MULTIPLE PRODUCTS

SYNOPSIS

1. **Scalar triple product**: If $\bar{a}$, $\bar{b}$ and $\bar{c}$ non-zero vectors then the scalar product of these vectors is given by $\bar{a}.(\bar{b}\times\bar{c})$ or $(\bar{a}\times\bar{b}).\bar{c}$ Or $(\bar{c}\times\bar{a}).\bar{b}$ etc..

i. $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}] = -[\bar{a}\ \bar{c}\ \bar{b}] = [\bar{b}\ \bar{a}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}]$

ii. If $\bar{a} = (a_1, a_2, a_3) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = (b_1, b_2, b_3) = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$, $\bar{c} = (c_1, c_2, c_3) = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$. Then $[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Note: All of the above relations hold for all non zero or some of non zero vectors.

iii. $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ if $\bar{a} = \bar{b}$ or $\bar{b} = \bar{c}$ or $\bar{c} = \bar{a}$

iv. If $[\bar{a}\ \bar{b}\ \bar{c}]$ = then vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

v. if $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0$ then vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non coplanar.

vi. $[\bar{a}\ \bar{b}\ \bar{c}] > 0$ then vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ triad of right handed system.

vii. $[\bar{a}\ \bar{b}\ \bar{c}] < 0$ then vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ triad of left handed system.

viii. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the coterminous edges of a parallelepiped then its volume is given as $\bar{V}$ $|[\bar{a}\ \bar{b}\ \bar{c}]|$ cu units.

ix. If $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$, and $\overline{OC} = \bar{c}$ then the volume of tetrahedron

OABC = $\frac{1}{6}|[\bar{a}\ \bar{b}\ \bar{c}]|$ cu units.

x. If A, B, C and D are the vectors of a tetrahedron whose P.V.s are $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ then its volume (V) is given as $\frac{1}{6}|[\overline{AB}\ \overline{AC}\ \overline{AD}]|$cu units.

$V = \frac{1}{6}|[\bar{a}-\bar{b}\ \ \bar{a}-\bar{c}\ \ \bar{a}-\bar{d}]|$cu units.

xi. If points $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are coplanar then $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{d}] + [\bar{c}\ \bar{a}\ \bar{d}] + [\bar{a}\ \bar{b}\ \bar{d}]$.

xii. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar then $[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}] = 0$.

xiii. If $\bar{a} + \bar{b} + \bar{c} = \bar{O}$ then $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar.

xiv. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are three mutually perpendicular unit vectors then $[\bar{a}\ \bar{b}\ \bar{c}] = \pm 1$.

xv. $[\bar{i}\ \bar{j}\ \bar{k}] = [\bar{j}\ \bar{k}\ \bar{i}] = [\bar{k}\ \bar{i}\ \bar{j}] = 1$

$[\bar{i}\ \bar{k}\ \bar{j}] = [\bar{k}\ \bar{j}\ \bar{i}] = [\bar{j}\ \bar{i}\ \bar{k}] = -1$

2. Cross Product of Three Vectors: For any three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ then cross product or vector product of these vectors are given as $\bar{a}\times(\bar{b}\times\bar{c})$, $(\bar{a}\times\bar{b})\times\bar{c}$ or $(\bar{b}\times\bar{c})\times\bar{a}$ etc.

i. $\bar{a}\times(\bar{b}\times\bar{c})$ is vector quantity and $|\bar{a}\times(\bar{b}\times\bar{c})| = |(\bar{b}\times\bar{c})\times\bar{a}|$

ii. In general $\bar{a}\times(\bar{b}\times\bar{c}) \neq (\bar{a}\times\bar{b})\times.\bar{c}$

iii. $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\times\bar{c}$ if $\bar{a}$ and $\bar{c}$ are collinear

iv. $\bar{a}\times(\bar{b}\times\bar{c}) = -(\bar{b}\times\bar{c})\times\bar{a}$

v. $(\bar{a}\times\bar{b})\times\bar{c} = -\bar{c}\times(\bar{a}\times\bar{b}) = (\bar{a}\ .\ \bar{c})\bar{b} - (\bar{a}\ .\ \bar{b})\bar{c} = \bar{a}\times(\bar{b}\times\bar{c})$

vi. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non zero vectors and $\bar{a}\times(\bar{b}\times\bar{c}) = \bar{O}$ then $\bar{b}$ and $\bar{c}$ are parallel (or collinear) vectors.

vii. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non zero and non parallel vectors then $\bar{a}\times(\bar{b}\times\bar{c})$, $\bar{b}\times(\bar{c}\times\bar{a})$ and $\bar{c}\times(\bar{a}\times\bar{b})$ are non collinear vectors.

viii. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are any three vectors then $\bar{a}(\bar{b}\times\bar{c}) + \bar{b}\times(\bar{c}\times\bar{a}) + \bar{c}\times(\bar{a}\times\bar{b}) = \bar{O}$

ix. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are any three vectors then $\bar{a}(\bar{b}\times\bar{c})$ + $\bar{b}\times(\bar{c}\times\bar{a})$ + $\bar{c}\times(\bar{a}\times\bar{b})$ are coplanar. [Since sum of these vectors is zero]

x. $\bar{a}(\bar{b}\times\bar{c})$ is vector lies in the plane of $\bar{b}$ and $\bar{c}$ or parallel to the plane of $\bar{b}$ and $\bar{c}$.

3. Dot product of four vectors: The dot product of four vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ is given as $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d})(\bar{b}.\bar{c})$

$$= \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix}$$

Some Meaning Less Products: $\bar{a}.(\bar{b}.\bar{c})$, $\bar{a}\times\bar{b}.(\bar{c}.\bar{d})$ $\bar{a}\times(\bar{b}.\bar{c})$ and

$(\bar{a}.\bar{b})\times(\bar{c}.\bar{d})$ Etc.

4. Cross Product Of Four Vectors: If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are any four vectors then $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a} = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

i. $[\bar{a}\ \bar{b}\ \bar{c}][\bar{l}\ \bar{m}\ \bar{n}] = \begin{vmatrix} \bar{a}.\bar{l} & \bar{b}.\bar{l} & \bar{c}.\bar{l} \\ \bar{a}.\bar{m} & \bar{b}.\bar{m} & \bar{c}.\bar{m} \\ \bar{a}.\bar{n} & \bar{b}.\bar{n} & \bar{c}.\bar{n} \end{vmatrix}$

5. Reciprocal System Of Vectors : If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non coplanar vectors then the triad $\bar{a}^{-1}$, $\bar{b}^{-1}$, $\bar{c}^{-1}$ of vectors given by $\bar{a}^{-1} = \frac{\bar{b}\times\bar{c}}{[\bar{a}\bar{b}\bar{c}]}$, $\bar{b}^{-1} = \frac{\bar{c}\times\bar{a}}{[\bar{a}\bar{b}\bar{c}]}$, $\bar{c}^{-1} = \frac{\bar{a}\times\bar{b}}{[\bar{a}\bar{b}\bar{c}]}$ is called reciprocal system vector triad of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

If $\bar{a}^{-1}$, $\bar{b}^{-1}$ and $\bar{c}^{-1}$ is reciprocal system of vectors triad of $\bar{a}$, $\bar{b}$ and $\bar{c}$ then

i. $\bar{a} \cdot \bar{a}^{-1} = \bar{b} \cdot \bar{b}^{-1} = \bar{c} \cdot \bar{c}^{-1} = 1$

ii. $\bar{a} \cdot \bar{b}^{-1} = \bar{b} \cdot \bar{c}^{-1} = \bar{c} \cdot \bar{a}^{-1} = 0.$

6. **Self Reciprocal System**: Let $\bar{a}, \bar{b}, \bar{c}$ be three vectors. If the reciprocal system of these vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is $\bar{a}, \bar{b}$ and $\bar{c}$ itself then the system of vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is called self reciprocal system of vectors.

7. **Vector Equation of a Plane Using Scalar Triple Product**:

- Vector equation of a plane passing through three non-collinear points having position vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is

  $|\bar{r} - \bar{a}\ \bar{b} - \bar{a}\ \bar{c} - \bar{a}| = 0$ (or)

  $\bar{r} \cdot (\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}) = [\bar{a}\ \bar{b}\ \bar{c}]$

- Vector equation of a plane passing through a given point with P.V. $\bar{a}$ and parallel to $\bar{b}, \bar{c}$ is $[\bar{r}-\bar{a}\ \bar{b}\ \bar{c}] = 0$ (or) $[\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$

- Vector equation of a plane passing through the points $\bar{a}, \bar{b}$ and parallel to $\bar{c}$ is $[\bar{r}-\bar{a}\ \bar{b}-\bar{a}\ \bar{c}] = 0.$

- Length of perpendicular from origin to the plane containing the points $\bar{a}, \bar{b}$ and $\bar{c}$ is

  $\dfrac{|[\bar{a}\bar{b}\bar{c}]|}{|\bar{b}\times\bar{c}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}|}.$

- Vector equation of a plane passing through origin and points $\bar{b}, \bar{c}$ is $[\bar{r}\ \bar{b}\ \bar{c}] = 0.$

- The shortest distance between the skew lines

  $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ is $\dfrac{|[\bar{a}-\bar{c}, \bar{b}, \bar{d}]|}{|\bar{b}\times\bar{d}|}$