overline{HA}+\overline{HB}+\overline{HC}=2\overline{HO}$

Sol. Let D be the mid point of BC.

i)



Take O as the origin,

Let $\overline{OA}=\overline{a},\overline{OB}=\overline{b}$ and $\overline{OC}=\overline{c}$

(see figure)

$\overline{OD}=\dfrac{\overline{b}+\overline{c}}{2}$

$\therefore \overline{OA}+\overline{OB}+\overline{OC}=\overline{OA}+2\overline{OD}=\overline{OA}+\overline{AH}=\overline{OH}$

(Observe that $\overline{AH}=2\overline{R}\cos A,\overline{OD}=\overline{R}\cos A,$

$\overline{R}$ is the circum radius of ΔABC and hence $\overline{AH}=2\overline{OD}$).

ii) $\overline{HA}+\overline{HB}+\overline{HC}=$

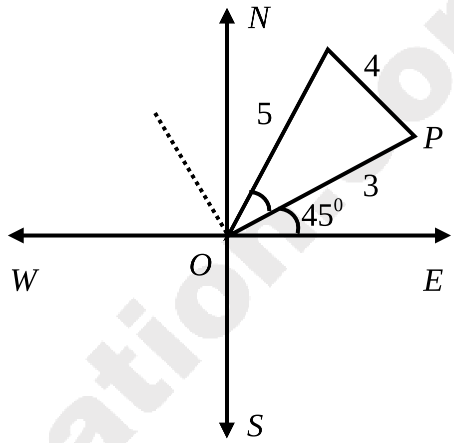
$\overline{HA}+2\overline{HD}=\overline{HA}+2(\overline{HO}+\overline{OD})$

$=\overline{HA}+2\overline{HO}+2\overline{OD}$

$=\overline{HA}+2\overline{HO}+\overline{AH}=2\overline{HO}$

36. In the Cartesian plane, O is the origin of the co-ordinate axes a person starts at 'o' and walks a distance of 3 units in the North –East direction and reaches the point P. From P he walks a distance 4 unit parallel to North-West direction and reaches the point Q. Express the vector OQ in terms of $\vec{c}$ and $\vec{j}$ observe that $\angle XOP = 45^0$

Sol: Given that OP=3



$\angle XOP = 45^0$

PQ=4

$\therefore OQ = 5$

Let $\angle POQ = \theta$

$\cos\theta = \frac{3}{5} \sin\theta = \frac{4}{5}$

$\theta = (5\cos(\theta + 45^0), 5\sin(\theta + 45^0))$

$= \left(\frac{5}{\sqrt{2}}(\cos\theta - \sin\theta), \frac{5}{\sqrt{2}}(\cos\theta + \sin\theta)\right)$

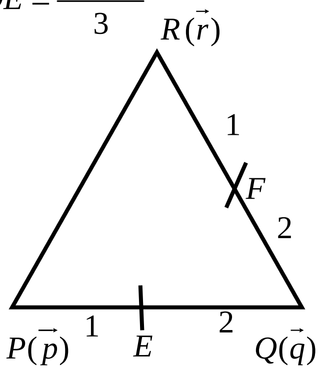
$= \left(\frac{\cancel{5}}{\sqrt{2}} \times \frac{1}{\cancel{5}}, \frac{\cancel{5}}{\sqrt{2}} \times \frac{7}{\cancel{5}}\right)$

$\overline{OQ} = -\frac{\vec{i}}{\sqrt{2}} + \frac{7\vec{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-\vec{i} + 7\vec{j})$

37. The point E divides the segment PQ internally in the ratio 1:2 and R is any point not on the line PQ. If F is a point on QR such that QF:FR=2:1 then show that EF is parallel to PR.

Sol: Let $\overline{OP} = \vec{p}$, $\overline{OQ} = \vec{q}$, $\overline{OR} = \vec{r}$ be the position vectors of P,Q, R, E divides PQ in the ratio 1:2

$\overline{OE} = \frac{\vec{q} + 2\vec{p}}{3}$



F divides QR in the ratio 2:1

$\overrightarrow{OF} = \overrightarrow{OF} - OE$

$= \dfrac{\vec{q}+2\vec{r}}{3} - \dfrac{q+2\overline{p}}{3}$

$= \dfrac{\vec{q}+2\vec{r}-q+2\overline{p}}{3}$

$= \dfrac{2}{3}(\vec{r} - \overline{p})$

$= \dfrac{2}{3}(\overrightarrow{OR} - \overrightarrow{OP})$

$\overrightarrow{EF} = \dfrac{2}{3}(\overrightarrow{PR})$

$\therefore \overrightarrow{EF}$ is parallel to $\overrightarrow{PR}$