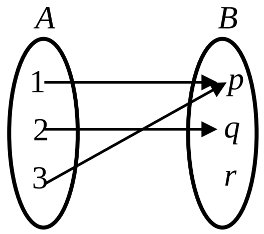
FUNCTIONS

Def 1: A relation f from a set A into a set B is said to be a function or mapping from A into B if for each $x \in A$ there exists a unique $y \in B$ such that $(x, y) \in f$. It is denoted by $f : A \to B$.

Note: Example of a function may be represented diagrammatically. The above example can be written diagrammatically as follows.



Def 2: A relation f from a set A into a set B is a said to be a function or mapping from a into B if i) $x \in A \Rightarrow f(x) \in B$ ii) $x_1, x_2 \in A,\ x_2 \Rightarrow f(x_1) = f(x_2)$

Def 3: If $f : A \to B$ is a function, then A is called domain, B is called codomain and $f(A) = \{f(x) : x \in A\}$ is called range of f.

Def 4: A function $f : A \to B$ if said to be one one function or injection from A into B if different element in A have different f-images in B.

Note: A function $f : A \to B$ is one one if f$(x_1, y) \in f, (x_2, y) \in f \Rightarrow x_1 = x_2$.

Note: A function $f : A \to B$ is one one iff $x_1, x_2 \in A,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Note: A function $f : A \to B$ is one one iff $x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Note: A function $f : A \to B$ which is not one one is called many one function

Note: If $f : A \to B$ is one one and A, B are finite then $n(A) \leq n(B)$.

Def 5: A function $f : A \to B$ is said to be onto function or surjection from A onto B if f(A) = B.

Note: A function $f : A \to B$ is onto if $y \in B \Downarrow \exists x \in A \ni f(x) = y$.

Note: A function $f : A \to B$ which is not onto is called an into function.

Note: If A, B are two finite sets and $f : A \to B$ is onto then $n(B) \leq n(A)$.

Note: If A, B are two finite sets and n (B) = 2, then the number of onto functions that can be defined from A onto B is $2^{n(A)} - 2$.

Def 6: A function $f : A \to B$ is said to be one one onto function or bijection from A onto B if $f : A \to B$ is both one one function and onto function.

Theorem: If $f: A \to B$, $g: B \to C$ are two functions then the composite relation gof is a function a into C.

Theorem: If $f: A \to B$, $g: B \to C$ are two one one onto functions then $gof: A \to C$ is also one one be onto.

Sol: i) Let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$.

$x_1, x_2 \in A,\ f: A \to B \Rightarrow f(x_1), f(x_2) \in B$

$f(x_1), f(x_2) \in B, \to C, f(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)] \Rightarrow (gof)(x_1) = (gof)(x_2)$

$x_1, x_2 \in A, (gof)(x_1) = (gof): A \to C$ is one one $\Rightarrow x_1 = x_2$

$\therefore x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$\therefore f: A \to B$ Is one one.

ii) Proof: let $z \in C, g: B \to C$ is onto $\exists y \in B \exists : g(y) = z$ $y \in B$ $f: A \to B$ is onto

$\therefore \exists x \in A \ni f(x) = y$

G {f(x)} = t

(g o f) x = t

$\forall z \in C \exists x \in A \ni (gof)(x) = z$.

$\therefore g$ is onto.

Def 7: Two functions $f: A \to B$, $g: C \to D$ are said to be equal if

i) A = C, B = D ii) $f(x) = g(x) \forall x \in A$. It is denoted by f = g

Theorem: If $f: A \to B, g: B \to C$, $h: C \to D$ are three functions, then $ho(gof) = (hof)of$

Theorem: if A is set, then the identify relation I on A is one one onto.

Def 8: If A is a set, then the function I on A defined by $I(x) = x \forall x \in A$, is called identify function on A. it is denoted by I_A.

Theorem: If $f: A \to B$ and I_A, I_B are identify functions on A, B respectively then $foI_A = I_B of = f$.

Proof: $I_A: A \to A$, $f: A \to B \Rightarrow foI_A: A \to B$

$f: A \to B$, $I_B: B \to B \Rightarrow I_B of: A \to B$

$(foI_A)(x) = f\{I_A(x)\} = f(x), \forall x \in A$. $\therefore f_0 I_A = f$

$(I_B of)(x) = I_B\{f(x)\} = f(x), \forall x \in A$ $\therefore I_B of = f$

$\therefore fo\, I_A = I_B of = f$

Def 9: If $f: A \to B$ is a function then $\{(y, x) \in B \times A : (x, y) \in f\}$ is called inverse of f. It is denoted by f^{-1}.

Def 10: If $f: A \to B$ is a bijection, then the function $f^{-1}: B \to A$ defined by $f^{-1}(y) = x$ iff $f(x) = y \forall y \in B$ is called inverse function of f.

Theorem: If $f: A \to B$ is a bijection, then $f^{-1} of = I_A,\ fof^{-1} = I_B$

Proof: Since $f: A \to B$ is a bijection $f^{-1}: B \to A$ is also a bijection and

$f^{-1}(y) = x \Leftrightarrow f(x) = y \forall y \in B$

$f: A \to B,\ f^{-1}: B \to A \Rightarrow f^{-1} of: A \to A$

Clearly $I_A: A \to A$ such that $I_A(x) = x,\ \forall x \in A$.

Let $x \in A$

$x \in A, f: A \to B \Rightarrow f(x) \in B$

Let y = f(x)

$y = f(x) \Rightarrow f^{-1}(y) = x$

$(f^{-1} of)(x) = f^{-1}[f(x) = f^{-1}(y) = x = I_A(x)$

$\therefore (f^{-1} of)(x) = I_A(x) \forall x \in A \qquad \therefore f^{1} of = I_A$

$f^{1}: B \to A,\ f: A \to B \to fof^{1}: B \to B$

Clearly $I_B: B \to B$ such that $I_B(y) = y \forall y \in B$

Let $y \in B$

$y \in B, f^{-1}: B \to A = f^{1}(y) \in A$

Let $f^{1}(y) = x$

$f^{1}(y) = x \Rightarrow f(x) = y$

$(fof^{1})(y) = f[f^{1}(y)] = f(x) = y = I_B(y)$

$\therefore (fof^{-1})(y) = I_B(y) \forall y \in B \qquad \therefore fof^{-1} = I_B$

Theorem: If $f: A \to B,\ g: B \to C$ are two bijections then $(gof)^{-1} = f^{-1} og^{-1}$.

Proof: $f: A \to B,\ g: B \to C$ are bijections $\Rightarrow gof: A \to C$ is bijection $\Rightarrow (gof)^{-1}: C \to A$ is a bijection.

$f: A \to B$ is a bijection $\Rightarrow f^{-1}: B \to A$ is a bijection

$g: B \to C$ Is a bijection $\Rightarrow g^{-1}: C \to B$ is a bijection

$g^{-1}: C \to B,\ g^{-1}: B \to A$ are bijections $\Rightarrow f^{-1} og^{-1}: C \to A$ is a bijection

Let $z \in C$

$z \in C, g: B \to C$ is onto $\Rightarrow \exists y \in B \ni g(y) = z \Rightarrow g^{-1}(z) = y$

$y \in B, f: A \to B$ is onto $\Rightarrow \exists x \in A \ni f(x) = y \Rightarrow f^{-1}(y) = x$

$(gof)(x) = g[f(x)] = g(y) = z \Rightarrow (gof)^{-1}(z) = x$

$\therefore (gof)^{-1}(z) = x = f^{-1}(y) = f^{-1}[g^{-1}(z)] = (f^{-1} og^{-1})(z) \qquad \therefore (gof)^{-1} = f^{-1} og^{-1}$

Theorem: If $f:A\to B$, $g:B\to A$ are two functions such that $gof=I_A$ and $fog=I_B$ then $f:A\to B$ is a bijection and $f^{-1}=g$.

Proof: Let $x_1,x_2\in A,\ f(x_1)=f(x_2)$

$x_1,x_2\in A,\ f:A\to B\Rightarrow f(x_1),f(x_2)\in B$

$f(x_1),f(x_2)\in B,\ f(x_1)=f(x_2)\ ,\ g=B\to A\Rightarrow g[f(x_1)]=g[f(x_2)]$

$\Rightarrow(gof)(x_1)=(gof)(x_2)\Rightarrow I_A(x_2)\Rightarrow x_1=x_2$

$\therefore x_1,x_2\in A,\ f(x_1)=f(x_2)\Rightarrow x_1=x_2$. $\therefore f:A\to B$ is one one

Let $y\in B$.

$y\in B, g:B\to A\Rightarrow g(y)\in A$

Def 11: A function $f:A\to B$ is said tobe a constant function if the range of f contain only one element i.e., $f(x)=c\forall x\in A$ where c is a fixed element of B

Def 12: A function $f:A\to B$ is said to be a real variable function if $A\subseteq R$.

Def 13: A function $f:A\to B$ is said to be a real valued function iff $B\subseteq R$.

Def 14: A function $f:A\to B$ is said to be a real function if $A\subseteq R, B\subseteq R$.

Def 15: If $f:A\to R$, $g:B\to R$ then $f+g:A\cap B\to R$ is defined as $(f+g)(x)=f(x)+g(x)\forall x\in A\cap B$

Def 16: If $f:A\to R$ and $k\in R$ then $kf:A\to R$ is defined as $(kf)(x)=kf(x),\forall x\in A$

Def 17: If $f:A\to B$, $g:B\to R$ then $fg:A\cap B\to R$ is defined as $(fg)(x)=f(x)g(x)\forall x\in A\cap B$.

Def 18: If $f:A\to R$, $g:B\to R$ then $\frac{f}{g}:C\to R$ is defined as $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$ $\forall x\in C$ where $C=\{x\in A\cap B: g(x)\neq 0\}$.

Def 19: If $f:A\to R$ then $|f|(x)=|f(x)|,\forall x\in A$

Def 20: If $n\in Z$,$n\geq 0, a_0,a_2,a_2,\ldots\ldots\ldots\ldots a_n\in R, a_n\neq 0$, then the function $f:R\to R$ defined by $f(x)=a_0+a_1x+a_2x^2+\ldots..+a_nx^n$ $\forall x\in R$ is called a polynomial function of degree n.

Def 21: If $f:R\to R$, $g:R\to R$ are two polynomial functions, then the quotient f/g is called a rational function.

Def 22: A function $f:A\to R$ is said to be bounded on A if there exists real numbers k_1,k_2 such that $k_1\leq f(x)\leq k_2\ \forall x\in A$

Def 23: A function $f:A\to R$ is said to be an even function if $f(-x)=f(x)\forall x\in A$

Def 24: A function $f:A\to R$ is said to be an odd function if $f(-x)=-f(x)\forall x\in A$.

Def 25: If $a \in R, a > 0$ then the function $f: R \to R$ defined as $f(x) = a^x$ is called an exponential function.

Def 26: If $a \in R$, $a > 0$, $a \neq 1$ then the function $f:(0,\infty) \to R$ defined as $f(x) = \log_a x$ is called a logarithmic function.

Def 27: The function $f: R \to R$ defined as f(x) = n where $n \in Z$ such that $n \le x < n+1$ ¸ $\forall x \in R$ is called step function or greatest integer function. It is denoted by f (x) = [x]

Def 28: The functions f(x) = sin x, cos x, tan x, cot x, sec x or cosec x are called trigonometric functions.

Def 29: The functions $f(x) = \sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \cot^{-1} x, \sec^{-1} x$ or $\cos ec^{-1} x$ are called inverse trigonometric functions.
Def 30: The functions f(x) = sinh x, cosh x, coth x, sech x or cosech x are called hyperbolic functions.

Def 31: The functions $f(x) = \sinh^{-1} x, \cos^{-1} x, \tanh^{-1} x, \coth^{-1} x, \sec h^{-1} x$ or $\cos ech^{-1} x$ are called iverse hyperbolic functions

	Function	**Domain**	**Range**
1.	a^x	R	$(0,\infty)$
2.	$\log_a x$	$(0,\infty)$	R
3.	[X]	R	Z
4.	[X]	R	$[0,\infty)$
5.	$\sqrt{x}$	$[0,\infty)$	$[0,\infty)$
6.	$\sin x$	R	[-1, 1]
7.	$\cos x$	R	[-1, 1]
8.	$\tan x$	$R - \{(2n+1)\frac{\pi}{2} : n \in Z\}$	R
9.	$\cot x$	$R - \{n\pi : n \in Z\}$	R
10.	$\sec x$	$R - \{(2n+1)\frac{\pi}{2} : n \in Z\}$	$(-\infty,-1] \cup [1,\infty)$
11.	$\cos ec\, x$	$R - \{n\pi : n \in Z\}$	$(-\infty,-1] \cup [1,\infty)$
12.	$Sin^{-1}x$	[-1 , 1]	$[-\pi/2, \pi/2]$
13.	$Cos^{-1} x$	[-1, 1]	$[0,\pi]$
14.	$Tan^{-1}x$	R	$(-\pi/2, \pi/2)$
15.	$Cot^{-1}x$	R	$(0,\pi)$
16.	$Sec^{-1}x$	$(-\infty,-1] \cup [1,\infty)$	$[0,\pi/2) \cup (\pi/2,\pi]$
17.	$Co\sec^{-1} x$	$(-\infty,-1] \cup [1,\infty)$	$[-\pi/2,0) \cup (0,\pi/2]$
18.	sinh x	R	R
19.	cosh x	R	$[1,\infty)$
20.	tanh x	R	$(-1,1)$
21.	coth x	$(-\infty,0) \cup (0,\infty)$	$(-\infty,-1) \cup (1,\infty)$
22.	sech x	R	(0, 1]
23.	cosech x	$(-\infty,0) \cup (0,\infty)$	$(-\infty,0) \cup (0,\infty)$

24.	$Sinh^{-1} x$	R	R
25.	$Cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
26.	$Tanh^{-1}x$	(-1, 1)	R
27.	$Coth^{-1} x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$
28.	$Sech^{-1} x$	(0, 1]	$[0,\infty)$
29.	$Coseh^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$

PROBLEMS

VSAQ'S

1. If : R – {0}→ is defined by $f(x)=x^3-\frac{1}{x^3}$, then show that $f(x)+f\left(\frac{1}{x}\right)=0$.

Sol. Given that $f(x)=x^3-\frac{1}{x^3}$

$$f\left(\frac{1}{x}\right)=\frac{1}{x^3}-x^3$$

$$\therefore f(x)+f\left(\frac{1}{x}\right)=x^3-\frac{1}{x^3}+\frac{1}{x^3}-x^3=0$$

2. If f : R – [±1] → R is defined by $f(x)=\log\left|\frac{1+x}{1-x}\right|$, then show that $f\left(\frac{2x}{1+x^2}\right)=2f(x)$.

Sol. $f(x)=\log\left|\frac{1+x}{1-x}\right|$

$$f\left(\frac{2x}{1+x^2}\right)=\log\left|\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right|$$

$$=\log\left|\frac{x^2+1+2x}{x^2+1-2x}\right|=\log\left|\frac{(1+x)^2}{(1-x)^2}\right|$$

$$=\log\left|\left(\frac{1+x}{1-x}\right)^2\right|=2\log\left|\frac{1+x}{1-x}\right|=2f(x)$$

3. If A = {–2, –1, 0, 1, 2} and f : A → B is a surjection defined by f(x) = x^2 + x + 1, then find B.

Sol. Given that

$f(x) = x^2 + x + 1$

$f(-2) = (-2)^2 - 2 + 1 = 4 - 2 + 1 = 3$

$f(-1) = (-1)^2 - 1 + 1 = 1 - 1 + 1 = 1$

$f(0) = (0)^2 - 0 + 1 = 1$

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = 2^2 + 2 + 1 = 7$

Thus range of f, f(A) = {1, 3, 7}

Since f is onto, f(A) = B

$\therefore$ B = {3, 1, 7}

4. If A = {1, 2, 3, 4} and f : A → R is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range of f.

Sol. Given that

$$f(x) = \frac{x^2 - x + 1}{x + 1}$$

$$f(1) = \frac{1^2 - 1 + 1}{1 + 1} = \frac{1}{2}$$

$$f(2) = \frac{2^2 - 2 + 1}{2 + 1} = \frac{3}{3} = 1$$

$$f(3) = \frac{3^2 - 3 + 1}{3 + 1} = \frac{7}{4}$$

$$f(4) = \frac{4^2 - 4 + 1}{4 + 1} = \frac{13}{5}$$

$\therefore$ Range of f is $\left\{\frac{1}{2}, 1, \frac{7}{4}, \frac{13}{5}\right\}$

5. If f(x + y) = f(xy) $\forall$x, y $\in$ R then prove that f is a constant function.

Sol. f(x + y) = f(xy)

Let f(0) = k

then f(x) = f(x + 0) = f(x · 0) = f(0) = k

$\Rightarrow$ f(x + y) = k

$\therefore$ f is a constant function.

6. Which of the following are injections or surjections or bijections? Justify your answers.

i) f : R → R defined by $f(x) = \frac{2x + 1}{3}$

ii) f : R → (0, ∞) defined by f(x) = 2^x.

iii) f : (0, ∞) → R defined by f(x) = $\log_e x$

iv) f : [0, ∞) → [0, ∞) defined by f(x) = x^2

v) f : R → [0, ∞) defined by f(x) = x^2

vi) f : R → R defined by f(x) = x^2

i) f : R → R defined by $f(x) = \frac{2x + 1}{3}$ is a bijection.

Sol. i) f : R → R defined by $f(x) = \frac{2x + 1}{3}$

a) To prove f : R → R is injection

Let $x_1, x_2 \in$ R and $f(x_1) = f(x_2)$

$\Rightarrow \frac{2x_1+1}{3} = \frac{2x_2+1}{3}$

$\Rightarrow 2x_1 + 1 = 2x_2 + 1$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

$\Rightarrow f : R \to R$ is injection

b) To prove $f : R \to R$ is surjection

Let $y \in R$ and $f(x) = y$

$\Rightarrow \frac{2x+1}{3} = y$

$\Rightarrow 2x + 1 = 3y$

$\Rightarrow 2x = 3y - 1$

$\Rightarrow x = \frac{3y-1}{2}$

Thus for every $y \in R$, $\exists$ an element $\frac{3y-1}{2} \in R$ such that

$$f\left(\frac{3y-1}{2}\right) = \frac{2\left(\frac{3y-1}{2}\right)+1}{3} = \frac{3y-1+1}{3} = y$$

$\therefore f : R \to R$ is both injection and surjection

$\therefore f : R \to R$ is a bijection.

ii) $f : R \to (0, \infty)$ defined by $f(x) = 2^x$.

a) To prove $f : R \to R^+$ is injection

Let $x_1, x_2 \in R$ and

$f(x_1) = f(x_2)$

$\Rightarrow 2^{x_1} = 2^{x_2}$

$\Rightarrow x_1 = x_2$

$\therefore f : R \to R^+$ is injection.

b) To prove $f : R \to R^+$ is surjection

Let $y \in R^+$ and $f(x) = y$

$\Rightarrow 2^x = y$

$\Rightarrow x = \log_2 y \in R$

Thus for every $y \in R^+$, $\exists$ an element

$\log_2 y \in$ such that

$f(\log_2 y) = 2^{\log_2 y} = y$

$\therefore f : R \to R^+$ is a surjection

Thus $f : R \to R^+$ is both injection and surjection.

$\therefore f : R \to R^+$ is a bijection.

iii) f : (0, ∞) → R defined by f(x) = $\log_e x$

Explanation :

a) To prove f : R^+ → R is injection

Let $x_1, x_2 \in R^+$ and

$f(x_1) = f(x_2)$

$\Rightarrow \log_e x_1 = \log_e x_2$

$\Rightarrow x_1 = x_2$

$\therefore f : R^+ \to R$ is injection.

b) **To prove** f : R^+ → R is surjection

Let $y \in R$ and f(x) = y

$\Rightarrow \log_e x = y$

$\Rightarrow x = e^y \in R^+$

Thus for every $y \in R$, ∃ an element $e^y \in R^+$ such that

$f(e^y) = \log_e e^y = y \log_e e = y$

$\therefore f : R^+ \to R$ is surjection

Thus f : R^+ → R is both injection and surjection.

∴ f : R^+ → R is a bijection.

iv) f : [0, ∞) → [0, ∞) **defined by f(x) =** x^2

Explanation :

a) To prove f : A → A is injection

Let $x_1, x_2 \in A$ and

$f(x_1) = f(x_2)$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2 (\because x_1 \geq 0, x_2 \geq 0)$

$\therefore f : A \to A$ is injection

b) To prove f : A → A is surjection

Let $y \in A$ and f(x) = y

$\Rightarrow x^2 = y$

$\Rightarrow x = \sqrt{y} \in A$

Thus for every $y \in A$, ∃ an element $\sqrt{y} \in A$

Such that $f\left[\sqrt{y}\right] = \left(\sqrt{y}\right)^2 = y$

∴ f : A → A is a surjection

Thus f : A → A is both injection and surjection.

∴ f : [0, ∞) → [0, ∞) is a bijection.

v) f : R → [0, ∞) defined by f(x) = x^2

Explanation :

a) **To prove** f : R → A is not a injection

Since distinct elements have not having distinct f-images

For example :

$f(2) = 2^2 = 4 = (-2)^2 = f(-2)$

But $2 \neq -2$

b) **To prove f : R → A is surjection**

Let $y \in A$ and $f(x) = y$

$\Rightarrow x^2 = y$

$\Rightarrow x = \pm\sqrt{y} \in R$

Thus for every $y \in A$, $\exists$ an element $\pm\sqrt{y} \in R$ such that

$f\left(\pm\sqrt{y}\right) = \left(\pm\sqrt{y}\right)^2 = y$

$\therefore$ f : R → A is a surjection

Thus f : R → A is surjection only.

vi) f : R → R defined by f(x) = x^2

a) To prove f : R → R is not a injection

Since distinct element in set R are not having distinct f-images in R.

For example :

$f(2) = 2^2 = 4 = (-2)^2 = f(-2)$

But $2 \neq -2$

$\therefore$ f : R → R is not a injection.

b) To prove f : R → R is not surjection

$-1 \in R$, suppose $f(x) = -1$

$x^2 = -1$

$x = \sqrt{-1} \notin R$

$\therefore$ f : R → R is not surjection.

7. If g = {(1, 1), (2, 3), (3, 5), (4, 7)} is a function from A = {1, 2, 3, 4} to B = {1, 3, 5, 7}? If this is given by the formula g(x) = ax + b, then find a and b.

Sol. Given that

A = {1, 2, 3,4} and B = {1, 3, 5, 7} and

g = {(1, 1), (2, 3), (3, 5), (4, 7)} …(1)

Clearly every element in set A has unique g-image in set B.

$\therefore$ g : A → B is a function.

Consider, g(x) = ax + b

g(1) = a + b

g(2) = 2a + b

g(3) = 3a + b

g(4) = 4a + b

$\therefore$ g = {(1, a + b), (2, 2a + b), (3, 3a + b), (4, 4a + b)} …(2)

Comparing (1) and (2)

$a + b = 1 \Rightarrow a = 1 - b \Rightarrow a = 1 + 1 = 2$

$2a + b = 3 \Rightarrow 2[1 - b] + b = 3$

$\Rightarrow 2 - 2b + b = 3 \Rightarrow 2 - b = 3 \Rightarrow b = -1$

8. If f(x) = 2, g(x) = x^2, h(x) = 2x for all x$\in$R, then find [fo(goh)(x)].

Sol. fo(goh)(x) = fog [h(x)]

= fog (2x)

= f [g(2x)]

= f $(4x^2)$ = 1

$\therefore$ fo(goh)(x) = 2.

9. Find the inverse of the following functions.

i) If a, b $\in$ R, f : R $\to$ R defined by f(x) = ax + b (a $\neq$ 0)

ii) f : R $\to$ (0, ∞) defined by f(x) = 5^x

iii) f : (0, ∞) $\to$ R defined by f(x) = $\log_2 x$.

Sol. i) Let f(x) = ax + b = y

$$\Rightarrow ax = y - b \Rightarrow x = \frac{y-b}{a}$$

Thus $f^{-1}(x) = \frac{x-b}{a}$

ii) Let f(x) = 5^x = y

$\Rightarrow x = \log_5 y$

Thus $f^{-1}(x) = \log_5 x$

iii) Let f(x) = $\log_2 x$ = y

$\Rightarrow x = 2^y$

$\Rightarrow f^{-1}(x) = 2^x$

10. If f(x) = 1 + x + x^2 + …… for |x| < 1 then show that $f^{-1}(x) = \frac{x-1}{x}$.

Sol. f(x) = 1 + x + x^2 + …… for |x| < 1

= $(1 - x)^{-1}$ by Binomial theorem for rational index

$$= \frac{1}{1-x} = y$$

$1 = y - xy$

$xy = y - 1$

$$x = \frac{y-1}{y}$$

$$f^{-1}(x) = \frac{x-1}{x}$$

11. If f : [1, ∞) → [1, ∞) defined by $f(x) = 2^{x(x-1)}$ then find $f^{-1}(x)$.

Sol. f(x) : [1 ……∞) → [1 ……∞)

$$f(x) = 2^{x(x-1)}$$

$$f(x) = 2^{x(x-1)} = y$$

$$x(x-1) = \log_2 y$$

$$x^2 - x - \log_2 y = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$

$$f^{-1}(x) = \frac{1 \pm \sqrt{1 + 4\log_2 x}}{2}$$

12. f(x) = 2x – 1, g(x) = $\frac{x+1}{2}$ for all x ∈ R, find gof(x).

Sol. gof(x) = g[f(x)] = g(2x – 1)

$$= \frac{2x - 1 + 1}{2} = \frac{2x}{2} = x$$

∴ gof(x) = x

13. Find the domain of the following real valued functions.

i) $f(x) = \frac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$

ii) $f(x) = \frac{1}{\log(2-x)}$

iii) $f(x) = \sqrt{4x - x^2}$

iv) $f(x) = \frac{1}{\sqrt{1 - x^2}}$

v) $f(x) = \sqrt{x^2 - 25}$

vi) $f(x) = \sqrt{x - [x]}$

vii) $f(x) = \sqrt{[x] - x}$

Sol. i) $f(x) = \frac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$

$$(x-1)(x-2)(x-3) \neq 0$$

$$\Rightarrow x - 1 \neq 0, x - 2 \neq 0, x - 3 \neq 0$$

$$\Rightarrow x \neq 1, x \neq 2, x \neq 3$$

$$\Rightarrow x \in R - \{1, 2, 3\}$$

$$\therefore \text{Domain of } f \text{ is } R - \{1, 2, 3\}$$

ii) $f(x) = \dfrac{1}{\log(2-x)}$

$2 - x > 0$ and $2 - x \neq 1$

$2 > x$ and $2 - 1 \neq x$

$x < 2$ and $x \neq 1$

$\therefore$ Domain of f is $(-\infty, 1) \cup (1, 2)$

iii) $f(x) = \sqrt{4x - x^2}$

$4x - x^2 \geq 0$

$x(4 - x) \geq 0$

$\Rightarrow 0 \leq x \leq 4$

Since the coefficient of x^2 is –ve

$\therefore$ Domain of f is [0, 4]

iv) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

$1 - x^2 > 0$

$\Rightarrow (1-x)(1+x) > 0$

$\Rightarrow -1 < x < 1$

Since the coefficient of x^2 is –ve

$\therefore$ Domain of f is (–1, 1).

v) $f(x) = \sqrt{x^2 - 25}$

$x^2 - 25 \geq 0$

$\Rightarrow (x-5)(x+5) \geq 0$

$\Rightarrow x \leq -5$ or $x \geq 5$

Since the coefficient of x^2 is +ve

$\therefore$ Domain of f is $(-\infty, -5] \cup [5, \infty)$

vi) $f(x) = \sqrt{x - [x]}$

$x - [x] \geq 0 \Rightarrow x \geq [x]$

It is true for all $x \in R$

$\therefore$ Domain of f is R.

vii) $f(x) = \sqrt{[x] - x}$

$\Rightarrow [x] - x \geq 0$

$\Rightarrow [x] \geq x$

It is true only when x is an integer

$\therefore$Domain of f is Z.

14. Find the ranges of the following real valued functions.

i) $\log|4-x^2|$ **ii)** $\sqrt{[x]-x}$

iii) $\frac{\sin\pi[x]}{1+[x]^2}$ **iv)** $\frac{x^2-4}{x-2}$

v) $\sqrt{9+x^2}$

Sol. i) f(x)= $\log|4-x^2|$

Domain of f is R – {–2, 2}

∴ Range = R

ii) f(x) = $\sqrt{[x]-x}$

Domain of f is Z

Range of f is {0}

iii) $\frac{\sin\pi[x]}{1+[x]^2}$

Domain of f is R

Range of f is {0}

Since sin nπ = 0, ∀ n ∈ Z.

iv) f(x) = $\frac{x^2-4}{x-2}$

Domain of f is R – {2}

Range of f is R – {4}

v) f(x) = $\sqrt{9+x^2}$

9 + x^2 > 0, ∀ x ∈ R

Domain of f is R

Range of f is [3, ∞)

SAQ'S

15. If the function f : R → R defined by $f(x)=\frac{3^x+3^{-x}}{2}$, then show that

f(x + y) + f(x – y) = 2f(x) f(y).

Sol. Given that

$f(x)=\frac{3^x+3^{-x}}{2}$ and $f(y)=\frac{3^y+3^{-y}}{2}$.

We have f(x + y) = $\frac{3^{x+y}+3^{-(x+y)}}{2}$

f(x – y) = $\frac{3^{x-y}+3^{-(x-y)}}{2}$

L.H.S. = f(x + y) + f(x – y)

$$= \frac{3^{x+y} + 3^{-(x+y)}}{2} + \frac{3^{x-y} + 3^{-(x-y)}}{2}$$

$$= \frac{1}{2}\left[3^{x+y} + 3^{-(x+y)} + 3^{x-y} + 3^{-(x-y)}\right] \ldots(1)$$

R.H.S. : 2 f(x) f(y) = $2\left[\frac{3^x + 3^{-x}}{2} \cdot \frac{3^y + 3^{-y}}{2}\right]$

$$= \frac{1}{2}\left[3^{x+y} + 3^{x-y} + 3^{y-x} + 3^{-x-y}\right]$$

$$= \frac{1}{2}\left[3^{x+y} + 3^{-(x-y)} + 3^{x-y} + 3^{-(x+y)}\right] \ldots(2)$$

From (1) and (2)

∴ L.H.S. = R.H.S.

f(x + y) + f(x – y) = 2 f(x) f(y)

16. If the function f : R → R defined by $f(x) = \frac{4^x}{4^x + 2}$, then show that f(1 – x) = 1 – f(x), and hence deduce the value of $f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)$.

Sol. Given that $f(x) = \frac{4^x}{4^x + 2}$

We obtain, f(1 – x) = $\frac{4^{1-x}}{4^{1-x} + 2}$

$$= \frac{\frac{4}{4^x}}{\frac{4}{4^x} + 2} = \frac{4}{4 + 2\cdot 4^x} = \frac{2}{2 + 4^x} \ldots(1)$$

$$1 - f(x) = 1 - \frac{4^x}{4^x + 2}$$

$$= \frac{4^x + 2 - 4^x}{4^x + 2} = \frac{2}{2 + 4^x} \ldots(2)$$

From (1) and (2) : f(1 – x) = 1 – f(x)

We have f(1 – x) = 1 – f(x)

Now, f(1 – x) + f(x)=1

Put x= ¼, then f(1 – 1/4) + f(1/4)=1

f(3/4) + f(1/4)=1-----------------------------(3)

f(1 – x) + f(x)=1 put x =1/2 then

f(1 – 1/2) + f(1/2)=1

f(1/2) + f(1/2)=1 => 2f(1/2) =1---------(4)

(3)+(4) => f(3/4) + f(1/4)+ 2f(1/2) = 2

Therefore, $f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+f\left(\frac{3}{4}\right)$. = **2.**

17. If the function f : {–1, 1} → {0, 2}, defined by f(x) = ax + b is a surjection, then find a and b.

Sol. Domain of f is {–1, 1} and

f(x) = ax + b

f(–1) = –a + b

f(1) = a + b

Case I : Suppose f = {(–1, 0), (1, 2)} …(1)

and f = {(–1, (–a + b)), (1, (a + b))} …(2)

Comparing (1) and (2)

–a + b = 0 ⇒ a = b

a + b = 2 ⇒ b + b = 2 (∵ a = b)

⇒ 2b = 2 ⇒ b = 1; a = 1.

Case II : Suppose f = {(–1, 2), (1, 0)} …(3)

and f = {(–1, (–a + b)), (1, (a + b))} …(4)

Comparing (3) and (4) we get

–a + b = 2 ⇒ a = b – 2

a + b = 0 ⇒ b = –a

Thus –a – a = 2

⇒ –2a = 2 ⇒ a = –1

⇒ b = –(–1) = 1

Thus a = –1, b = 1.

18. If f(x) = cos (log x), then show that $f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]=0$.

Sol. Given that f(x) = cos (log x)

Consider,

$$f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right)=\cos\left(\log\frac{1}{x}\right)\cos\left(\log\frac{1}{y}\right)$$

$$=\cos(\log x^{-1})\cos(\log y^{-1})$$

$$=[-\cos(\log x)][-\cos(\log y)]$$

$$=\cos(\log x)\cos(\log y)$$

$$\therefore f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right)=\cos(\log x)\cos(\log y) \quad \ldots(1)$$

Again

$$\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]=\frac{1}{2}\left[\cos\left(\log\frac{x}{y}\right)+\cos\log(xy)\right]$$

$$=\frac{1}{2}[\cos(\log x-\log y)+\cos(\log x+\log y)]$$

$$=\frac{1}{2}\cdot 2\cos(\log x)\cos(\log y)$$

$= \cos(\log x)\cos(\log y)$ $[\because \cos(A-B)+\cos(A+B)=2\cos A\cos B]$

$$\therefore \frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]=\cos(\log x)\cos(\log y)\quad \ldots(2)$$

(1) – (2) :

$$f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]=0.$$

19. If $f(y)=\frac{y}{\sqrt{1-y^2}}$ and $g(y)=\frac{y}{\sqrt{1+y^2}}$ then show that (fog)(y) = y.

Sol. Given that

$$f(y)=\frac{y}{\sqrt{1-y^2}} \text{ and } g(y)=\frac{y}{\sqrt{1+y^2}}$$

$$\therefore fog(y)=f[g(y)]=f\left[\frac{y}{\sqrt{1-y^2}}\right]$$

$$=\frac{y}{\sqrt{1+y^2}}\Bigg/\sqrt{1-\left(\frac{y}{\sqrt{1+y^2}}\right)^2}$$

$$=\frac{y}{\sqrt{1+y^2}}\times\frac{\sqrt{1+y^2}}{1+y^2-y^2}=y$$

$$\therefore fog(y)=y$$

20. If f : R → R and g : R → R are defined by f(x) = $2x^2$ + 3 and g(x) = 3x – 2 then find
(i) (fog)(x) (ii) gof(x)
(iii) fof (0) (iv) go(fof)(3)

Sol. i) fog(x) = f[g(x)]
= f(3x – 2)
= $2(3x-2)^2+3$
= $2[9x^2+4-12x]+3$
= $18x^2+8-24x+3$
= $18x^2-24x+11$

$\therefore$ (fog)(x) = $18x^2-24x+11$

ii) gof(x) = g[f(x)]
= $g(2x^2+3)$
= $3(2x^2+3)-2$
= $6x^2+9-2$

$= 6x^2 + 7$

$\therefore$ (gof)(x) $= 6x^2 + 7$

iii) fof(0) = f[f(0)]

$= f[2(0)^2 + 3]$

$= f(3) = 2(3)^2 + 3$

$= 2 \times 9 + 3 = 18 + 3 = 21$

$\therefore$ fof(0) = 21

iv) go(fof)(3) = gof[f(3)]

= gof(21)

= g[f(21)]

$= g[2(21)^2 + 3]$

= g[2 (441) + 3]

= g[882 + 3]

= g(885) = 3(885) – 2

= 2655 – 2 = 2653

$\therefore$ go(fof)(3) = 2653.

21. If f : R → R, g : R → R are defined by f(x) = 3x – 1, g(x) = x^2 + 1, then find (i) fof(x^2 + 1) (ii) fog(2), (iii) gof(2a – 3).

Sol. i) fof(x^2 + 1) $= f[f(x^2 + 1)]$

$= f[3(x^2 + 1) - 1]$

$= f[3x^2 + 3 - 1]$

$= f[3x^2 + 2]$

$= 3(3x^2 + 2) - 1$

$= 9x^2 + 6 - 1$

$= 9x^2 + 5$

fof(x^2 + 1) $= 9x^2 + 5$

ii) log(2) = f[g(2)]

$= f(2^2 + 1)$

= f(5)

= 3(5) – 1

= 15 – 1 = 14

$\therefore$ fog(2) = 14

iii)(gof)(2a – 3) = g [f(2a – 3)]

= g[3(2a – 3) – 1]

= g[6a – 9 – 1]

= g[6a – 10]

$= (6a - 10)^2 + 1$

$= 36a^2 + 100 - 120a + 1$

$= 36a^2 - 120a + 101$

(gof)(2a – 3) $= 36a^2 - 120a + 101$

22. If $f(x)=\frac{x-1}{x+1}, x \neq \pm 1$, show that fof^{-1}(x)=x.

Sol. Given that $f(x)=\frac{x-1}{x+1}$

Let y = f(x)

$$\Rightarrow y=\frac{x-1}{x+1} \Rightarrow x=\frac{1+y}{1-y}$$

$$f^{-1}(y)=\frac{1+y}{1-y}$$

$$\therefore f^{-1}(x)=\frac{1+x}{1-x}$$

$$\therefore fof^{-1}(x)=f[f^{-1}(x)]$$

$$=f\left[\frac{1+x}{1-x}\right]=\frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}$$

$$=\frac{1+x-1+x}{1+x+1-x}=\frac{2x}{2}=x$$

$$\therefore fof^{-1}(x)=x$$

23. If f : R → R, g : R → R defined by f(x) = 3x – 2, g(x) = x^2 + 1 then find (i) gof^{-1}(2), (ii) gof(x – 1).

Sol. i) Given that f(x) = 3x – 2

Let y = f(x)

y = 3x – 2

$$x=\frac{y+2}{3}$$

$$\therefore f^{-1}(x)=\frac{x+2}{3}$$

$$\therefore gof^{-1}(2)=g[f^{-1}(2)]$$

$$=g\left(\frac{2+2}{3}\right)=g\left(\frac{4}{3}\right)$$

$$=\left(\frac{4}{3}\right)^2+1=\frac{16}{9}+1=\frac{25}{9}$$

ii) gof(x – 1) = g[f(x – 1)]

$$=g[3(x-1)-2]$$

$$=g[3x-3-2]$$

$$=g[3x-5]$$

$$=(3x-5)^2+1$$

$$=9x^2+25-30x+1$$

$$=9x^2-3-x+26$$

24. Let f = {(1, a), (2, c), (4, d), (3, b)} and g^{-1} = {(2, a), (4, b), (1, c), (3, d)}, then show that $(gof)^{-1} = f^{-1}og^{-1}$.

Sol. Given that,

f = {(1, a), (2, c), (4, d), (3, b)}

$\Rightarrow f^{-1}$ = {(a, 1), (c, 2), (d, 4), (b, 3)}

g^{-1} = {(2, a), (4, b), (1, c), (3, d)}

$\Rightarrow$ g = {(a, 2), (b, 4), (c, 1), (d, 3)}

L.H.S. : gof = {(1, 2), (2, 1), (4, 3), (3, 4)}

$(gof)^{-1}$ = {(2, 1), (1, 2), (3, 4), (4, 3)}

R.H.S. :

$f^{-1}og^{-1}$ = {(2, 1), (4, 3), (1, 2), (3, 4)}

L.H.S. = R.H.S.

25. Let f : R → R, g : R → R are defined by f(x) = 2x – 3, g(x) = x^3 + 5 then find $(fog)^{-1}(x)$.

Sol. Given that,

f(x) = 2x – 3 and g(x) = $x^3 + 5$

$$\begin{aligned} fog(x) &= f[g(x)] \\ &= f(x^3+5) \\ &= 2(x^3+5)-3 \\ &= 2x^3+10-3 \\ &= 2x^3+7 \end{aligned}$$

$\therefore$ fog(x) = $2x^3 + 7$

Let y = fog(x)

$$y = 2x^3+7$$

$$x^3 = \frac{y-7}{2}$$

$$x = \sqrt[3]{\frac{y-7}{2}}$$

$$\therefore (fog)^{-1}(x) = \sqrt[3]{\frac{x-7}{2}}$$

$$\therefore (fog)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$$

26. If $f(x) = \frac{x+1}{x-1} (x \neq \pm 1)$ then find (fofof)(x) and (fofofof)(x).

Sol. Given that, $f(x) = \frac{x+1}{x-1}$

(fofof)(x) = (fof)[f(x)]

$$= fof\left(\frac{x+1}{x-1}\right) = f\left[f\left(\frac{x+1}{x-1}\right)\right]$$

$$= f\left[\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}\right] = f\left[\frac{x+1+x-1}{x+1-x+1}\right]$$

$$= f\left(\frac{2x}{2}\right) = f(x) = \frac{x+1}{x-1}$$

$$\therefore (fofof)(x) = \frac{x+1}{x-1}$$

$$(fofofof)(x) = f[(fofof)(x)]$$

$$= f\left(\frac{1+x}{1-x}\right)$$

$$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

$$\therefore (fofofof)(x) = x$$

27. If f and g are real valued functions defined by f(x) = 2x – 1 and g(x) = x^2 then find

(i) (3f – 2g)(x) **(ii) (fg)(x)** **(iii)** $\left(\frac{\sqrt{f}}{g}\right)(x)$ **(iv) (f + g + 2)(x)**

Sol. Given that f(x) = 2x – 1, g(x) = x^2

i) 3f = 3(2x – 1) = 6x – 3

$g(x) = x^2 \Rightarrow 2g = 2x^2$

$\therefore$ (3f – 2g)(x) = 3f(x) – 2g(x)

$= 6x - 3 - 2x^2$

$= -2x^2 + 6x - 3$

$= -[2x^2 - 6x + 3]$

ii) $(fg)(x) = f(x)g(x) = (2x-1)x^2 = 2x^3 - x^2$

iii) $\left(\frac{\sqrt{f}}{g}\right)(x) = \frac{\sqrt{f(x)}}{g(x)} = \frac{\sqrt{2x-1}}{x^2}$

iv) $(f+g+2)(x) = f(x) + g(x) + 2$

$= 2x - 1 + x^2 + 2$

$= x^2 + 2x + 1$

$= x^2 + x + x + 1$

$= x(x+1) + 1(x+1)$

$= (x+1)(x+1) = (x+1)^2$

28. If f = {(1, 2), (2, –3), (3, –1)} then find (i) 2f, (ii) 2 + f, (iii) f^2, (iv) $\sqrt{f}$.

Sol. Given that

f = {(1, 2), (2, –3), (3, –1)}

i) 2f = {(1, 2 × 2), (2, –3× 2), (3, –1 × 2)}

= {(1, 4), (2, –6), (3, –2)}

ii) 2 + f = {(1, 2+2), (2, –3+2), (3, –1+2)}

= {(1, 4), (2, –1), (3, 1)}

iii) f^2 = {(1, 2^2), (2, $(–3)^2$), (3, $(–1)^2$)}

= {(1, 4), (2, 9), (3, 1)}

iv) $\sqrt{f} = \left\{(1, \sqrt{2})\right\}$

29. Find the domains at the following real valued functions.

i) $f(x) = \sqrt{x^2 - 3x + 2}$

ii) $f(x) = \log(x^2 - 4x + 3)$

iii) $f(x) = \dfrac{\sqrt{2+x} + \sqrt{2-x}}{x}$

iv) $f(x) = \dfrac{1}{\sqrt[3]{x-2}\,\log_{(4-x)} 10}$

v) $f(x) = \sqrt{\dfrac{4-x^2}{[x]+2}}$

vi) $f(x) = \sqrt{\log_{0.3}(x - x^2)}$

vii) $f(x) = \dfrac{1}{x + |x|}$

Sol. i) $f(x) = \sqrt{x^2 - 3x + 2}$

$x^2 - 3x + 2 \ge 0$

$\Rightarrow (x-1)(x-2) \ge 0$

$\Rightarrow x \le 1 \text{ or } x \ge 2$

Since the coefficient of x^2 is +ve

Domain of f is (–∞, 1] ∪ [2, ∞)

ii) $f(x) = \log(x^2 - 4x + 3)$

$x^2 - 4x + 3 > 0$

$(x-1)(x-3) > 0$

$x < 1 \text{ or } x > 3$

Since the coefficient of x^2 is +ve

Domain of f is R – [1, 3]

iii) $f(x)=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{x}$

$$\begin{array}{c|c|c} 2+x\ge 0 & 2-x\ge 0 & x\ne 0 \\ \Rightarrow x\ge -2 & \Rightarrow 2\ge x & x\ne 0 \\ & \Rightarrow x\le 2 & \end{array}$$

$\therefore$ Domain of f is $[-2, 2]-\{0\}$

iv) $f(x)=\dfrac{1}{\sqrt[3]{x-2}\,\log_{(4-x)}10}$

$x-2\ne 0 \Rightarrow x\ne 2$

$4-x>0\ \ \&\ 4-x\ne 1 \Rightarrow 4-x\ne 1 \Rightarrow x\ne 3$

$\Rightarrow 4>x \Rightarrow x<4$

$\therefore$ Domain of f is $(-\infty, 2)\cup(2, 3)\cup(3, 4)$

or

Domain of f is $(-\infty, 4)-\{2, 3\}$

v) $f(x)=\sqrt{\dfrac{4-x^2}{[x]+2}}$

Case I :

$4-x^2\ge 0$

$(2+x)(2-x)\ge 0$

$\Rightarrow x\in[-2, 2]$ …(1)

Since the coefficient of x^2 is –ve

Also

$[x]+2>0$

$[x]>-2$

$x\in[-1, \infty)$ …(2)

From (1) and (2)

$x\in[-1, 2]$

Case II :

$4-x^2\le 0$

$x^2-4\le 0$

$(x+2)(x-2)\ge 0$

$x\in(-\infty, -2]\cup[2, \infty)$ …(3)

Since the coefficient of x^2 is +ve

Also $[x]+2<0$

$[x]<-2$

$x\in(-\infty, -2)$ …(4)

From (3) and (4)

$x\in(-\infty, -2)$

From case-I and case-II

Domain of f is $(-\infty, -2)\cup[-1, 2]$

vi) $f(x)=\sqrt{\log_{0.3}(x-x^2)}$

$\log_{0.3}(x-x^2)\geq 0$

$\Rightarrow (x-x^2)\leq (0.3)^0$

$\Rightarrow x-x^2\leq 1$

$\Rightarrow 0\leq x^2-x+1$

$\Rightarrow x^2-x+1\geq 0$

$\Rightarrow x^2-x+1>0,\ \forall x\in R$ …(1)

$x-x^2>0$

$\Rightarrow x^2-x<0$

$\Rightarrow x(x-1)<0$

$\Rightarrow 0<x<1$

Since the coefficient of x^2 is +ve

$\therefore\ x\in (0, 1)$ …(2)

From (1) and (2)

Domain of f is $R\cap(0, 1) = (0, 1)$

(or) Domain of f is (0, 1)

vii) $f(x)=\dfrac{1}{x+|x|}$

$x + |x| \neq 0$

$x \neq -|x|$

It is not holds good when $x\in(-\infty, 0]$

$\therefore$ Domain of f is $(0, \infty) = R^+$.

30. Prove that the real valued function $f(x)=\dfrac{x}{e^x-1}-\dfrac{x}{2}+1$ is an even function on R – {0}.

Sol. $f(x)=\dfrac{x}{e^x-1}-\dfrac{x}{2}+1$ …(1)

Let $x\in R-\{0\}$

Consider

$f(x)=\dfrac{-x}{e^{-x}-1}+\dfrac{x}{2}+1$

$=\dfrac{-x}{\dfrac{1}{e^x}-1}+\dfrac{x}{2}+1$

$=\dfrac{-xe^x}{1-e^x}+\dfrac{x}{2}+1=\dfrac{-xe^x}{-(e^x-1)}+\dfrac{x}{2}+1$

$=\dfrac{xe^x}{e^x-1}+\dfrac{x}{2}+1$ …(2)

Consider f(x) – f(–x)

$$= \frac{x}{e^x - 1} - \frac{x}{2} + 1 - \frac{xe^x}{e^x - 1} - \frac{x}{2} - 1$$

$$= \frac{x - xe^x}{e^x - 1} - \frac{2x}{2}$$

$$= \frac{x(e^x - 1)}{(e^x - 1)} - x$$

$$= x - x = 0$$

$f(x) - f(-x) = 0$

$\Rightarrow f(-x) = f(x)$

$\therefore$ f is an even function.

31. Find the domain and range of the following functions.

i) $f(x) = \frac{\tan \pi[x]}{1 + \sin \pi[x] + [x^2]}$

ii) $f(x) = \frac{x}{2 - 3x}$

iii) $f(x) = |x| + |1 + x|$

Sol. i) $f(x) = \frac{\tan \pi[x]}{1 + \sin \pi[x] + [x^2]}$

Domain of f is R ($\because$ tan nπ = 0, $\forall n \in Z$)

Range of f is {0}

ii) $f(x) = \frac{x}{2 - 3x}$

$2 - 3x \neq 0$

$2 \neq 3x$

$x \neq \frac{2}{3}$

Domain of f is $R - \left\{\frac{2}{3}\right\}$

$\frac{x}{2 - 3x} = y$

$\Rightarrow x = y(2 - 3x)$

$\Rightarrow x = 2y - 3yx$

$\Rightarrow x + 3yx = 2y$

$\Rightarrow x(1 + 3y) = 2y$

$\Rightarrow x = \frac{2y}{1 + 3y}$

$\Rightarrow 1 + 3y \neq 0$

$\Rightarrow 3y \neq -1$

$y \neq -\frac{1}{3}$

$\therefore$ Range of f is $R - \left\{-\frac{1}{3}\right\}$.

iii) $f(x) = |x| + |1+x|$

Domain of f is R

Range of f is [1, ∞)

32. Determine whether the function f : R → R defined by $f(x) = \begin{cases} x \text{ if } x > 2 \\ 5x - 2 \text{ if } x \le 2 \end{cases}$ is an injection or a surjection or a bijection.

Sol. Since 3 > 2, we have f(3) = 3

Since 1 < 2, we have f(1) = 5(1) – 2 = 3

$\therefore$ 1 and 3 have same f image.

Hence f is not an injection.

Let y ∈ R then y > 2 (or) y ≤ 2

If y > 2 take x = y ∈ R so that f(x) = x = y

If y ≤ 2 take

$$x = \frac{y+2}{5} \in R \text{ and } x = \frac{y+2}{5} < 1$$

$$\therefore f(x) = 5x - 2 = 5\left(\frac{y+2}{5}\right) - 2 = y$$

$\therefore$ f is a surjection

Since f is not an injection, it is not a bijection.

33. If f : R → R, g : R → R are defined by f(x) = 4x – 1 and g(x) = x^2 + 2 then find

(i) (gof)(x) (ii) $(gof)\left(\frac{a+1}{4}\right)$ (iii) fof(x) (iv) go(fof)(0)

Sol. i) (gof)(x) = g(f(x))

$$= g(4x-1)$$
$$= (4x-1)^2 + 2$$
$$= 16x^2 + 1 - 8x + 2$$
$$= 16x^2 - 8x + 3$$

ii) $$(gof)\left(\frac{a+1}{4}\right) = g\left[f\left(\frac{a+1}{4}\right)\right]$$
$$= g\left[4\left(\frac{a+1}{4}\right) - 1\right]$$
$$= g(a) = a^2 + 2$$

iii) fof(x) = f[f(x)]

$$= f(4x-1) = 4[4x-1] - 1$$
$$= 16x - 4 - 1$$
$$= 16x - 5$$

iv) go(fof)(0) $= go(fof)$

$= g[16\times 0 - 5]$

$= g[-5]$

$= (-5)^2 + 2$

$= 25 + 2 = 27$

34. If f : Q → Q is defined by f(x) = 5x + 4 for all x ∈ Q, show that f is a bijection and find f^{-1}.

Sol. Let $x_1, x_2 \in Q$, $f(x_1) = f(x_2)$

$\Rightarrow 5x_1 + 4 = 5x_2 + 4$

$\Rightarrow 5x_1 = 5x_2$

$\Rightarrow x_1 = x_2$

∴ f is an injection.

Let y ∈ Q, then $x = \frac{y-4}{5} \in Q$ and

$f(x) = f\left(\frac{y-4}{5}\right) = 5\left(\frac{y-4}{5}\right) + 4 = y$

∴ f is a surjection, f is a bijection.

∴ $f^{-1} : Q \to Q$ is a bijection.

We have $fof^{-1}(x) = 1(x)$

$f[f^{-1}(x)] = x$

$5f^{-1}(x) + 4 = x$

$f^{-1}(x) = \frac{x-4}{5}$ for all $x \in Q$

35. Find the domains of the following real valued functions.

i) $f(x) = \frac{1}{\sqrt{|x| - x}}$ **ii)** $f(x) = \sqrt{|x| - x}$

Sol. i) $f(x) = \frac{1}{\sqrt{|x| - x}} \in R$

$\Rightarrow |x| - x > 0$

$\Rightarrow |x| > x$

$\Rightarrow x \in (-\infty, 0)$

∴ Domain of f is $(-\infty, 0)$

ii) $f(x) = \sqrt{|x| - x}$

$\Rightarrow |x| - x \geq 0$, which is true $\forall\, x \in R$

∴ Domain of f is R.

36. If f = {(4, 5), (5, 6), (6, –4)} and g = {(4, –4), (6, 5), (8, 5)} then find

(i) f + g (ii) f – g (iii) 2f + 4g (iv) f + 4 (v) fg (vi) $\frac{f}{g}$

(vii) |f| (viii) $\sqrt{f}$ (ix) f^2 (x)f^3.

Sol. Domain of f = A = {4, 5, 6}

Domain of g = B = {4, 6, 8}

Domain of f ± g = A ∩ B = {4, 6}

i) f + g = {(4, 5 – 4), (6, –4 + 5)]

= {(4, 1), (6, 1)}

(ii) f – g = {(4, 5 + 4), (6, –4 – 5)]

= {(4, 9), (6, –9)}

(iii) 2f = {(4, 2×5), (5, 6×2), (6, –4×2)}

= {(4, 10), (5, 12), (6, –8)}

4g = {(4, –4×4), (6, 5×4), (8, 5×4)}

= {(4, –16), (6, 20), (8, 20)}

Domain of 2f + 4g = {4, 6}

∴2f+4g = {(4, 10, –16), (6, –8+20)}

= {(4, –6), (6, 12)}

(iv) f + 4 = {(4, 5+4), (5, 6+4), (6, –4+4)}

= {(4, 9), (5, 10), (6, 0)}

(v) fg = {(4, (5×–4)), (6, –4×5)}

= {(4, –20), (6, –20)}

(vi) $\frac{f}{g} = \left\{\left(4, \frac{-5}{4}\right), \left(6, \frac{-4}{5}\right)\right\}$

(vii) | f | = {(4, |5|), (5, |6|), (6, |–4|)}

= {(4, 5), (5, 6), (6, 4)}

(viii) $\sqrt{f} = \left\{(4, \sqrt{5}), (5, \sqrt{6})\right\}$

(ix) f^2 = {(4, 25), (5, 36), (6, 16)}

(x) f^3 = {(4, 125), (5, 216), (6, –64)}

37. Find the domain of the following real valued functions.

i) $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 2}}$ **ii)** $f(x) = \log(x - |x|)$

iii) $f(x) = \sqrt{\log_{10}\left(\frac{3-x}{x}\right)}$ **iv)** $f(x) = \sqrt{x+2} + \frac{1}{\log(1-x)}$

v) $f(x) = \frac{\sqrt{3+x} + \sqrt{3-x}}{x}$

Sol. i) $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 2}} \in R$

$\Leftrightarrow [x]^2 - [x] - 2 > 0$

$\Rightarrow ([x]+1)([x]-2) > 0$

$\Rightarrow [x] < -1$ (or) $[x] > 0$

But $[x] < -1$

$\Rightarrow [x] = -2, -3, -4, ...$

$\Rightarrow x < -1$

$[x] > 2 \Rightarrow [x] = 3, 4, ...$

$\Rightarrow x \geq 3$

$\therefore$ Domain of f = $(-\infty, -1) \cup [3, \infty)$ = R – [–1, 3)

ii) $f(x) = \log(x - |x|) \in R$

$\Leftrightarrow x - [x] > 0 \Leftrightarrow x > [x]$

$\Rightarrow$ x is a non-integer $\therefore$ Domain of f is R – Z.

iii) $f(x) = \sqrt{\log_{10}\left(\frac{3-x}{x}\right)} \in R$

$\log_{10}\left(\frac{3-x}{x}\right) \geq 0$ and $\frac{3-x}{x} > 0$

$\Rightarrow \frac{3-x}{x} \geq 10^0 = 1$ and $3 - x > 0, x > 0$

$\Rightarrow 3 - x \geq x$ and $0 < x < 3$

$\Rightarrow x \leq \frac{3}{2}$ and $0 < x < 3$

$\Rightarrow x \in \left(-\infty, \frac{3}{2}\right] \cap (0, 3) = \left(0, \frac{3}{2}\right]$

$\therefore$ Domain of f is $\left(0, \frac{3}{2}\right]$.

iv) $f(x) = \sqrt{x+2} + \frac{1}{\log(1-x)} \in R$

$x + 2 \geq 0$ and $1 - x > 0$ and $1 - x \neq 1$

$\Rightarrow x \geq -2$ and $1 > x$ and $x \neq 0$

$\Rightarrow x \in [-2, \infty) \cap (-\infty, 1) - \{0\}$

$\Rightarrow x \in [-2, 1) - \{0\}$

$\therefore$ Domain of f is $[-2, 1) - \{0\}$.

v) $f(x) = \frac{\sqrt{3+x} + \sqrt{3-x}}{x} \in R$

$\Leftrightarrow 3 + x \geq 0, 3 - x \geq 0, x \neq 0$

$\Rightarrow -3 \leq x \leq 3, x \neq 0$

$\Rightarrow x \in [-3, 3] - \{0\}$

$\therefore$ Domain of f is $[-3, 3] - \{0\}$.