

2.MATHEMATICAL INDUCTION

Principle of finite Mathematical Induction:

Let $\{P(n) / n \in N\}$ be a set of statements. If (i) $p(1)$ is true

(ii) $p(m)$ is true $\Rightarrow p(m+1)$ is true ; then $p(n)$ is true for every $n \in N$.

Principle of complete induction:

Let $\{P(n) / n \in N\}$ be a set of statements. If $p(1)$ is true and $p(2), p(3), \dots, p(m-1)$ are true $\Rightarrow p(m)$ is true, then $p(n)$ is true for every $n \in N$.

Note:

- (i) The principle of mathematical induction is a method of proof of a statement.
- (ii) We often use the finite mathematical induction, hence or otherwise specified the mathematical induction is the finite mathematical induction.

Some important formulae:

$$1. \sum n = \frac{n(n+1)}{2}$$

$$2. \sum n^2 = \frac{n(n+1)(2n+1)}{6};$$

$$3. \sum n^3 = \frac{n^2(n+1)^2}{4}$$

4. $a, (a+d), (a+2d), \dots$ are in a.p

$$\text{n}^{\text{th}} \text{ term } t_n = a + (n-1)d, \text{ sum of } n \text{ terms } S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a+l],$$

a = first term, l = last term.

5. a, ar, ar^2, \dots is a g.p.

$$\text{Nth term } t_n = a.r^{n-1}. \quad a = 1^{\text{st}} \text{ term}, \quad r = \text{common ratio.}$$

$$\text{Sum of } n \text{ terms } s_n = a \frac{(r^n - 1)}{r - 1}; r > 1, \quad = a \left(\frac{1 - r^n}{1 - r} \right); r < 1$$

Using mathematical induction prove the following

1. $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Sol: Let S_n be the given statement

For $n = 1$ L.H.S = 1

$$R.H.S = \frac{1(1+1)(2+1)}{6} = 1$$

L.H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding $(k+1)^{th}$ term on both sides i.e. $(k+1)^2$ on both sides

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \left\{ \frac{k(2k+1) + 6(k+1)}{6} \right\} \\ &= \frac{(k+1)\{2k^2 + 7k + 6\}}{6} \\ &= \frac{(k+1)(k+2)(k+3)}{6} \\ &= \frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6} \end{aligned}$$

$\therefore S_{k+1}$ is true

$\therefore S_n$ is true

2. **Prove that $2.3 + 3.4 + 4.5 + \dots$ Up to n terms $\frac{n(n^2 + 6n + 11)}{3}$**

Sol: 2, 3, 4..... n terms $t_n = 2 + (n-1)1 = n+1$

3, 4, 5..... n terms $t_n = 3 + (n-1)1 = n+2$

$$2.3 + 3.4 + 4.5 + \dots + (n+1)(n+2) = \frac{n(n^2 + 6n + 11)}{3}$$

Let S_n be the given statement

For $n = 1$ L.H.S = $2.3 = 6$

$$R.H.S = \frac{1(1+6+11)}{3} = 6$$

L. H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$\begin{aligned} \therefore 2.3 + 3.4 + 4.5 + \dots + (k+1)(k+2) &= \frac{k(k^2 + 6k + 11)}{3} + (k+2)(k+3) \\ &= \frac{k(k^2 + 6k + 11) + 3(k^2 + 5k + 6)}{3} \end{aligned}$$

$$\begin{aligned}
 &= \frac{k^3 + 9k^2 + 26k + 18}{3} \\
 &= \frac{(k+1)\{k^2 + 8k + 18\}}{3} \quad k = -1 \\
 &\quad \begin{array}{r|rrr}
 1 & 9 & 26 & 18 \\
 0 & -1 & -8 & -18 \\
 \hline
 1 & 8 & 18 & 0
 \end{array} \\
 &= \frac{(k+1)\{(k+1)^2 + 6(k+1) + 11\}}{3}
 \end{aligned}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

3. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Sol: Let $S_{(n)}$ be the given statement

For $n = 1 \quad L.H.S \quad \frac{1}{1.3} = \frac{1}{3}$

$$R.H.S = \frac{1}{2+1} = \frac{1}{3}$$

Assume S_k is true

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

Adding $(k+1)^{th}$ term i.e. $\frac{1}{(2k+1)(2k+3)}$ on both sides

$$\begin{aligned}
 &\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
 &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
 &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\
 &= \frac{k+1}{2k+3}
 \end{aligned}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

4. $4^3 + 8^3 + 12^3 + \dots \text{ up to } n \text{ terms} = 16n^2(n+1)^2$

Sol: Let $S_{(n)}$ be the given statements

For $n = 1 \quad L.H.S = 4^3 = 64$

$$R.H.S = 16(1)^2(1+1)^2 = 64$$

$\therefore L.H.S = R.H.S$

Hence $S_{(1)}$ is true

Assume $S_{(k)}$ is true

$$\therefore 4^3 + 8^3 + 12^3 + \dots + (4k)^3 = 16k^2(k+1)^2$$

Adding $\{4(k+1)\}^3$ on both sides

$$\begin{aligned}
 4^3 + 8^3 + 12^3 + \dots + (4k)^3 + \{4(k+1)\}^3 &= 16k^2 (k+1)^2 + 64(k+1)^3 \\
 4^3 + 8^3 + 12^3 + \dots + \{4(k+1)\}^3 &= 16(k+1)^2 \{k^2 + 4k + 4\} \\
 &= 16(k+1)^2 (k+2)^2
 \end{aligned}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

5. **a + (a+d) + (a+2d) + up to n terms** $a = \frac{n}{2}[2a + (n-1)d]$

Sol: $a + (a+d) + (a+2d) + \dots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$

Let $S_{(n)}$ be the given statement

For $n = 1$

L.H.S. = a

$$\text{R.H.S} = \frac{1}{2}[2a + (1-1)d] = a$$

L.H.S. = R.H.S

Assume S_k is true

$$a + (a+d) + (a+2d) + \dots + a + (k-1)d + (a+kd) = \frac{k}{2}[2a + (k-1)d] + (a+kd)$$

$$a + (a+d) + \dots + a + kd = ak + \frac{k}{2}(k-1)d + a + kd$$

$$= a(k+1) + d \left\{ \frac{k}{2}(k-1) + k \right\}$$

$$= a(k+1) + kd \left\{ \frac{k-1+2}{2} \right\}$$

$$= \left(\frac{k+1}{2} \right) \{2a + (kd)\}$$

$\therefore S_{k+1}$ is true

Hence S_n is true for all $n \in N$

6. **$a + ar + ar^2 + \dots$ upto n terms** $= \frac{a(r^n - 1)}{r-1}$ ($r \neq 1$)

Sol: $a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r-1}$

Let $S_{(n)}$ be the given statement

For $n = 1$

L.H.S = a

$$\text{R.H.S} = \frac{a(r-1)}{(r-1)} = a$$

\therefore L.H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$\therefore a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r-1}$$

Adding $a.r^k$ on both sides

$$\begin{aligned} a + ar + ar^2 + \dots + ar^{k-1} + ar^k &= \frac{a(r^k - 1)}{r - 1} + ar^k \\ a + ar + ar^2 + \dots + ar^k &= \frac{a\{r^k - 1 + r^{k+1} - r^k\}}{r - 1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} \end{aligned}$$

Hence S_{k+1} is true

$\therefore S_n$ is true $\forall n \in N$

7. $2 + 7 + 12 + \dots + (5n - 3) = \frac{n(Sn - 1)}{2}$

Sol: Let $S_{(n)}$ be the given statement

For $n = 1$

L.H.S = 2

$$\text{R.H.S} = \frac{1(5-1)}{2} = 2$$

$\therefore \text{L.H.S} = \text{R.H.S}$ for $n = 1$

Hence $S_{(1)}$ is true

Assume $S_{(k)}$ is true

$$\begin{aligned} 2 + 7 + 12 + \dots + (5k - 3) &= \frac{k(5k - 1)}{2} \\ &= \frac{5k^2 + 9k + 4}{2} = \frac{(k+1)\{5(k+1)-1\}}{2} \end{aligned}$$

$\Rightarrow S_{k+1}$ is true

Hence $S_{(n)}$ is true $\forall n \in N$

8. $4^n - 3n - 1$ is divisible by 9

Sol: Let $S_{(n)}$ be the given statement i.e. $S_{(n)} = 4^n - 3n - 1$

For $n = 1 \Rightarrow S_{(1)} = 4 - 3 - 1 = 0$ is divisible by 9

Assume $S_{(k)}$ is true i.e. $4^k - 3k - 1$ is divisible by 9

$$\begin{aligned} S_{k+1} &= 4^{k+1} - 3^{k+1} - 1 = 4^k \cdot 4 - 3k - 4 \\ &= \{9m + 3k + 1\}4 - 3k - 4 \\ &= 36m + 12k + 4 - 3k - 4 \\ &= 36m + 9k = 9\{4m + k\} \text{ this is} \end{aligned}$$

Divisible by 9

Hence S_{k+1} is divisible by 9

$\therefore S_n$ is divisible by 9 $\forall n \in N$

9. Show that $3.5^{2n+1} + 2^{3n+1} + 2^{3n+1}$ is divisible by 17

Sol: Let $S_{(n)} = 3.5^{2n+1} + 2^{3n+1} + 2^{3n+1}$ be the given statement

$$S_{(1)} = 3.5^3 + 2^4 = 375 + 16 = 391 = 17 \times 23$$

This is divisible by 17

Assume S_k is true

$$S_k = 3.5^{2k+1} + 2^{3k+1} + 2^{3k+1} \text{ is divisible by 17}$$

$$\text{Let } 3.5^{2k+1} + 2^{3k+1} = 17m$$

$$3.5^{2k+1} = 17m - 2^{3k+1}$$

$$S_{k+1} = 3.5^{2k+3} + 2^{3k+4}$$

$$= 3.5^{2k+1} \cdot 2^2 + 2^{3k+1} \cdot 2^3$$

$$= 25 \{17m - 2^{3k+1}\} + 2^{3k+1} \cdot 8$$

$$= 25 \times 17m - 17(2^{3k+1}) = 17 \{25m - 2^{3k+1}\} \text{ is divisible by 17}$$

Hence S_{k+1} is true

$\therefore S_n$ is true for all $n \in N$

10. $1.2.3+2.3.4+3.4.5+\dots\dots\dots\text{up to n terms} = \frac{n(n+1)(n+2)(n+3)}{4}$

Sol: $1.2.3 + 2.3.4 + 3.4.5 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Let $S_{(n)}$ be the given statement

For $n = 1$

$$\text{L.H.S} = 1.2.3 = 6$$

$$\text{R.H.S} = \frac{1(2)(3)(4)}{4} = 6$$

Assume $S_{(k)}$ is true

$$1.2.3 + 2.3.4 + 3.4.5 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

Adding $(k+1)(k+2)(k+3)$ on both sides

$$\therefore 1.2.3 + 2.3.4 + 3.4.5 + \dots + k(k+1)(k+2) + (k+2)(k+2)(k+3) = \frac{k(k+1)(k+2)(k+3)}{4}$$

S_{k+1} is true

Hence $S_{(n)}$ is true $\forall n \in N$

11. $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots \text{upto n terms} = \frac{n}{24}[2n^2 + 9n + 13]$

Sol: $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots + \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{1+3+5+\dots+(2n-1)} = \frac{n}{24}[2n^2 + 9n + 13]$

Let $S_{(n)}$ be the given statement

For $n = 1$

$$\text{L.H.S} = \frac{1^3}{1} = 1$$

$$\text{R.H.S} = \frac{1}{24}[2+9+13] = 1$$

L.H.S = R.H.S

Hence $S_{(1)}$ is true

Assume S_k is true

$$\therefore \frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots + \frac{1^3 + 2^3 + 3^3 + \dots + k^3}{1+3+5+\dots+2(k-1)} = \frac{k}{24} [2k^2 + 9k + 13]$$

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots + \frac{k^2(k+1)^2}{4k^2} = \frac{k}{24} [2k^2 + 9k + 13]$$

Adding $\frac{(k+2)^2}{4}$ on both sides

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots + \frac{(k+1)^2}{4} + \frac{(k+2)^2}{4} = \frac{k}{24} \{2k^2 + 9k + 13\} + \frac{(k+2)^2}{4}$$

$$\begin{aligned} \frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots + \frac{(k+2)^2}{4} &= \frac{k \{2k^2 + 9k + 13\} + 6(k^2 + 4k + 4)}{24} \\ &= \frac{2k^2 + 9k + 13 + 6k^2 + 24k + 24}{24} \end{aligned}$$

$$= \frac{2k^2 + 15k^2 + 37k + 24}{24}$$

$$k = -1 \left| \begin{array}{cccc} 2 & 15 & 37 & 2 \\ 0 & -2 & -13 & -24 \\ \hline 2 & 13 & 24 & 0 \end{array} \right.$$

$$= \frac{(k+1) \{ (2k^2 + 13k + 24) \}}{24}$$

$$= \frac{(k+1) \{ 2(k^2 + 2k + 1) + 9(k+1) + 13 \}}{24}$$

$$= \frac{(k+1) \{ 2(k+1)^2 + 9(k+1) + 13 \}}{24}$$

$\therefore S_{k+1}$ is true

$\therefore S_n$ is true $\forall n \in N$

12. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots \text{ up to } n \text{ terms} = \frac{n(n+1)^2(n+2)}{12}$

Sol: $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + (1^2 + 2^2 + 3^2 + \dots + n^2) = \frac{n(n+1)^2(n+2)}{12}$

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots = \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)^2(n+2)}{12}$$

Let $S_{(n)}$ be the given statement

For $n = 1$

L.H.S = 1

$$\text{R.H.S} = \frac{1(1+1)^2(1+2)}{12}$$

Assume S_k is true

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + \frac{k(k+1)(2k+1)}{6} = \frac{k(k+1)^2(k+2)}{12}$$

Adding $(k+1)^{\text{th}}$ term i.e. $= \frac{(k+1)(k+2)(2k+3)}{6}$ on both sides

$$\begin{aligned} 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + &= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{k(k+1)^2(k+2)}{12} + \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{k(k+1)^2(k+2) + 2(k+1)(k+2)(2k+3)}{12} \\ &= \frac{(k+1)(k+2)\{k^2 + k + 4k + 6\}}{12} \\ &= \frac{(k+1)(k+2)\{k^2 + 5k + 6\}}{12} \\ &= \frac{(k+1)(k+2)(k+2)(k+3)}{12} \\ &= \frac{(k+1)(k+2)^2(k+3)}{12} \end{aligned}$$

13. Using mathematical induction, prove the following statements, for all $n \in \mathbb{N}$.

$$\left(1 + \frac{3}{1}\right) + \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2$$

Sol. Let $S(n): \left(1 + \frac{3}{1}\right) + \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2$ be the given statement.

Let $n=1$

$$\text{L.H.S.} = 1 + 3 = 4$$

$$\text{R.H.S.} = (1 + 1)^2 = 2^2 = 4$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Therefore, $S(1)$ is true.

Let us assume that $S(k)$ is true.

$$\text{i.e. } \left(1 + \frac{3}{1}\right) + \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) = (k+1)^2$$

To prove $S(k+1)$ is true.

We know that $(k+1)$ th factor is :

$$1 + \frac{2(k+1)+1}{(k+1)^2} = \left[1 + \frac{2k+3}{(k+1)^2}\right]$$

$$\text{Consider } \left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) = (k+1)^2$$

$$\text{Multiplying both sides by } \left(1 + \frac{2k+3}{(k+1)^2}\right)$$

$$\begin{aligned}
 & \left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \dots \dots \quad \left(1 + \frac{2k+1}{k^2}\right) \left(1 + \frac{2k+3}{(k+1)^2}\right) \\
 &= (k+1)^2 \left(1 + \frac{2k+3}{(k+1)^2}\right) \\
 &= (k+1)^2 \frac{[(k+1)^2 + 2k+3]}{(k+1)^2} \\
 &= k^2 + 1 + 2k + 2k + 3 \\
 &= k^2 + 4k + 4 \\
 &= (k+2)^2 = [(k+1)+1]^2
 \end{aligned}$$

Thus $S(k+1)$ is true

By the principle of mathematical induction $S(n)$ is true for all $n \in N$.

14. If x and y are natural numbers and $x \neq y$, using mathematical induction show that

$x^n - y^n$ is divisible by $x - y$ for all $n \in N$.

Sol. Let $S(n)$: $x^n - y^n$ is divisible by $x - y$ be the given statement.

Put $n = 1$, $x^1 - y^1 = x - y$

$x^n - y^n$ is divisible by $x - y$

$\therefore S(1)$ is true for $n = 1$

Put $n = k$, $x^k - y^k = (x - y)p$ ($\because p$ is an integer)

$\therefore x^k - y^k = (x - y)p \dots (1)$

$S(k)$ is true for $n = k$

We know that,

$$\begin{aligned}
 x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\
 &= x^k \cdot x - x^k y + x^k y - y^k \cdot y \\
 &= x^k(x - y) + y(x^k - y^k) \\
 &= x^k(x - y) + y(x - y)p \quad (\because \text{from(1)}) \\
 &= (x - y)(x^k + yp) \\
 &= (x - y)q \quad (\because q \text{ is an integer})
 \end{aligned}$$

Since p is a polynomial in x and y , so is q .

Hence $x^{k+1} - y^{k+1}$ is divisible by $(x - y)$

$\therefore S(k+1)$ is true for $n = k + 1$

By the principle of mathematical induction, $S(n)$ is true for all $n \in N$.

15. **Show that $49^n + 16n - 1$ is divisible by 64 for all positive integers n.**

Sol. Let $S(n)$: $49^n + 16n - 1$ is divisible by 64 be the statement.

Since $49^1 + 16 \cdot 1 - 1 = 64$ is divisible by 64.

$\therefore S(n)$ is true for $n = 1$

Assume that the statement $S(n)$ is true for $n = k$

i.e. $49^n + 16n - 1$ is divisible by 64

Then $49^k + 16k - 1 = 64 M \quad \dots(1) (\because M \text{ is an integer})$

We show that the statement $S(n)$ is true for $n = k + 1$

i.e. we show that $49^{k+1} + 16(k+1) - 1$ is divisible by 64.

From (1), we have

$$49^k + 16k - 1 = 64 M$$

$$49^k = 64 M - 16k + 1$$

$$49^k \times 49 = (64 M - 16k + 1) \times 49$$

$$49^{k+1} + 16(k + 1) - 1 = (64M - 16k + 1)49 + 16(k + 1) - 1$$

$$= 64 \times 49 M - 49 \times 16k + 49 + 16k + 16 - 1$$

$$= 64 \times 49 M - 48 \times 16k + 64$$

$$= 64 \times 49 M - 64 \times 12k + 64$$

$$= 64(49 M - 12k + 1)$$

$$= 64 N [\because N \text{ is an integer}]$$

$\therefore S(n)$ is true for $n = k + 1$

\therefore By the principle of mathematical induction, $S(n)$ is true for all $n \in N$.